Math 2173 Spring 2021 Recitation Handout 11 Solutions

Group Member 1: ______________________________

Group Member 2: ______________________________

Group Member 3: ______________________________

Group Member 4: ______________________________

Group Member 5: ______________________________

Group Member 6: ______________________________

Below is a checklist of instructions to follow when completing this assignment. Failure to follow these directions will result in penalty on your final score and/or in some problems not being graded. If multiple directions are not followed, then it is also possible that the assignment will not be accepted for any credit at all. Please contact your TA or make a post on the discussion boards for this course if you have any questions about this assignment or these directions.

Sohail Farhangi: farhangi.3@osu.edu, Pan Yan: yan.669@osu.edu, Yilong Zhang: zhang.6100@osu.edu

	Checklist of Instructions
	Please clearly write the names of all group members working on this assignment in the spaces allotted above.
	This assignment must be completed by a group of 3, 4, 5, or 6 members.
	This assignment is to be uploaded to gradescope as a pdf file no later than 11:59 PM EST on Sunday, April 11.
	The assignment will be uploaded by 1 group member, and that group member will be responsible for manually entering the names of all other collaborators into gradescope.
	This assignment must be completed using this template. You may either print this template to write on it and then scan it (pages ordered correctly) into a pdf file, or you may write directly on the template using programs such as notability.
	If you need more space than what is given to solve a given problem, then you will find blank pages provided at the end of this template. At the end of each problem section of this assignment you will find a space in which to indicate on what page your work is continued in case you used additional pages to complete your solution. You must provide the page number on which your work is continued in the alloted space, or write 'N/A' incase you did not use any additional pages.
	On the additional pages, you will also find space in which to indicate which problem the page is being used for, and if the page is used then that space must also be filled.
	To complete this handout, you may use your textbook, class notes, discussions with your TA and group members, and any resources that are available on Carmen. You should not receive any help from the MSLC or people outside of your group when solving these problems. You may discuss these problems on the Carmen discussion boards, but you should not provide your entire solution when answering a such question, you should only give a hint or a helpful idea.

Ungraded Optional Problem 15.3.52: Conservation of energy. Suppose an object with mass m moves in a conservative force field given by $\vec{F} = -\nabla\phi$, where ϕ is a potential function in a region R. The motion of the object is governed by Newton's second law of Motion, $\vec{F} = m\vec{a}$, where $\vec{a}$ is the acceleration. Suppose the object moves (either in the plane or in space) from point A to point B in R.
(a). Show that the equation of the motion is

$$m\frac{d\vec{v}}{dt} = -\nabla\phi.$$

(b). Show that

$$\frac{d\vec{v}}{dt} \cdot \vec{v} = \frac{1}{2}\frac{d}{dt}(\vec{v} \cdot \vec{v}).$$

(c). Take the dot product of both sides of the equation in part (a) with $\vec{v}(t) = \frac{d\vec{r}(t)}{dt}$ and integrate along a curve between A and B. Use part (b) and the fact that $\vec{F}$ is conservative to show that the total energy (kinetic plus potential) $\frac{1}{2}m|\vec{v}|^2 + \phi$ is the same at A and B. Conclude that because A and B are arbitrary, energy is conserved in R.

Ungraded Optional Problem 16.1.41: Initial value problems. Solve the following initial value problem using the given general solution.

$$y'' - y' - 20y = 0;\ y(0) = -3, y'(0) = 3;$$

General solution $y = c_1 e^{5t} + c_2 e^{-4t}$.

Problem 1. (10 Points): Find the work required to move an object along the line segment from $(1,1,1)$ to $(8,4,2)$ through the force field $\vec{F}$ given by

$$\vec{F} = \frac{\langle x, y, z\rangle}{x^2+y^2+z^2}.$$

Solution 1: We note that for $\varphi = \frac{1}{2}\ln(x^2+y^2+z^2)$ we have $\nabla\varphi = \vec{F}$, so we may use the Fundamental Theorem for Line Integrals as follows:

(1) $$\text{Work} = \int_C \vec{F}\cdot d\vec{r} = \int_C \nabla\varphi\cdot d\vec{r} = \varphi\left((8,4,2)\right) - \varphi\left((1,1,1)\right)$$

(2) $$= \frac{1}{2}\ln(8^2+4^2+2^2) - \frac{1}{2}\ln(1^2+1^2+1^2) = \frac{1}{2}\ln(84) - \frac{1}{2}\ln(3) = \boxed{\frac{1}{2}\ln(28)}.$$

Solution 2: Firstly, we recall that one method of parameterizing the line segment that starts at $\vec{p}$ and ends at $\vec{q}$ is to use the parameterization

(3) $$\vec{r}(t) = (1-t)\vec{p} + t\vec{q} = \vec{p} + t(\vec{q}-\vec{p}), \quad 0 \le t \le 1.$$

It follows that

(4) $$\vec{r}(t) = \langle 1,1,1\rangle + t\left(\langle 8,4,2\rangle - \langle 1,1,1\rangle\right) = \langle 1+7t, 1+3t, 1+t\rangle, \quad 0 \le t \le 1,$$

is a parameterization of the line segment from $(1,1,1)$ to $(8,4,2)$. We now see that

(5) $$\text{Work} = \int_C \vec{F}\cdot d\vec{r} = \int_0^1 \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)dt$$

. . . .

(6) $$= \int_0^1 \underbrace{\frac{\langle 1+7t, 1+3t, 1+t\rangle}{(1+7t)^2+(1+3t)^2+(1+t)^2}}_{\vec{F}(\vec{r}(t))} \cdot \underbrace{\langle 7,3,1\rangle dt}_{d\vec{r}}$$

. . . .

$$= \int_0^1 \frac{(1+7t)\cdot 7 + (1+3t)\cdot 3 + (1+t)\cdot 1}{1+14t+49t^2+1+6t+9t^2+1+2t+t^2}dt \tag{7}$$

$$= \int_0^1 \frac{11+59t}{3+22t+59t^2}dt = \int_0^1 \frac{t+\frac{11}{59}}{t^2+\frac{22}{59}t+\frac{3}{59}}dt = \int_0^1 \frac{t+\frac{11}{59}}{(t+\frac{11}{59})^2+\frac{56}{3481}}dt \tag{8}$$

$$= \frac{1}{2}\ln\left((t+\frac{11}{59})^2+\frac{56}{3481}\right)\Big|_0^1 = \boxed{\frac{1}{2}\ln(28)}. \tag{9}$$

Problem 15.3.27. (10 points): Evaluate the line integral $\int_C \nabla\phi \cdot d\vec{r}$ for $\phi(x,y) = xy$ and $C : \vec{r}(t) = \langle \cos(t), \sin(t)\rangle$, for $0 \le t \le \pi$ in two ways.
(a). Use a parametric description of C and evaluate the integral directly;
(b). Use the Fundamental Theorem for line integrals.

Solution to a: We see that $\nabla\phi(x,y) = \langle y, x\rangle$, so

$$\int_C \nabla\phi \cdot d\vec{r} = \int_0^\pi \nabla\phi(\vec{r}(t)) \cdot \vec{r}\,'(t)dt \tag{10}$$

$$\int_0^\pi \nabla\phi(\cos(t), \sin(t)) \cdot \langle -\sin(t), \cos(t)\rangle dt \tag{11}$$

$$= \int_0^\pi \langle \sin(t), \cos(t)\rangle \cdot \langle -\sin(t), \cos(t)\rangle dt \tag{12}$$

$$= \int_0^\pi (-\sin^2(t) + \cos^2(t))dt = \int_0^\pi \cos(2t)dt = \frac{1}{2}\sin(2t)\Big|_0^\pi = \boxed{0}. \tag{13}$$

Solution to b: We see that

$$\int_C \nabla\phi \cdot d\vec{r} = \int_0^\pi \nabla\phi(\vec{r}(t)) \cdot \vec{r}\,'(t)dt \tag{14}$$

$$= \phi(\vec{r}(\pi)) - \phi(\vec{r}(0)) = \phi(-1,0) - \phi(1,0) = 0 - 0 = \boxed{0}. \tag{15}$$

Problem 16.1.66. (10 points): As will be shown in Section 16.4, the equation $y''+py'+qy=f(t)$, where p and q are constants and f is a specified function, is used to model both the mechanical oscillators and electrical circuits. Depending on the values of p and q, the solutions to this equation display a wide variety of behavior.
Consider the equation

$$y''+9y=8\sin(t).$$

(a). **(4 points)** Verify that the following equations have the given general solutions

$$y=c_1\sin(3t)+c_2\cos(3t)+\sin t.$$

(b). **(4 points)** Solve the initial value problem with the given initial conditions $y(0)=0$, $y'(0)=2$.
(c). **(2 points)** Graph the solutions to the initial value problem, for $t\geq 0$.

Solution to a: We see that for $y(t)=c_1\sin(3t)+c_2\cos(3t)+\sin(t)$ we have

$$y''+9y=(\underbrace{c_1\sin(3t)+c_2\cos(3t)+\sin(t)}_{y(t)})''+9(\underbrace{c_1\sin(3t)+c_2\cos(3t)+\sin(t)}_{y(t)}) \tag{16}$$

$$=\underbrace{-9c_1\sin(3t)-9c_2\cos(3t)-\sin(t)}_{y''(t)}+\underbrace{9c_1\sin(3t)+9c_2\cos(3t)+9\sin(t)}_{9y(t)} \tag{17}$$

$$=8\sin(t). \tag{18}$$

Solution to b: Noting that $y'(t)=3c_1\cos(3t)-3c_2\sin(3t)+\cos(t)$, we see that

$$\begin{array}{rcccl} 0 &=& y(0) &=& c_2 \\ 2 &=& y'(0) &=& 3c_1+1 \end{array} \rightarrow (c_1,c_2)=(\frac{1}{3},0). \tag{19}$$

$$\rightarrow \boxed{y(t)=\frac{1}{3}\sin(3t)+\sin(t)}. \tag{20}$$

Solution to c:

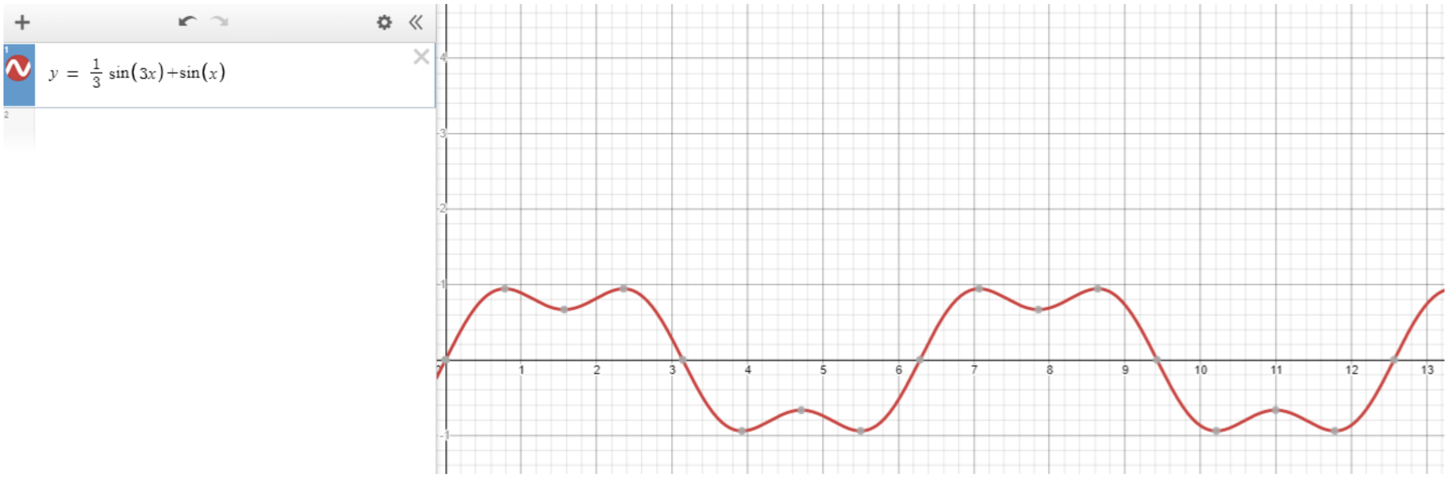

FIGURE 1. A graph of the solution to the initial value problem.

(Modified) Problem 15.3.55 (20 points): Let $\vec{F}$ be the vector field

$$\vec{F} = \langle f(x,y), g(x,y)\rangle = \langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\rangle.$$

It is a rotational vector field with the graph below

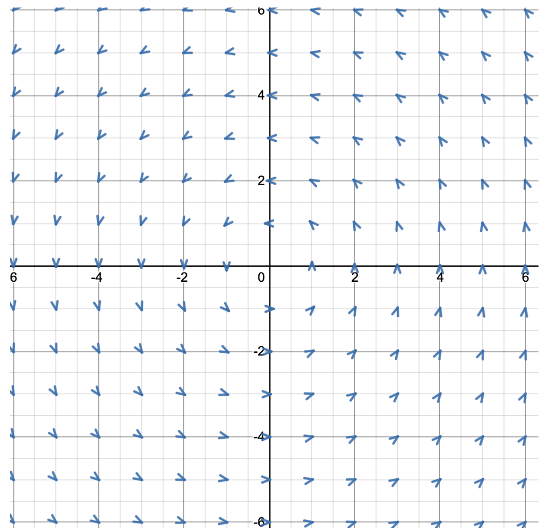

FIGURE 2. vector field $\vec{F}$

(1) **(2 points)** Find the domain R of $\vec{F}$.
(2) **(4 points)** Is the domain R connected? Is R simply connected?
(3) **(4 points)** Show that $\frac{\partial g}{\partial x} = \frac{\partial f}{\partial y}$.

Solution to part 1: The domain of $\vec{F}$ consists of all points in $\mathbb{R}^2$ at which $\vec{F}$ is defined. We see that the only time that $\vec{F}$ is undefined is when $x^2+y^2=0$, as we cannot divide by 0, but $x^2+y^2=0$ is only satisfed by $(x,y)=(0,0)$, so the domain of $\vec{F}$ is $\boxed{R = \mathbb{R}^2 \setminus \{(0,0)\}}$.

Solution to part 2: The domain of R is connected since it is actually path connected[1]. Given any 2 points in R, there exists a path consisting of either 1 or 2 straight line segments that connects the 2 points.

Solution to part 3: We see that

$$\frac{\partial g}{\partial x} = \frac{\partial}{\partial x}(\frac{x}{x^2+y^2}) = -\frac{x}{(x^2+y^2)^2}\cdot 2x + \frac{1}{x^2+y^2} \tag{21}$$

[1]Being path connected is a stronger condition than just being connected, but you probably won't study the difference between the 2 notions unless you go on to take a course in real analysis or topology.

. . . .

$$= -\frac{2x^2}{(x^2+y^2)^2} + \frac{x^2+y^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}, \text{ and} \tag{22}$$

. . . .

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(\frac{-y}{x^2+y^2}) = -\frac{-y}{(x^2+y^2)^2} \cdot 2y + \frac{-1}{x^2+y^2} \tag{23}$$

. . . .

$$= \frac{2y^2}{(x^2+y^2)^2} - \frac{x^2+y^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2} = \frac{\partial f}{\partial x}. \tag{24}$$

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(4) **(4 points)** Let C_a be the parameterized circle $\vec{r}(t) = \langle a\cos(t), a\sin(t)\rangle$, $0 \leq t < 2\pi$ of radius $a > 0$. Show that the integral

$$\int_{C_a} \vec{F} \cdot d\vec{r} = 2\pi.$$

Solution to part 4: We see that

$$\int_{C_a} \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)dt = \int_0^{2\pi} \vec{F}(a\cos(t), a\sin(t)) \cdot \langle -a\sin(t), a\cos(t)\rangle dt \tag{25}$$

. . . .

$$= \int_0^{2\pi} \langle \frac{-a\sin(t)}{(a\cos(t))^2 + (a\sin(t))^2}, \frac{a\cos(t)}{(a\cos(t))^2 + (a\sin(t))^2} \rangle \cdot \langle -a\sin(t), a\cos(t)\rangle dt \tag{26}$$

. . . .

$$= \int_0^{2\pi} \langle -\sin(t), \cos(t)\rangle \cdot \langle -\sin(t), \cos(t)\rangle dt = \int_0^{2\pi} (\sin^2(t) + \cos^2(t))dt \tag{27}$$

. . . .

$$= \int_0^{2\pi} 1dt = \boxed{2\pi}. \tag{28}$$

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(5) **(3 points)** Is $\vec{F}$ a conservative vector field on R? If so, please explain. Otherwise, please explain why it doesn't contradict the result in (3).

Solution to part 5: Since C_a is a closed loop inside of R for any radius $a > 0$, and $\int_{C_a} \vec{F} \cdot d\vec{r} = 2\pi \neq 0$, we see (Theorem 15.6 on page 1118) that $\vec{F}$ is not a conservative vector field on R. Our calcuations in part (3) cannot be used alongside Theorem 15.3 (on page 1113) to conclude that the vector field $\vec{F}$ is conservative, because Theorem 15.3 requires that the vector field $\vec{F}$ be defined on a simply connected region D.

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(6) **(3 points)** Let R_1 be the region $R_1 = \{1 \leq x \leq 2, 1 \leq y \leq 2\}$. Is $\vec{F}$ a conservative vector field on R_1? Please explain.

Solution to part 6: Since the region R_1 is simply connected (it has no holes), we may use the result of part (3) alongside Theorem 15.3 to conclude that the vector field $\vec{F}$ is conservative on the region R_1.