MATH 2173 – Engineering Calculus B – Practice Final Exam

Name: Sohail Farhangi April 25, 2021

Procedures for the exam: Please log on to the Main Lecture Zoom room by **10:15 am ET**. You will be split into a breakout room with one of the moderators.
https://osu.zoom.us/j/96249977566?pwd=Wk5jN1k5c2JmZkZEUTRVOHNhOUJDdz09
Zoom Meeting ID: 962 4997 7566, Password: 650807
At **10:20 am ET**, the exam will be made available on the Carmen Homepage. At this time, you may download the exam and immediately begin working. You may download, complete, and submit the assignment using an ipad or other tablet device. You may also do your work on paper and then scan and submit your work. If you choose the first option, please use the exam template for your work. If you choose the second option, be sure to clearly label your work. Unlike the recitation handouts and midterm 1 if you require additional pages to show your work for a given problem, include those pages in order immediately after the problem they pertain to. When you submit your exam to gradescope you are then required to mark the pages of your exam that correspond to the various problems on the exam. Video instructions for how to do this can be found here for the afternoon lecture and here for the morning lecture. If you have questions throughout the exam, you can direct message your moderator using the chat feature in Zoom.
When you have completed the exam, you should send a message to your moderator letting them know that you are no longer writing and are beginning to submit. You may then submit to Gradescope. You must submit the exam to Gradescope by **11:15 am ET**. If you have not finished by 11:10 am ET, an announcement will be made letting you know that you must now begin submitting. In other words, you should spend approximately 1 hour and 40 minutes working on the exam with the remaining five minutes left for submitting the exam.

Exam rules: In order to get credit for the exam, you must be in the Main Lecture Zoom room for the duration of the exam with your webcam on. No books, no notes, no calculators, and no internet resources may be used to complete the exam.
You must show your work. Work on the scrap work page will not be graded unless you indicate otherwise. Your work must be legible, and your final answers must be reasonably simplified.
On some problems, you are asked to use a specific method to solve the problem. On all other problems, you may use any method we've covered. You may not use methods we have not covered.
If at any point you experience technical difficulties, you must immediately e-mail both Dr. Skipper and your recitation instructor. Your email **must** include a complete copy of your exam, even if that is just in the form of photographs taken on your phone.
Good luck!

Problem 1 (20 points): Please determine whether the following statements are true or false. Provide some justification for each of your answers.

(a) The function $e^{\frac{1}{x}} + \sin(y^x) - 1$ attains its absolute maximum value on the region $R = \{(x, y) \mid (x-1)^2 + y^2 \leq 1\}$.

Solution to (a): False. We see that $e^{\frac{1}{x}} + \sin(y^x) - 1$ is not continuous when $x = 0$. Since $(0, 0) \in R$, we are unable to use the Extreme Value Theorem. In fact, noting that $-1 \leq \sin(y^x) \leq 1$ for all $(x, y) \in \mathbb{R}^2$, and that $\lim_{x\to 0} e^{\frac{1}{x}} = \infty$, we see that $e^{\frac{1}{x}} + \sin(y^x) - 1$ is unbounded on the region R. In particular, letting $y = 0$, we can approach the point $(0, 0)$ along the line segment $\{(x, 0) \mid 0 \leq x \leq 2\}$, which is contained in the region R, and see that

$$\lim_{x\to 0} e^{\frac{1}{x}} + \sin(0^x) - 1 = \lim_{x\to\infty} e^{\frac{1}{x}} - 1 = \infty. \tag{1}$$

(b) The vector field $\vec{F} = \langle ax+by, cx+ay\rangle$ is conservative for any real numbers a, b, and c.

Solution to (b): False. Letting $m(x, y) = ax + by$ and $n(x, y) = cx + ay$, we see that $\vec{F} = \langle m, n\rangle$. We also see that $\frac{\partial m}{\partial y} = b$ and $\frac{\partial n}{\partial x} = c$. In order for $\vec{F}$ to be conservative, we need $\frac{\partial m}{\partial y} = \frac{\partial n}{\partial x}$, so $\vec{F}$ is conservative if and only if $b = c$.

(c) For any 3 continuous functions $f, g, h : \mathbb{R} \to \mathbb{R}$ we have

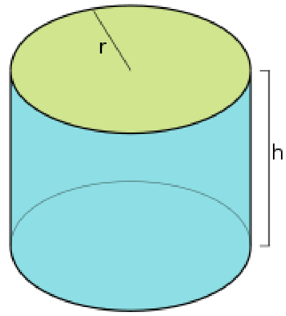
$$\int_0^1 \int_0^z \int_{-1}^0 f(x)g(y)h(z)dxdydz = \left(\int_0^1 f(x)dx\right)\left(\int_0^z g(y)dy\right)\left(\int_{-1}^0 h(z)dz\right). \tag{2}$$

Solution to (c): False. We see that the left hand side will result in a scalar value when evaluated, but the right hand side will result in a function of z when evaluated for any function $g(y) \neq 0$.

(d) The equation $yy' = e^t$ is a first order linear nonhomogeneous differential equation.

Solution to (d): False. The term yy' is a nonlinear term, so the given equation is a first order nonlinear nonhomogeneous differential equation.

Problem 2 (20 points): Use Lagrange multipliers to find the dimensions of the right circular cylinder of minimum surface area (including the circular ends) with a volume of 32π in^3.



Solution: We recall that a cylinder of radius r and height h has a volume of $V = \pi r^2 h$ and a surface area (including the 2 circular ends) of $S = 2\pi r^2 + 2\pi rh$. It follows that we want to optimize the function $f(r, h) = 2\pi r^2 + 2\pi rh$ subject to the constraint $0 = g(r, h) = \pi r^2 h - 32\pi$. Since

$$\nabla f(r,h) = \langle 4\pi r + 2\pi h, 2\pi r\rangle \text{ and } \nabla g(r,h) = \langle 2\pi rh, \pi r^2\rangle, \text{ we obtain} \tag{3}$$

. . . .

$$\begin{matrix} 4\pi r + 2\pi h & = & 2\pi\lambda rh \\ 2\pi r & = & \pi\lambda r^2 \\ \pi r^2 h & = & 32\pi \end{matrix} \overset{r\neq 0}{\to} \begin{matrix} 2r + h & = & \lambda rh \\ 2 & = & \lambda r \\ r^2 h & = & 32 \end{matrix} \to \begin{matrix} 2r + h & = & 2h \\ 2 & = & \lambda r \\ r^2 h & = & 32 \end{matrix} \tag{4}$$

. . . .

$$\to \begin{matrix} 2r & = & h \\ 2 & = & \lambda r \\ r^2 h & = & 32 \end{matrix} \to \begin{matrix} 2r & = & h \\ 2 & = & \lambda r \\ 2r^3 & = & 32 \end{matrix} \to r = \sqrt[3]{16} = 2\sqrt[3]{2} \to h = 4\sqrt[3]{2}. \tag{5}$$

Since the cylinder does not have a maximum surface area when subjected to the constraint $V = 32\pi$, we see that the critical point that we found has to correspond to a local minimum. The extreme/boundary cases occur when either $r \to \infty$ or $h \to \infty$, in which case we also have $S \to \infty$. It follows that $f(r, h)$ attains a minimum value of $24\pi\sqrt[3]{4}$ when $(r, h) = \boxed{(2\sqrt[3]{2}, 4\sqrt[3]{2})}$.

Problem 3 (20 points): Use triple integration in Cartesian coordinates to find the volume of the tetrahedron S that has its vertices at $(0,0,0)$, $(a,0,0)$, $(0,b,0)$, and $(0,0,c)$, where $a,b,c>0$.
Hint: One of the faces of the tetrahedron lies on the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$.

Solution: We see that an alternative description of S is that it is the solid bound between the planes $x=0$, $y=0$, $z=0$, and $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$.

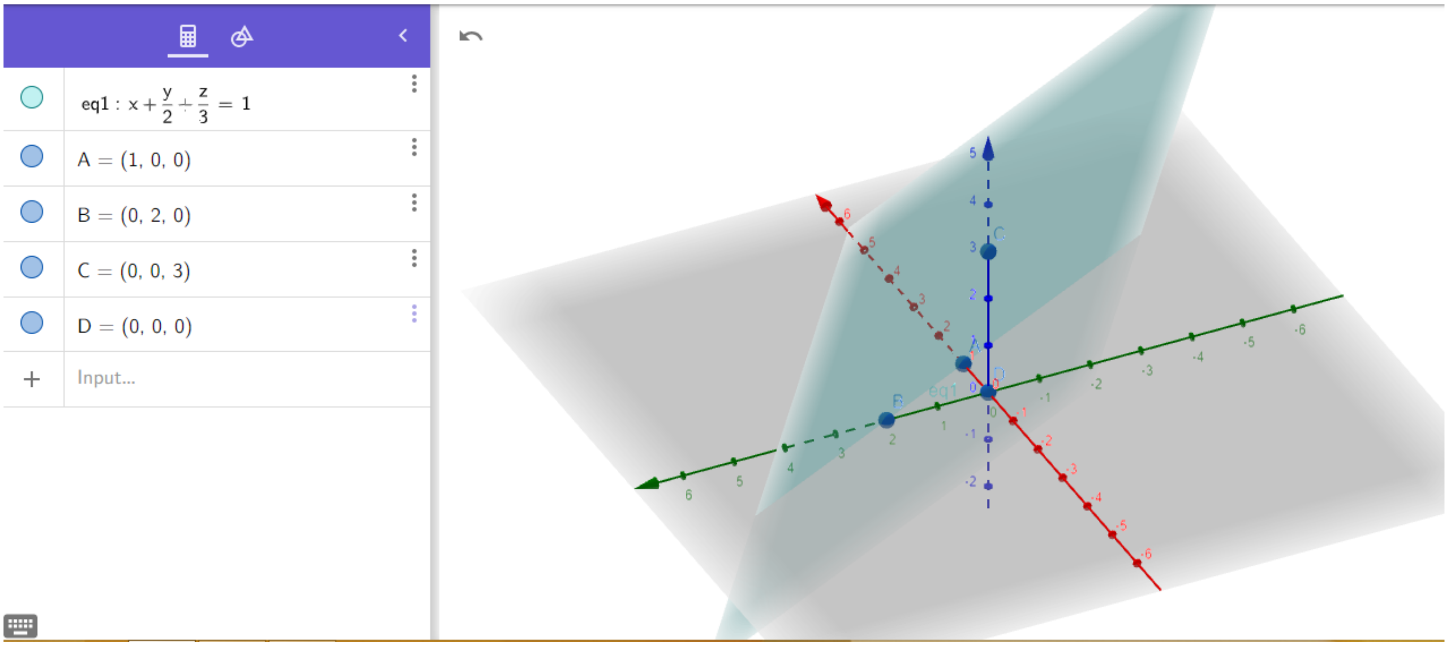


FIGURE 1. A picture of the solid S when $a=1$, $b=2$, and $c=3$.

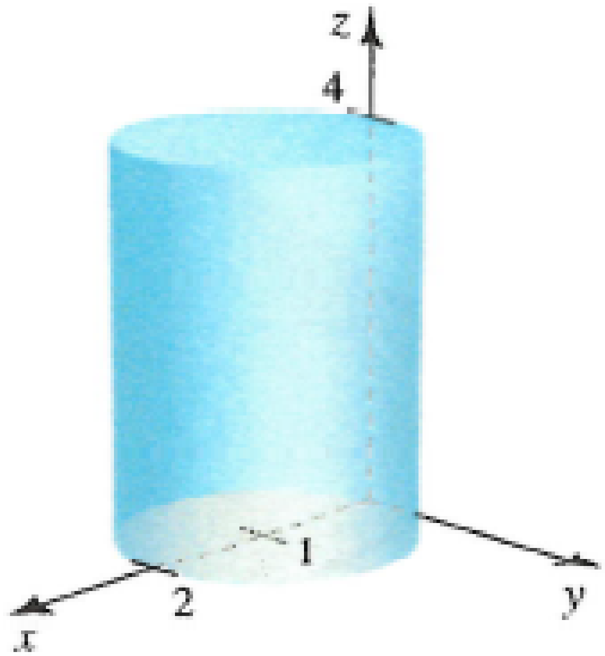
$$\text{Volume of } S = \iiint_S 1dV = \int_0^c \int_0^{b(1-\frac{z}{c})} \int_0^{a(1-\frac{z}{c}-\frac{y}{b})} 1dxdydz \tag{6}$$

$$= \int_0^c \int_0^{b(1-\frac{z}{c})} a(1-\frac{z}{c}-\frac{y}{b})dydz = a\int_0^c (y-\frac{z}{c}y-\frac{1}{2b}y^2\Big|_{y=0}^{b(1-\frac{z}{c})})dz \tag{7}$$

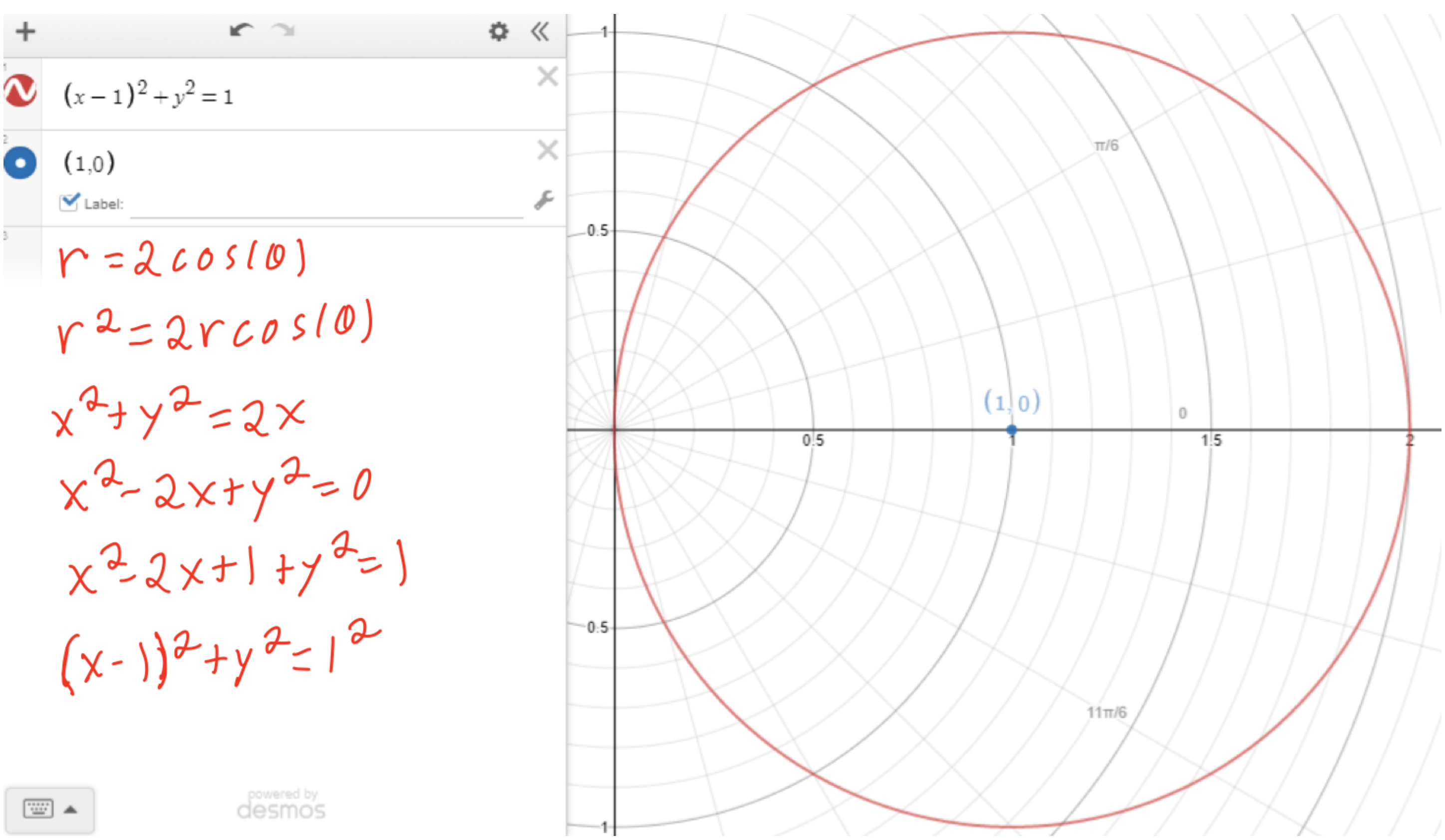
$$= a\int_0^c \Big(\underbrace{b(1-\frac{z}{c})-\frac{z}{c}b(1-\frac{z}{c})}_{b(1-\frac{z}{c})^2} - \frac{1}{2b}b^2(1-\frac{z}{c})^2\Big)dz = \frac{ab}{2}\int_0^c (1-\frac{z}{c})^2dz \tag{8}$$

$$= \frac{ab}{2}(-\frac{c}{3}(1-\frac{z}{c})^3\Big|_{z=0}^{c}) = \boxed{\frac{abc}{6}}. \tag{9}$$

Problem 4 (20 points): Use triple integration in cylindrical coordinates to find the volume of the solid cylinder E whose height is 4 and whose base in the xy-plane is the disk $\{(r, \theta) : 0 \leq r \leq 2\cos(\theta)\}$.



Solution: We first look at the cross section of E in the xy−plane to help us determine our bounds.



$$\text{Volume}(E) = \iiint_E 1dV = \int_0^4 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\cos(\theta)} r dr d\theta dz \tag{10}$$

$$= \int_0^4 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2} r^2 \Big|_0^{2\cos(\theta)} d\theta dz = \int_0^4 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2\cos^2(\theta) d\theta dz \tag{11}$$

$$= \int_0^4 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos(2\theta) + 1) d\theta dz = \int_0^4 (\frac{1}{2}\sin(2\theta) + \theta)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dz \tag{12}$$

$$= \int_0^4 \pi dz = \boxed{4\pi}. \tag{13}$$

Problem 5 (20 points): Find the average value of the function $f(x,y,z) = x$ over the curve $\mathcal{C}$ that is paramterized by

$$\vec{r}(t) = \langle 20\sin(\frac{t}{4}), 20\cos(\frac{t}{4}), \frac{t}{2}\rangle, 0 \le t \le 4\pi. \tag{14}$$

Solution: We begin by evaluating $\int_{\mathcal{C}} f ds$ and finding the arclength of $\mathcal{C}$. Since $ds = |\vec{r}\,'(t)|dt$, we first observe that

$$\vec{r}\,'(t) = \langle 5\cos(\frac{t}{4}), -5\sin(\frac{t}{4}), \frac{1}{2}\rangle \tag{15}$$

$$\to |\vec{r}\,'(t)| = \sqrt{(5\cos(\frac{t}{4}))^2 + (5\sin(\frac{t}{4}))^2 + (\frac{1}{2})^2} \tag{16}$$

$$= \sqrt{25\cos^2(\frac{t}{4}) + 25\sin^2(\frac{t}{4}) + \frac{1}{4}} = \sqrt{25 + \frac{1}{4}} = \frac{1}{2}\sqrt{101}. \tag{17}$$

We now see that

$$\int_{\mathcal{C}} f ds = \int_0^{4\pi} f(\vec{r}(t)) \cdot |\vec{r}\,'(t)| dt \tag{18}$$

$$= \int_0^{4\pi} f(20\sin(\frac{t}{4}), 20\cos(\frac{t}{4}), \frac{t}{2})\frac{1}{2}\sqrt{101}dt = \int_0^{4\pi} \underbrace{20\sin(\frac{t}{4})}_{f(x,y,z)=x} \cdot \frac{1}{2}\sqrt{101}dt \tag{19}$$

$$= 10\sqrt{101}\int_0^{4\pi} \sin(\frac{t}{4})dt = 10\sqrt{101}\Big(-4\cos(\frac{t}{4})\Big|_0^{4\pi}\Big) = 80\sqrt{101}, \text{ and} \tag{20}$$

$$\text{Arclength of } \mathcal{C} = \int_{\mathcal{C}} 1ds = \int_0^{4\pi} |\vec{r}\,'(t)|dt = \int_0^{4\pi} \frac{1}{2}\sqrt{101}dt = 2\sqrt{101}\pi. \tag{21}$$

. . . .

Putting everything together, we see that

$$\text{Average value of } f \text{ over } \mathcal{C} = \frac{\int_{\mathcal{C}} f ds}{\text{Arclength of } \mathcal{C}} = \frac{80\sqrt{101}}{2\sqrt{101}\pi} = \boxed{\frac{40}{\pi}}. \tag{22}$$

Problem 6 (20 points): Evaluate

$$\int_C \langle \sqrt[4]{x+6} + \ln(\ln(\ln(e^{e^e} + 4 + x))) - 1, y^3 + 2 + e^{y^2} \rangle \cdot d\vec{r}, \tag{23}$$

where C is the curve that is shown in the picture below.

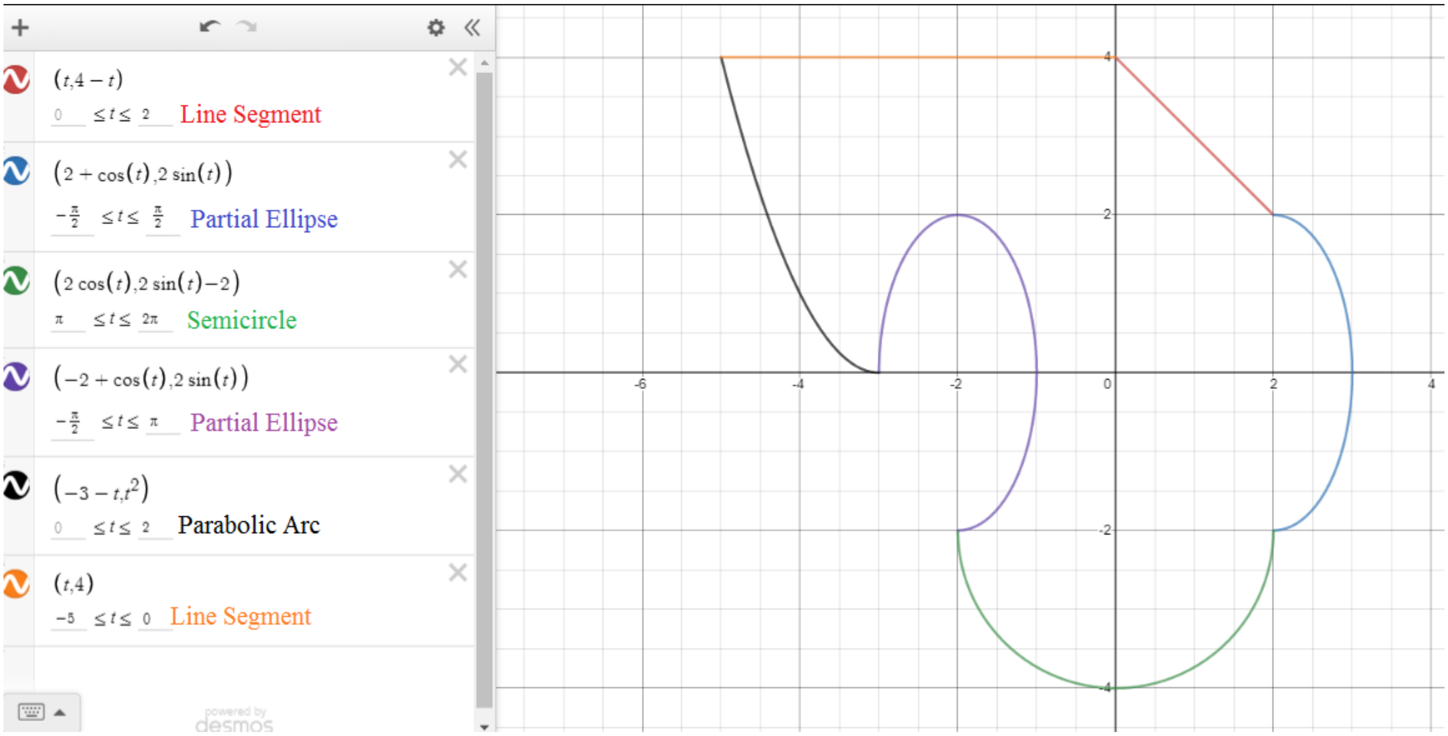


FIGURE 2.

Be sure to show all of your work and give a complete justification of your answer.

Solution: Letting

$$m(x,y,z) = \sqrt[4]{x+6} + \ln(\ln(\ln(e^{e^e} + 4 + x))) - 1, \text{ and} \tag{24}$$

$$n(x,y,z) = y^3 + 2 + e^{y^2}, \text{ we see that} \tag{25}$$

$$\vec{F} := \langle m, n \rangle, \text{ satisfies} \tag{26}$$

$$\frac{\partial m}{\partial y} = 0 = \frac{\partial n}{\partial x} \tag{27}$$

so $\vec{F}$ is a conservative vector field. We also see that

$$\int_C \langle \sqrt[4]{x+6} + \ln(\ln(\ln(e^{e^e} + 4 + x))) - 1, y^3 + 2 + e^{y^2} \rangle \cdot d\vec{r} = \int_C \vec{F} \cdot d\vec{r}. \tag{28}$$

Since $\vec{F}$ is conservative on a simply connected domain that contains the curve C, and C is a (simple piecewise smooth oriented) closed curve, we see that

$$\int_C \vec{F} \cdot d\vec{r} = \boxed{0}. \tag{29}$$

Challenge for the brave: Letting C once again denote the curve in figure 2, evaluate

$$\int_C \langle y, 0 \rangle \cdot d\vec{r}. \tag{30}$$

Problem 7 (20 points): Let a be a real number.

(a) Find the general solution to equation (31) in terms of a.

$$y'' - (a+2)y' + 2ay = 0. \tag{31}$$

(b) Solve the initial value problem given in (32).

$$y'' - 25y' + 46y; \quad y(0) = 0,\ y'(0) = 21. \tag{32}$$

Solution to (a): By examining the characteristic equation of equation (31), we see that

$$0 = r^2 - (a+2)r + 2a = (r-a)(r-2) \tag{33}$$

$$\rightarrow y(t) = \boxed{\begin{cases} c_1e^{at} + c_2e^{2t} & \text{if } a \neq 2 \\ c_2e^{2t} + c_2te^{2t} & \text{if } a = 2 \end{cases}}. \tag{34}$$

Solution to (b): We saw in part (a) that the general solution to equation (32) is $y(t) = c_1e^{23t} + c_2e^{2t}$. It follows that $y'(t) = 23c_2e^{23t} + 2c_2e^{2t}$. Making use of the given initial conditions, we see that

$$\begin{array}{rcccccc} 0 & = & y(0) & = & c_1e^{23\cdot 0} + c_2e^{2\cdot 0} & = & c_1 + c_2 \\ 21 & = & y'(0) & = & 23c_1e^{23\cdot 0} + 2c_2e^{2\cdot 0} & = & 23c_1 + 2c_2 \end{array} \tag{35}$$

$$\rightarrow c_1 = \frac{1}{21}\cdot 21c_1 = \frac{1}{21}\Big((23c_1 + 2c_2) - 2(c_1 + c_2)\Big) = \frac{1}{21}(21 - 2\cdot 0) = 1 \tag{36}$$

$$\rightarrow c_2 = -c_1 = -1 \rightarrow y(t) = \boxed{e^{23t} - e^{2t}}. \tag{37}$$

Problem 8 (20 points): Find a particular solution to equation (38).

$$y'' + 4y = t\sin(2t). \tag{38}$$

Solution: We begin by finding the general solutions of the corresponding homogeneous equation, which is the equation

$$y_c'' + 4y_c = 0. \tag{39}$$

By examining the characteristic equation for equation (39), we see that

$$r^2 + 4 = 0 \to r = \pm 2i \to y_c(t) = c_1\cos(2t) + c_2\sin(2t). \tag{40}$$

Normally, the method of undetermined coefficients would tell us to use the trial solution $y_p(t) = (At + B)\cos(2t) + (Ct + D)\sin(2t)$, since $\{\cos(2t), \sin(2t), t\cos(2t), t\sin(2t)\}$ is a linearly independent set of functions that can 'generate' any function that is a derivative of $t\sin(2t)$. However, since $\cos(2t)$ and $\sin(2t)$ are a solutions to equation (39), we see that plugging $y_p(t)$ into the left hand side of equation (38) will result in an expression of the form $E\cos(2t) + F\sin(2t)$, which is not what we want. We consequently have to adjust our trial solution and use $y_p(t) = ty_p(t) = (At^2 + Bt)\cos(2t) + (Ct^2 + Dt)\sin(2t)$. We now see that

$$y_p'(t) = (2At + B)\cos(2t) - 2(At^2 + Bt)\sin(2t) + (2Ct + D)\sin(2t) + 2(Ct^2 + Dt)\cos(2t) \tag{41}$$

$$= (2Ct^2 + 2At + 2Dt + B)\cos(2t) + (-2At^2 + 2Ct - 2Bt + D)\sin(2t), \text{ and} \tag{42}$$

$$y_p''(t) = (4Ct + 2A + 2D)\cos(2t) - 2(2Ct^2 + 2At + 2Dt + B)\sin(2t) \\ +(-4At + 2C - 2B)\sin(2t) + 2(-2At^2 + 2Ct - Bt + D)\cos(2t) \tag{43}$$

$$= (-4At^2 + 8Ct - 4Bt + 2A + 4D)\cos(2t) + (-4Ct^2 - 8At - 4Dt + 2C - 4B)\sin(2t). \tag{44}$$

We may now plug $y_p(t)$, $y_p'(t)$, and $y_p''(t)$ into equation (38) in order to solve for A, B, C and D.

$$t\sin(2t) = y_p''(t) + 4y_p(t) \tag{45}$$

$$= \underbrace{(-4At^2 + 8Ct - 4Bt + 2A + 4D)\cos(2t) + (-4Ct^2 - 8At - 4Dt + 2C - 4B)\sin(2t)}_{y_p''(t)} + 4(\underbrace{(At^2 + Bt)\cos(2t) + (Ct^2 + Dt)\sin(2t)}_{y_p(t)}) \tag{46}$$

$$= (8Ct + 2A + 4D)\cos(2t) + (-8At + 2C - 4B)\sin(2t) \tag{47}$$

$$\rightarrow \begin{aligned} 8C &= 0 \text{ (by considering the } t\cos(2t) \text{ term)} \\ 2A + 4D &= 0 \text{ (by considering the } \cos(2t) \text{ term)} \\ -8A &= 1 \text{ (by considering the } t\sin(2t) \text{ term)} \\ 2C - 4B &= 0 \text{ (by considering the } \sin(2t) \text{ term)} \end{aligned} \rightarrow \tag{48}$$

$$\begin{aligned} C &= 0 \\ A &= -\tfrac{1}{8} \end{aligned} \rightarrow \begin{aligned} D &= -\tfrac{1}{2}A = \tfrac{1}{16} \\ B &= \tfrac{1}{2}C = 0 \end{aligned} \tag{49}$$

$$\rightarrow y_p(t) = \boxed{-\frac{1}{8}t^2\cos(2t) + \frac{1}{16}t\sin(2t)}. \tag{50}$$