Math 2173 Spring 2021 Recitation Handout 7 Solutions

Group Member 1: Sohail Farhangi

Group Member 2: ____________________

Group Member 3: ____________________

Group Member 4: ____________________

Group Member 5: ____________________

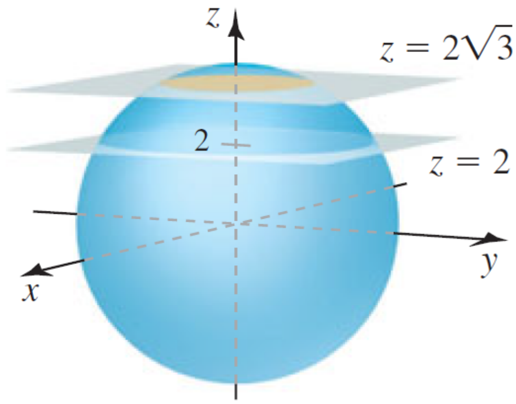
Group Member 6: ____________________

Below is a checklist of instructions to follow when completing this assignment. Failure to follow these directions will result in penalty on your final score and/or in some problems not being graded. If multiple directions are not followed, then it is also possible that the assignment will not be accepted for any credit at all. Please contact your TA or make a post on the discussion boards for this course if you have any questions about this assignment or these directions.

Sohail Farhangi: farhangi.3@osu.edu, Pan Yan: yan.669@osu.edu, Yilong Zhang: zhang.6100@osu.edu

Checklist of Instructions	
	Please clearly write the names of all group members working on this assignment in the spaces allotted above.
	This assignment must be completed by a group of 3, 4, 5, or 6 members.
	This assignment is to be uploaded to gradescope as a pdf file no later than 11:59 PM EST on Sunday, March 7.
	The assignment will be uploaded by 1 group member, and that group member will be responsible for manually entering the names of all other collaborators into gradescope.
	This assignment must be completed using this template. You may either print this template to write on it and then scan it (pages ordered correctly) into a pdf file, or you may write directly on the template using programs such as notability.
	If you need more space than what is given to solve a given problem, then you will find blank pages provided at the end of this template. At the end of each problem section of this assignment you will find a space in which to indicate on what page your work is continued in case you used additional pages to complete your solution. You must provide the page number on which your work is continued in the alloted space, or write 'N/A' incase you did not use any additional pages.
	On the additional pages, you will also find space in which to indicate which problem the page is being used for, and if the page is used then that space must also be filled.
	To complete this handout, you may use your textbook, class notes, discussions with your TA and group members, and any resources that are available on Carmen. You should not receive any help from the MSLC or people outside of your group when solving these problems. You may discuss these problems on the Carmen discussion boards, but you should not provide your entire solution when answering a such question, you should only give a hint or a helpful idea.

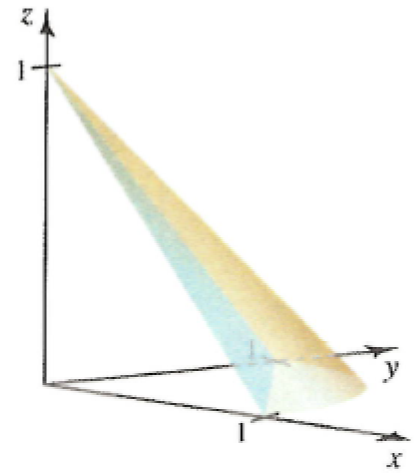
Ungraded Optional Problem 14.5.51: Let S be the solid region inside of the ball $\rho \leq 4$ that lies between the planes $z = 2$ and $z = 2\sqrt{3}$.



a. Find the volume of S using triple integration in spherical coordinates.
b. Find the volume of S using triple integration in cylindrical coordinates.

Note: You final answers for both parts of this problem should agree.

Problem 14.4.24 (10 points): Find the volume of the solid S in the first octant that is bounded by the cone $z = 1 - \sqrt{x^2 + y^2}$ and the plane $x + y + z = 1$.



Solution 1: We see that the horizontal cross sections of S are easy to study, so we choose an order of integration with dz on the outside and note that $0 \leq z \leq 1$. By examining the diagrams of the horizontal cross sections,

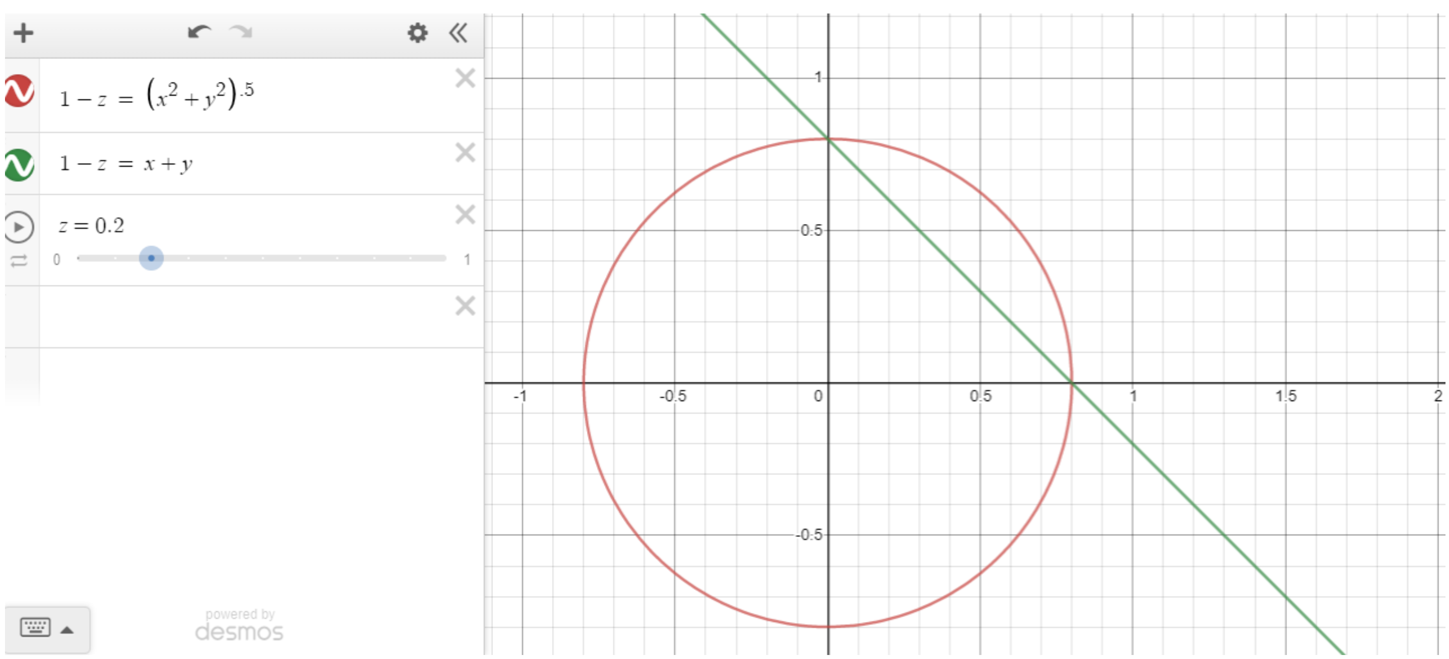


FIGURE 1. The cross section of S at a particular height z.

we see that

$$\text{Volume}(S) = \iiint_S 1dV = \int_0^1 \int_0^{1-z} \int_{1-z-y}^{\sqrt{(1-z)^2-y^2}} 1dxdydz \tag{1}$$

$$= \int_0^1 \int_0^{1-z} x\Big|_{1-z-y}^{\sqrt{(1-z)^2-y^2}} dydz \tag{2}$$

$$= \int_0^1 \int_0^{1-z} \left(\sqrt{(1-z)^2-y^2} - (1-z-y)\right) dydz. \tag{3}$$

We see that evaluating (the difficult part of) the inner integral in (3) is tantamount to evaluating

$$\int \sqrt{1-y^2}dy, \tag{4}$$

which is certainly possible, but it is difficult and computationally intensive, so we will evaluate the volume by an alternative method. If we more closely examine the integrals in (1), then we see that

$$\int_0^{1-z} \int_{1-z-y}^{\sqrt{(1-z)^2-y^2}} 1dxdy \tag{5}$$

calculates the area of the cross section C_z shown in figure 1. Using elementary Euclidena geometry, we see that

$$\int_0^{1-z} \int_{1-z-y}^{\sqrt{(1-z)^2-y^2}} 1dxdy = \text{Area}(C_z) \tag{6}$$
$$= \frac{1}{4}\pi(1-z)^2 - \frac{1}{2}(1-z)^2 = \frac{\pi-2}{4}(1-z)^2.$$

It follows that

$$(7)\quad \int_0^1\int_0^{1-z}\int_{1-z-y}^{\sqrt{(1-z)^2-y^2}} 1dxdydz = \int_0^1 \frac{\pi-2}{4}(1-z)^2dz$$
$$= -\frac{\pi-2}{12}(1-z)^3\Big|_0^1 = \boxed{\frac{\pi-2}{12}}.$$

Solution 2: Let C be the portion of the cone $z = 1-\sqrt{x^2+y^2}$ that is in the first quadrant and let T be the tetrahedron with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$. We see that S is simply the solid C with the solid T removed from it. Recalling that the volume of a cone of radius r and height h is $\frac{1}{3}\pi r^2h$, and that the volume of a tetrahedron with height h and a base of area b is $\frac{1}{3}bh$, we see that

$$(8)\quad \text{Vol}(S) = \text{Vol}(C) - \text{Vol}(T) = \frac{1}{3}\pi\cdot 1^2\cdot 1\underbrace{\cdot\frac{1}{4}}_{QI} - \frac{1}{3}\cdot\underbrace{(\frac{1}{2}\cdot 1\cdot 1)}_{\text{Area of base}}\cdot 1 = \boxed{\frac{\pi-2}{12}}.$$

Solution 3: We proceed as we did in Solution 2, but we will now derive the formula for the volume of C and T by using a triple integral in cylindrical coordinates for C and a triple integral in Cartesian coordinates for T. Recalling that the Cartesian equation $z = 1-\sqrt{x^2+y^2}$ is rewritten as $z = 1-r$ in cylindrical coordinates, we see that

$$(9)\quad \text{Vol}(S) = \text{Vol}(C) - \text{Vol}(T)$$

. . . .

$$(10)\quad = \int_0^{\frac{\pi}{2}}\int_0^1\int_0^{1-r} rdzdrd\theta - \int_0^1\int_0^{1-z}\int_0^{1-z-y} dxdydz$$

. . . .

$$(11)\quad = \int_0^{\frac{\pi}{2}}\int_0^1 rz\Big|_0^{1-r} drd\theta - \int_0^1\int_0^{1-z} x\Big|_0^{1-z-y} dydz$$

. . . .

$$= \int_0^{\frac{\pi}{2}} \int_0^1 (r - r^2) dr d\theta - \int_0^1 \int_0^{1-z} (1 - z - y) dy dz \tag{12}$$

$$= \int_0^{\frac{\pi}{2}} (\frac{1}{2}r^2 - \frac{1}{3}r^2 \Big|_0^1) d\theta - \int_0^1 ((1 - z)y - \frac{1}{2}y^2 \Big|_{y=0}^{1-z}) dy dz \tag{13}$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{6} d\theta - \int_0^1 \frac{1}{2}(1 - z)^2 dz \tag{14}$$

$$= \frac{1}{6}\theta \Big|_0^{\frac{\pi}{2}} - \frac{-}{1}6(1 - z)^3 \Big|_0^1 = \boxed{\frac{\pi - 2}{12}}. \tag{15}$$

Problem 14.4.46 (10 points): Evaluate

$$\int_1^4 \int_z^{4z} \int_0^{\pi^2} \frac{\sin(\sqrt{yz})}{x^{\frac{3}{2}}} dy dx dz. \tag{16}$$

Hint: Try a different order of integration.

Solution: We see that trying to evaluate the inner integral in the current order of integration is tantamount to evaluating

$$\int c_1 \sin(c_2\sqrt{y}) dy, \tag{17}$$

which is very difficult, so we decide to change the order of integration in hopes that the inner integral becomes easier to evaluate. We see that integrating with respect to z in the inner integral is not any easier since z and y are symmetric in the integrand, so we decide to integrate with respect to x in the inner integral in our new order of integration. Since z and y are symmetric in the integrand, the difficulty of the integrations doesn't seem to change if we use $dxdydz$ or $dxdzdy$, so we will use the order $dxdydz$ in order to reduce our workload by only changing the order of dx and dy instead of changing the order of dx, dy, and dz. We see that the bounds that we have in (16) tell us that

$$\begin{array}{lrcccl} \text{First pick a} & 1 & \le & z & \le & 4 \\ \text{Then pick a} & z & \le & x & \le & 4z \\ \text{Then pick a} & 0 & \le & y & \le & \pi^2 \end{array} \rightarrow \begin{array}{lrcccl} \text{First pick a} & 1 & \le & z & \le & 4 \\ \text{Then pick a} & 0 & \le & y & \le & \pi^2 \\ \text{Then pick a} & z & \le & x & \le & 4z \end{array}. \tag{18}$$

Thankfully, we didn't have to do any work to interchange the order of dx and dy since the bounds for y in the first order of integration were independent of x. In particular, we see that each of the horizontal cross sections represented by the bounds of the $dxdy$ double integral are rectangles (whose widths depend on z), and when we integrate over a rectangular region, we can switch the order of the integrals and the differentials without changing any of the bounds. We now see that

(19) $$\int_1^4\int_z^{4z}\int_0^{\pi^2}\frac{\sin(\sqrt{yz})}{x^{\frac{3}{2}}}dydxdz = \int_1^4\int_0^{\pi^2}\int_z^{4z}\sin(\sqrt{yz})x^{-\frac{3}{2}}dxdydz$$

. . . .

(20) $$= \int_1^4\int_0^{\pi^2} -2\sin(\sqrt{yz})x^{-\frac{1}{2}}\Big|_{x=z}^{4z}dydz$$

. . . .

(21) $$= \int_1^4\int_0^{\pi^2}\left(-2\sin(\sqrt{yz})(4z)^{-\frac{1}{2}} + 2\sin(\sqrt{yz})z^{-\frac{1}{2}}\right)dydz$$

. . . .

(22)
$$= \int_1^4\int_0^{\pi^2}\left(-\frac{\sin(\sqrt{yz})}{z^{\frac{1}{2}}} + 2\frac{\sin(\sqrt{yz})}{z^{\frac{1}{2}}}\right)dydz = \int_1^4\int_0^{\pi^2}\frac{\sin(\sqrt{yz})}{z^{\frac{1}{2}}}dydz.$$

We see that evaluating the inner integral at the end of (22) is again tantamount to evaluating the integral in (17), so we decide to change the order of integration once again. Note that this is equivalent to having decided to use the order $dxdzdy$ from the beginning, but we were not able to see that $dxdzdy$ was the best order of integration until now. Nonetheless, our initial change in the order of integration did allow us to make progress despite not being the best possible order of integration.

(23) $$\int_1^4\int_0^{\pi^2}\frac{\sin(\sqrt{yz})}{z^{\frac{1}{2}}}dydz = \int_0^{\pi^2}\int_1^4\frac{\sin(\sqrt{yz})}{z^{\frac{1}{2}}}dzdy.$$

Recalling that y does not change when evaluating the inner integral with respect to z, we treat y as a constant (relative to z) to perform the u-substituion

(24) $$u = \sqrt{yz}, du = \frac{\sqrt{y}}{2\sqrt{z}}dz, dz = \frac{2\sqrt{z}}{\sqrt{y}}du.$$

We now see that

$$\int_0^{\pi^2}\int_1^4 \frac{\sin(\sqrt{yz})}{z^{\frac{1}{2}}}dzdy = \int_0^{\pi^2}\int_{z=1}^4 \frac{2\sin(u)}{\sqrt{y}}dudy \tag{25}$$

. . . .

$$= \int_0^{\pi^2} \frac{-2\cos(u)}{\sqrt{y}}\Big|_{z=1}^4 dy = \int_0^{\pi^2} \frac{-2\cos(\sqrt{yz})}{\sqrt{y}}\Big|_{z=1}^4 dy \tag{26}$$

. . . .

$$= \int_0^{\pi^2}\left(\frac{-2\cos(\sqrt{4y})}{\sqrt{y}} + \frac{2\cos(\sqrt{y})}{\sqrt{y}}\right)dy \tag{27}$$

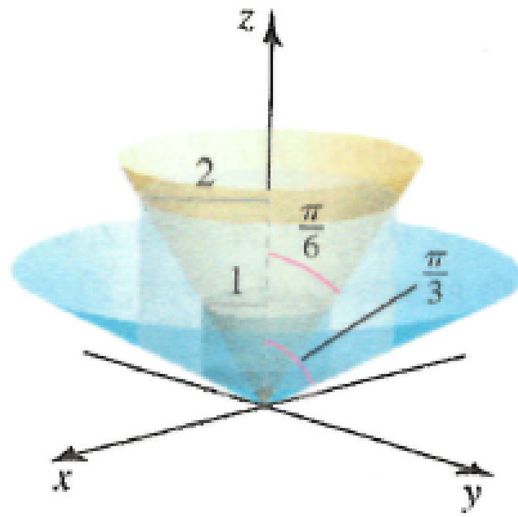
. . . .

$$\overset{u=\sqrt{y}}{=} \int_{y=0}^{\pi^2} \left(-4\cos(2u) + 4\cos(u)\right) du = \left(-2\sin(2u) + 4\sin(u)\right)\Big|_{y=0}^{\pi^2} \tag{28}$$

. . . .

$$= \left(-2\sin(2\sqrt{y}) + 4\sin(\sqrt{y})\right)\Big|_{y=0}^{\pi^2} = \boxed{0}. \tag{29}$$

Problem 14.5.50 (20 points): Let S be the solid region that is bounded by the cylinders $r = 1$ and $r = 2$, and the cones $\varphi = \frac{\pi}{6}$ and $\varphi = \frac{\pi}{3}$.



a. Find the volume of S using triple integration in spherical coordinates.
b. Find the volume of S using triple integration in cylindrical coordinates.

Note: You final answers for both parts of this problem should agree. You should also draw a separate diagram for your cross sections for each part of this problem.

Solution to (a): We will proceed by using spherical coordinates. Due to the symmetry of our solid with respect to θ we begin by taking a cross section with the xz-plane. Since we are working in spherical coordinates, the cross section will be in coordinates similar to polar coordinates. This time we will focus on the right of the z-axis (y-axis) in order to only see the part of the solid corresponding to $\theta = 0$.

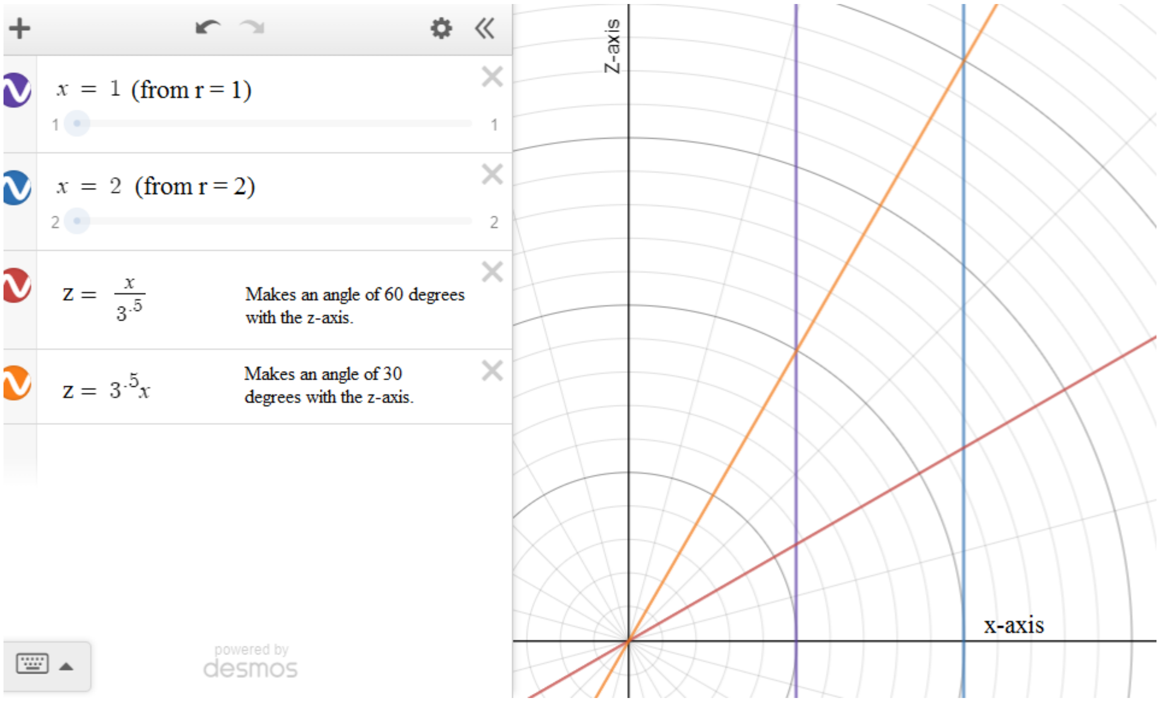


FIGURE 2. The xz-plane cross section in spherical coordinates.

We see that for any $0 \leq \theta < 2\pi$ we have $\frac{\pi}{6} \leq \varphi \leq \frac{\pi}{3}$. Noting that $r = \rho\sin(\varphi)$, we see that when $r = 1$ we have $\rho = \csc(\varphi)$ and when $r = 2$ we have $\rho = 2\csc(\varphi)$. It follows that once φ is also chosen we have $\csc(\varphi) \leq \rho \leq 2\csc(\varphi)$. We now see that the volume of the solid is given by

$$\text{(30)} \qquad \text{Volume}(S) = \iiint_S 1dV = \int_0^{2\pi}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\int_{\csc(\varphi)}^{2\csc(\varphi)} \rho^2\sin(\varphi)d\rho d\varphi d\theta$$

$$\text{(31)} \qquad = \int_0^{2\pi}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{3}\rho^3\sin(\varphi)\Big|_{\rho=\csc(\varphi)}^{2\csc(\varphi)} d\varphi d\theta = \int_0^{2\pi}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{7}{3}\csc^2(\varphi)d\varphi d\theta$$

$$\text{(32)} \qquad = \int_0^{2\pi} -\frac{7}{3}\cot(\varphi)\Big|_{\frac{\pi}{6}}^{\frac{\pi}{3}} d\theta = \int_0^{2\pi} \frac{14}{3\sqrt{3}}d\theta = \boxed{\frac{28\pi}{3\sqrt{3}}}.$$

Solution to (b): We will proceed by using cylindrical coordinates. Due to the symmetry of our solid with respect to θ we begin by taking a cross section with the xz-plane. Since we are working in spherical coordinates, the cross section will be in coordinates similar to Cartesian coordinates with (r, z) taking the place of (x, y). This time we will focus on the right of the z-axis (y-axis) in order to only see the part of the solid corresponding to $\theta = 0$.

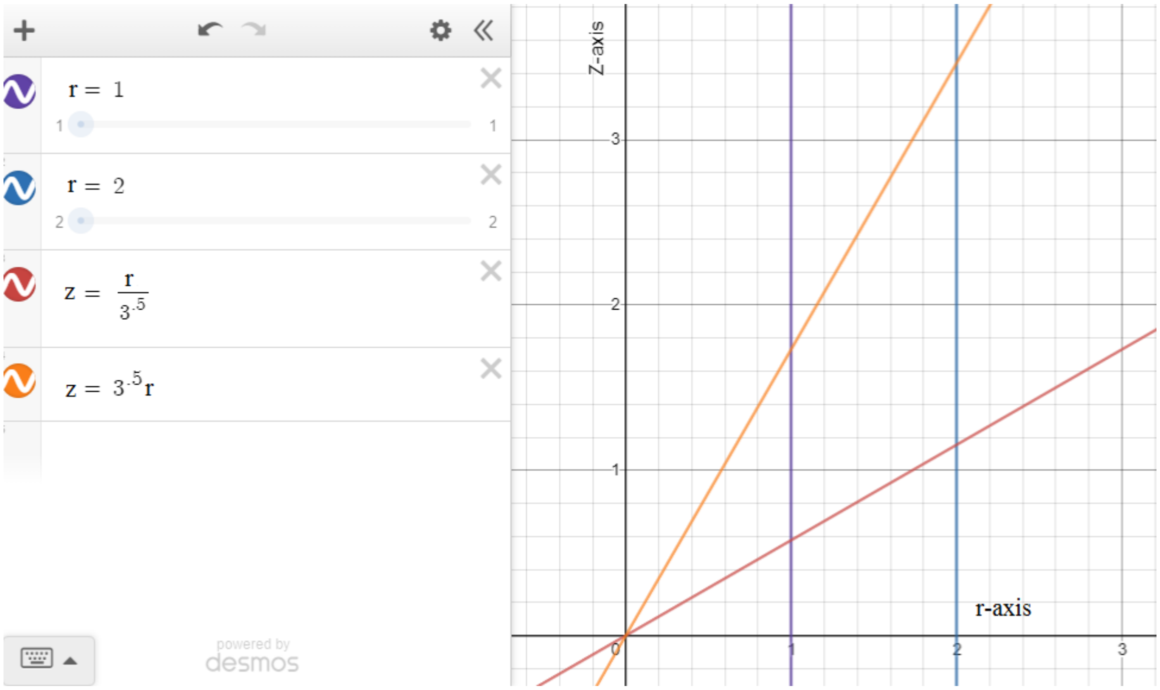


FIGURE 3. The xz-plane cross section in cylindrical coordinates.

We note that for any $0 \leq \theta < 2\pi$ we have $1 \leq r \leq 2$. Once r is also chosen, we see that $\frac{1}{\sqrt{3}}r \leq z \leq r\sqrt{3}$. We now see that the volume of the solid is given by

$$\text{Volume}(S) = \iiint_S 1dV = \int_0^{2\pi} \int_1^2 \int_{\frac{1}{\sqrt{3}}r}^{r\sqrt{3}} rdzdrd\theta \tag{33}$$

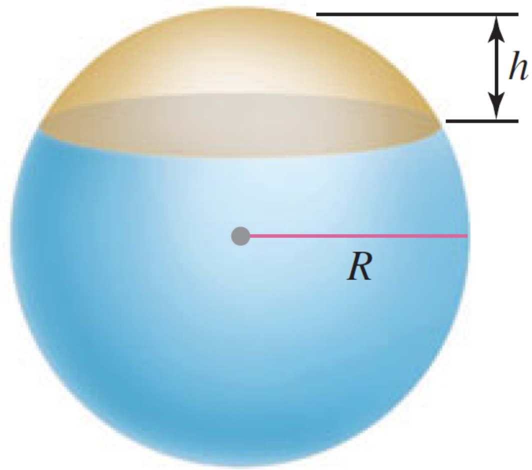
. . . .

$$= \int_0^{2\pi} \int_1^2 rz\Big|_{z=\frac{1}{\sqrt{3}}r}^{r\sqrt{3}} drd\theta = \int_0^{2\pi} \int_1^2 \frac{2}{\sqrt{3}}r^2drd\theta = \int_0^{2\pi} \frac{2}{3\sqrt{3}}r^3\Big|_1^2 d\theta \tag{34}$$

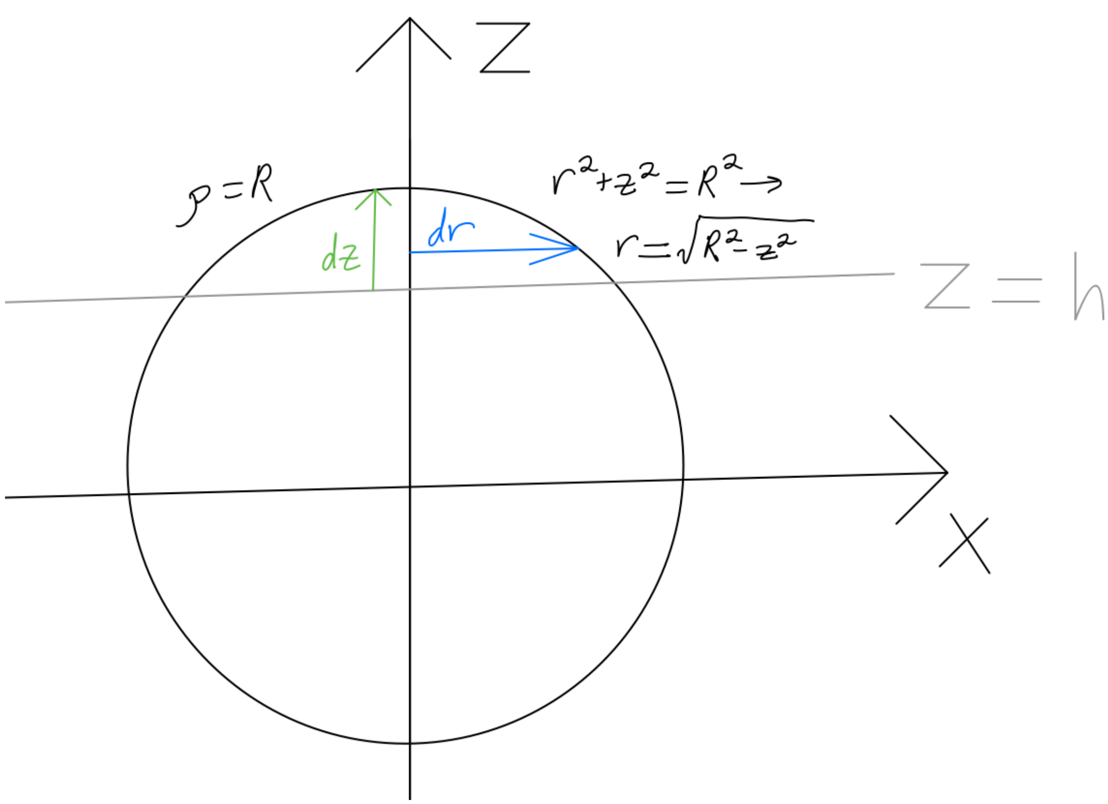
. . . .

$$= \int_0^{2\pi} \frac{14}{3\sqrt{3}}d\theta = \boxed{\frac{28\pi}{3\sqrt{3}}}. \tag{35}$$

Problem 14.5.78 (10 points): Find the volume of S, the cap of a sphere of radius R with thickness h.



Solution 1: We will first solve this problem using cylindrical coordinates. Due to the symmetry of our solid with respect to θ we begin by taking a cross section with the xz-plane, which corresponds to the $\theta = 0$ and $\theta = \pi$ cross sections combined. Since we are working in cylindrical coordinates, the cross section will be handled in coordinates similar to Cartesian coordinates.



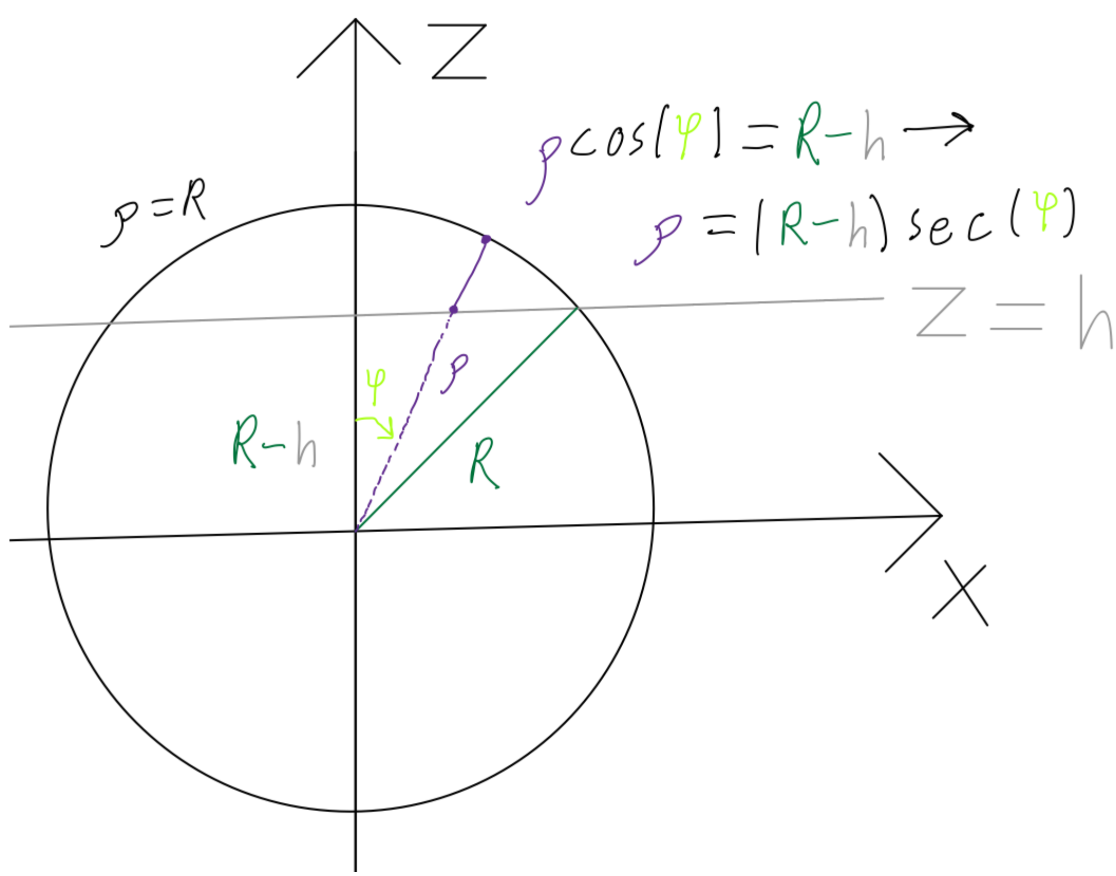
$$\text{Vol}(S) = \int_0^{2\pi}\int_{R-h}^{R}\int_0^{\sqrt{R^2-z^2}} r\,dr\,dz\,d\theta = \int_0^{2\pi}\int_{R-h}^{R} \frac{1}{2}r^2\Big|_{r=0}^{\sqrt{R^2-z^2}} dz\,d\theta \tag{36}$$

$$= \frac{1}{2}\int_0^{2\pi}\int_{R-h}^{R}(R^2-z^2)\,dz\,d\theta = \frac{1}{2}\int_0^{2\pi}\left(R^2z-\frac{1}{3}z^3\Big|_{z=R-h}^{R}\right)d\theta \tag{37}$$

$$= \frac{1}{2}\int_0^{2\pi}\left(R^3-\frac{1}{3}R^3-\left(R^2(R-h)-\frac{1}{3}(R-h)^3\right)\right)d\theta = \frac{1}{2}\int_0^{2\pi}\left(Rh^2-\frac{1}{3}h^3\right)d\theta \tag{38}$$

$$= \pi\left(Rh^2-\frac{1}{3}h^3\right) = \boxed{\frac{\pi}{3}h^2(3R-h)}. \tag{39}$$

Solution 2: We will now solve this problem using spherical coordinates. Due to the symmetry of our solid with respect to θ we once again begin by taking a cross section with the xz-plane. Since we are working in spherical coordinates, the cross section will be handled in coordinates similar to polar coordinates.



(40)
$$\text{Vol}(S) = \int_0^{2\pi}\int_0^{\cos^{-1}(\frac{R-h}{R})}\int_{(R-h)\sec\phi}^{R}\rho^2\sin(\varphi)d\rho d\varphi d\theta$$

(41)
$$= \int_0^{2\pi}\int_0^{\cos^{-1}(\frac{R-h}{R})}\frac{1}{3}\rho^3\sin(\varphi)\Big|_{\rho=(R-h)\sec\phi}^{R}d\varphi d\theta$$

(42)
$$= \frac{1}{3}\int_0^{2\pi}\int_0^{\cos^{-1}(\frac{R-h}{R})}(R^3\sin(\varphi) - (R-h)^3\sin(\varphi)\sec^3(\varphi))d\varphi d\theta$$

(43)
$$= \frac{2\pi}{3}\int_0^{\cos^{-1}(\frac{R-h}{R})}(R^3\sin(\varphi) - (R-h)^3\sin(\varphi)\sec^3(\varphi))d\varphi$$

(44)
$$= \frac{2\pi}{3}\left(\int_0^{\cos^{-1}(\frac{R-h}{R})}R^3\sin(\varphi)d\varphi - \int_0^{\cos^{-1}(\frac{R-h}{R})}(R-h)^3\sin(\varphi)\sec^3(\varphi))d\varphi\right)$$

(45)
$$= \frac{2\pi}{3}\left(-R^3\cos(\varphi)\Big|_0^{\cos^{-1}(\frac{R-h}{R})} - \int_0^{\cos^{-1}(\frac{R-h}{R})}(R-h)^3\tan(\varphi)\sec^2(\varphi))d\varphi\right)$$

(46)
$$\overset{u=\tan(\varphi)}{=} \frac{2\pi}{3}\left(-R^3(\frac{R-h}{R}) - (-R^3\cdot 1) - \frac{1}{2}(R-h)^3\tan^2(\varphi)\Big|_0^{\cos^{-1}(\frac{R-h}{R})}\right)$$

(47)
$$= \frac{2\pi}{3}\left(R^2h - \frac{1}{2}(R-h)^3\frac{1-\cos^2(\varphi)}{\cos^2(\varphi)}\Big|_0^{\cos^{-1}(\frac{R-h}{R})}\right)$$

$$= \frac{2\pi}{3}\left(R^2h - \frac{1}{2}(R-h)^3\frac{1-(\frac{R-h}{R})^2}{(\frac{R-h}{R})^2}\right) \tag{48}$$

. . . .

$$= \frac{2\pi}{3}\left(R^2h - \frac{1}{2}(R-h)^3\frac{R^2-(R-h)^2}{(R-h)^2}\right) \tag{49}$$

. . . .

$$= \frac{2\pi}{3}\left(R^2h - \frac{1}{2}(R-h)(2Rh-h^2)\right) \tag{50}$$

. . . .

$$= \frac{\pi}{3}(2R^2h - 2R^2h + 2Rh^2 + Rh^2 - h^3) = \boxed{\frac{\pi}{3}h^2(3R-h)}. \tag{51}$$

Remark: In both solutions we can easily check our final answer by noting that $h = 0$ results in a volume of 0, $h = R$ results in a volume of $\frac{2\pi}{3}R^3$ which is indeed the volume of a hemisphere of radius R, and $h = -R$ results in a volume of $\frac{4}{3}R^3$ which is indeed the volume of a sphere of radius R.