Math 2173 Spring 2021 Recitation Handout 12 Solutions

Group Member 1: Sohail Farhangi

Group Member 2: ______________________________

Group Member 3: ______________________________

Group Member 4: ______________________________

Group Member 5: ______________________________

Group Member 6: ______________________________

Below is a checklist of instructions to follow when completing this assignment. Failure to follow these directions will result in penalty on your final score and/or in some problems not being graded. If multiple directions are not followed, then it is also possible that the assignment will not be accepted for any credit at all. Please contact your TA or make a post on the discussion boards for this course if you have any questions about this assignment or these directions.

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	Checklist of Instructions
	Please clearly write the names of all group members working on this assignment in the spaces allotted above.
	This assignment must be completed by a group of 3, 4, 5, or 6 members.
	This assignment is to be uploaded to gradescope as a pdf file no later than 11:59 PM EST on Sunday, April 18.
	The assignment will be uploaded by 1 group member, and that group member will be responsible for manually entering the names of all other collaborators into gradescope.
	This assignment must be completed using this template. You may either print this template to write on it and then scan it (pages ordered correctly) into a pdf file, or you may write directly on the template using programs such as notability.
	If you need more space than what is given to solve a given problem, then you will find blank pages provided at the end of this template. At the end of each problem section of this assignment you will find a space in which to indicate on what page your work is continued in case you used additional pages to complete your solution. You must provide the page number on which your work is continued in the alloted space, or write 'N/A' incase you did not use any additional pages.
	On the additional pages, you will also find space in which to indicate which problem the page is being used for, and if the page is used then that space must also be filled.
	To complete this handout, you may use your textbook, class notes, discussions with your TA and group members, and any resources that are available on Carmen. You should not receive any help from the MSLC or people outside of your group when solving these problems. You may discuss these problems on the Carmen discussion boards, but you should not provide your entire solution when answering a such question, you should only give a hint or a helpful idea.

Ungraded Optional Problem 16.2.29: Solve the following initial value problem.

$$y'' - 2y' + 5y = 0; \quad y(0) = 1, y'(0) = -1. \tag{1}$$

Draw the graph of the solution.

Ungraded Optional Problem 16.2.43: Express the function

$$y = -3\sin(4t) + 3\cos(4t)$$

in the form

$$y = A\sin(\omega t + \varphi).$$

Draw the graph the function and explain the roles A, ω and φ that play in the graph.

Solution: Firstly, we use the angle-addition formula for sin to see that

(2) $$A\sin(\omega t + \varphi) = A\sin(\omega t)\cos(\varphi) + A\sin(\varphi)\cos(\omega t), \text{ so}$$

(3) $$-3\sin(4t) + 3\cos(4t) = A\cos(\varphi)\sin(\omega t) + A\sin(\varphi)\cos(\omega t).$$

We now see that $\omega = 4$, and that

(4) $$\begin{aligned} A\cos(\varphi) &= -3 \\ A\sin(\varphi) &= 3 \end{aligned}$$

(5) $$\to A^2 = A^2\cos^2(\varphi) + A^2\sin^2(\varphi) = (-3)^2 + 3^2 = 18 \to A = \pm 3\sqrt{2}$$

(6) $$\to \begin{aligned} \cos(\varphi) &= \mp\tfrac{1}{\sqrt{2}} \\ \sin(\varphi) &= \pm\tfrac{1}{\sqrt{2}} \end{aligned} \to \varphi = \frac{3\pi}{4}, -\frac{\pi}{4}$$

(7) $$\to -3\sin(4t) + 3\cos(4t) = \underbrace{\boxed{3\sqrt{2}\sin(4t + \frac{3\pi}{4})}}_{\text{This is amplitude-phase form since A is positive.}} = \boxed{-3\sqrt{2}\sin(4t - \frac{\pi}{4})}.$$

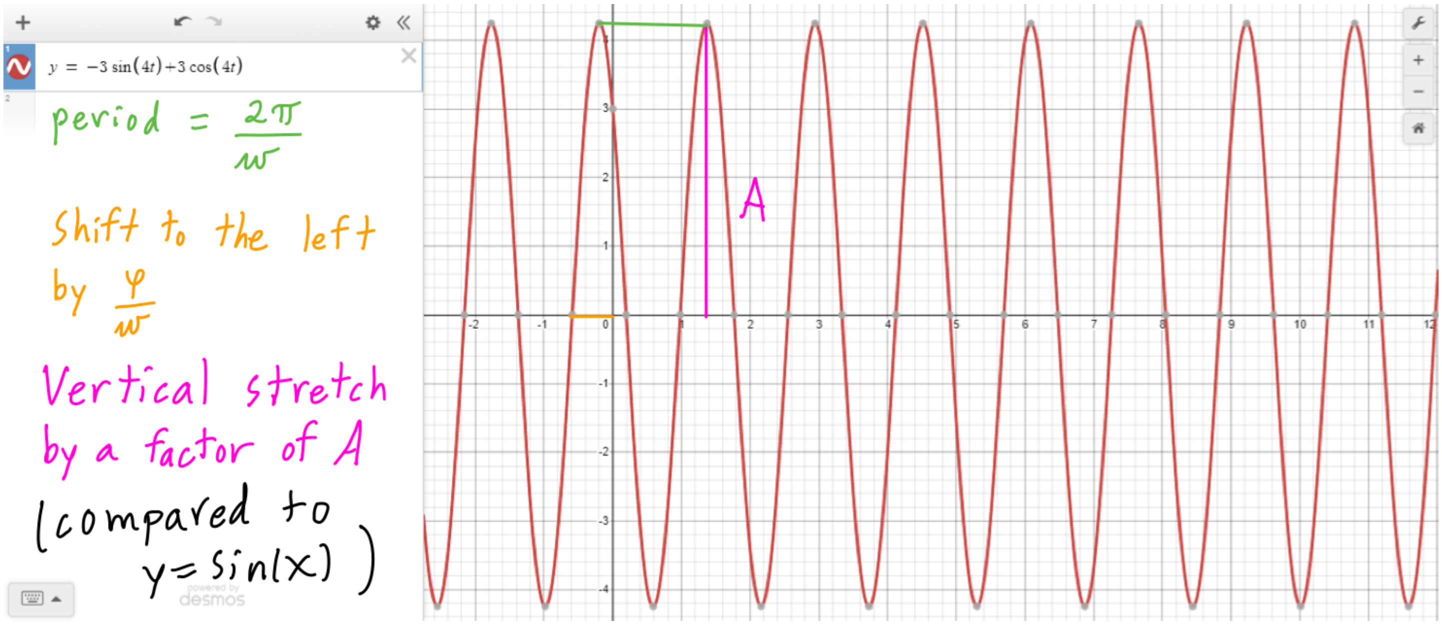

FIGURE 1. The graph of $y = -3\sin(4t) + 3\cos(4t)$ showing the Amplitude of A, Phase Shift of φ, and the period of $\frac{2\pi}{\omega}$.

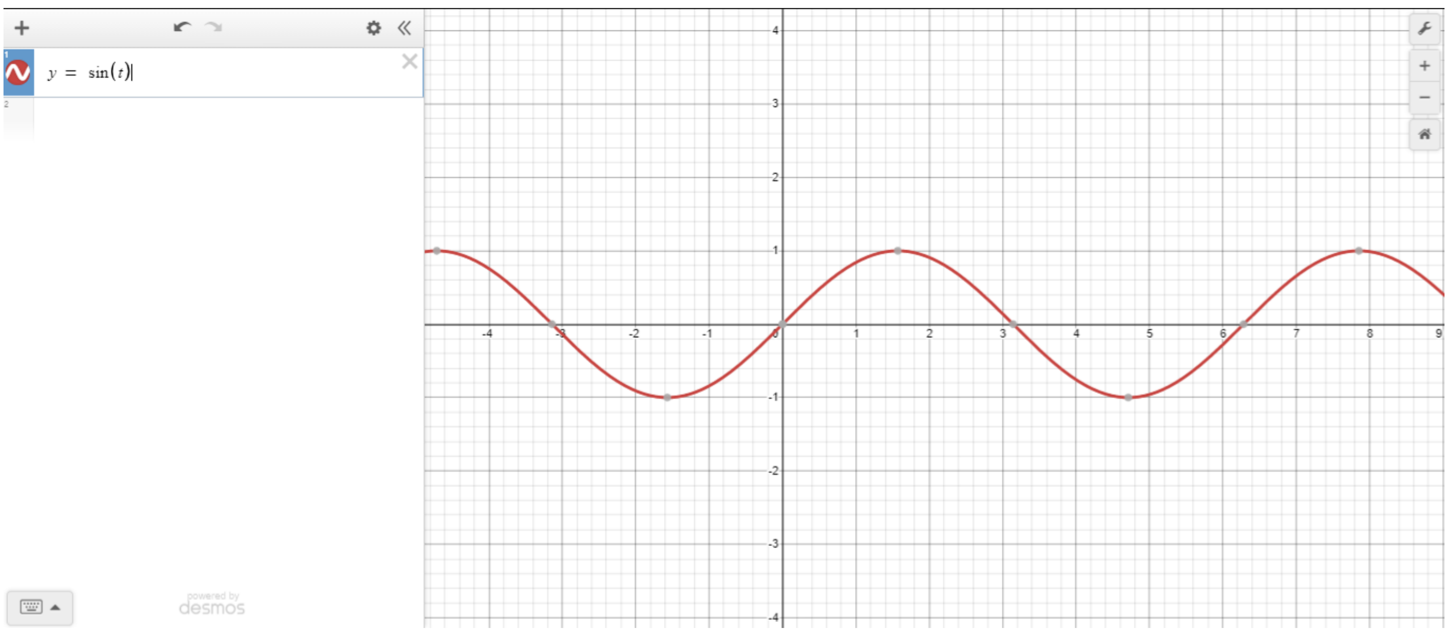

FIGURE 2. A graph of $y = \sin(t)$ for comparison.

Problem 1 (4+4+4=12 points): Let $z = -1 + i$ and $w = 1 + i\sqrt{3}$ be the two complex numbers.
(1) Compute directly $z \cdot w$ and $\frac{z}{w}$ and express the answer in Cartesian form, i.e., the form $x + iy$, where x and y are real numbers.
(2) Express z and w in polar form. Compute $z \cdot w$ and $\frac{z}{w}$ in polar forms. Compare your answer with part (1).
(3) Draw the four complex numbers w, z, $z \cdot w$ and $\frac{z}{w}$ in the following coordinate. Explain what multiplication by w and division by w do to the complex number z in terms of argument and modulus.

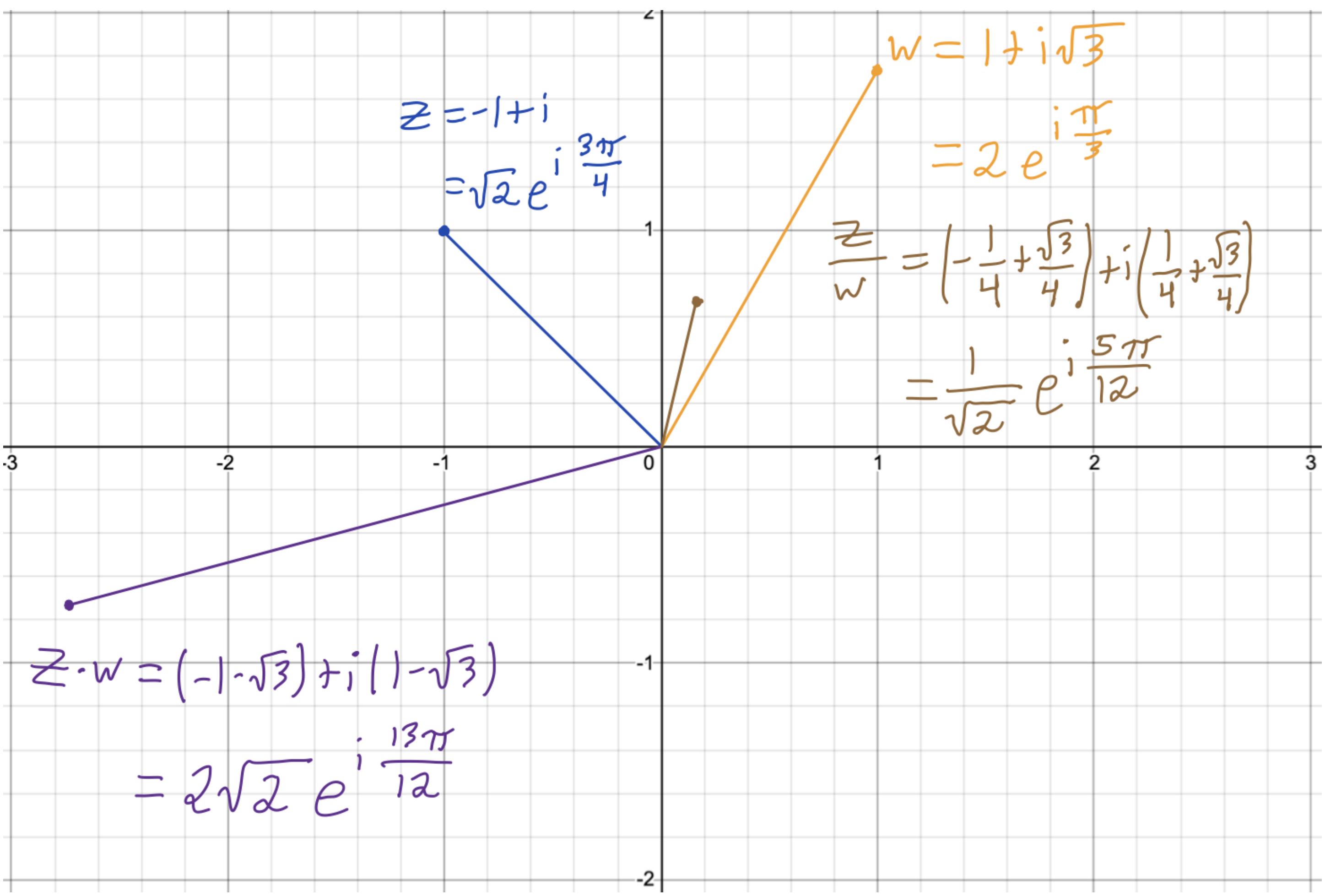

FIGURE 3. A grid to help you plot $w, z, z \cdot w$, and $\frac{z}{w}$.

Solution to Part (1): Firstly, we see that

$$z \cdot w = (-1+i)(1+i\sqrt{3}) = -1\cdot 1 + -1\cdot i\sqrt{3} + i\cdot 1 + \underbrace{i\cdot i}_{i^2=-1}\sqrt{3} \tag{8}$$

$$= -1 - i\sqrt{3} + i - \sqrt{3} = \boxed{(-1-\sqrt{3}) + i(1-\sqrt{3})}, \text{ and} \tag{9}$$

$$\frac{z}{w} = \frac{z\overline{w}}{w\overline{w}} = \frac{z\overline{w}}{|w|^2} = \frac{(-1+i)(1-i\sqrt{3})}{1^2+\sqrt{3}^2} \tag{10}$$

$$= \frac{1}{4}(-1\cdot 1 + (-1)\cdot(-i\sqrt{3}) + i\cdot 1 + \underbrace{i\cdot(-i}_{-i^2=1}\sqrt{3})) = \frac{1}{4}(-1+i\sqrt{3}+i+\sqrt{3}) \tag{11}$$

$$= \boxed{(-\frac{1}{4}+\frac{\sqrt{3}}{4}) + i(\frac{1}{4}+\frac{\sqrt{3}}{4})}. \tag{12}$$

Solution to Part (2): We see that $|z| = \sqrt{1^2+1^2} = \sqrt{2}$, and that $|w| = \sqrt{1^2+\sqrt{3}^2} = 2$. It follows that

Since z is in the third quadrant and w is in the first quadrant, we see that

$$\theta_z = \tan^{-1}(\frac{\text{Im(z)}}{\text{Re}(z)}) + \pi = \tan^{-1}(\frac{1}{-1}) + \pi = \tan^{-1}(-1) + \pi = \frac{3\pi}{4}, \text{ and} \tag{13}$$

$$\theta_w = \tan^{-1}(\frac{\text{Im}(w)}{\text{Re}(w)}) = \tan^{-1}(\frac{\sqrt{3}}{1}) = \tan^{-1}(\sqrt{3}) = \frac{\pi}{3}. \tag{14}$$

Recalling that $z = |z|e^{i\theta_z}$ and $w = |w|e^{i\theta_w}$ are the polar forms of z and w respectively, we see that $z = \sqrt{2}e^{\frac{3\pi i}{4}}$ and $w = 2e^{\frac{\pi i}{3}}$. We now see that

$$z\cdot w = \sqrt{2}e^{\frac{3\pi i}{4}} \cdot 2e^{\frac{\pi i}{3}} = 2\sqrt{2}e^{\frac{3\pi i}{4}+\frac{\pi i}{3}} = \boxed{2\sqrt{2}e^{\frac{13\pi i}{12}}}, \text{ and} \tag{15}$$

$$\frac{z}{w} = \frac{\sqrt{2}e^{\frac{3\pi i}{4}}}{2e^{\frac{\pi i}{3}}} = \frac{1}{\sqrt{2}}e^{\frac{3\pi i}{4}-\frac{\pi i}{3}} = \boxed{\frac{1}{\sqrt{2}}e^{\frac{5\pi i}{12}}}. \tag{16}$$

Using a computer algebra system such as wolfram alpha, we can confirm that

$$(17)\quad z\cdot w = 2\sqrt{2}e^{\frac{13\pi i}{12}} = 2\sqrt{2}(\cos(\frac{13\pi}{12})+i\sin(\frac{13\pi}{12})) = (-1-\sqrt{3})+i(1-\sqrt{3})$$

and

$$(18)\quad \frac{z}{w} = \frac{1}{\sqrt{2}}e^{\frac{5\pi i}{12}} = \frac{1}{\sqrt{2}}(\cos(\frac{5\pi}{12})+i\sin(\frac{5\pi}{12})) = (-\frac{1}{4}+\frac{\sqrt{3}}{4})+i(\frac{1}{4}+\frac{\sqrt{3}}{4}).$$

Solution to Part (3): The calcuations from Part (2) show us that when we multiply 2 complex numbers we multiply their magnitudes and add their arguments, and when we divide 2 complex numbers, we divide their magnitudes and subtract their arguments. In particular, if we multiply a complex number (such as z) by w, then we will double the magnitude and add $\frac{5\pi}{12}$ to the argument, and if we divide a complex number (such as z) by w, then we will half the magnitude and subtract $\frac{5\pi}{12}$ from the argument.

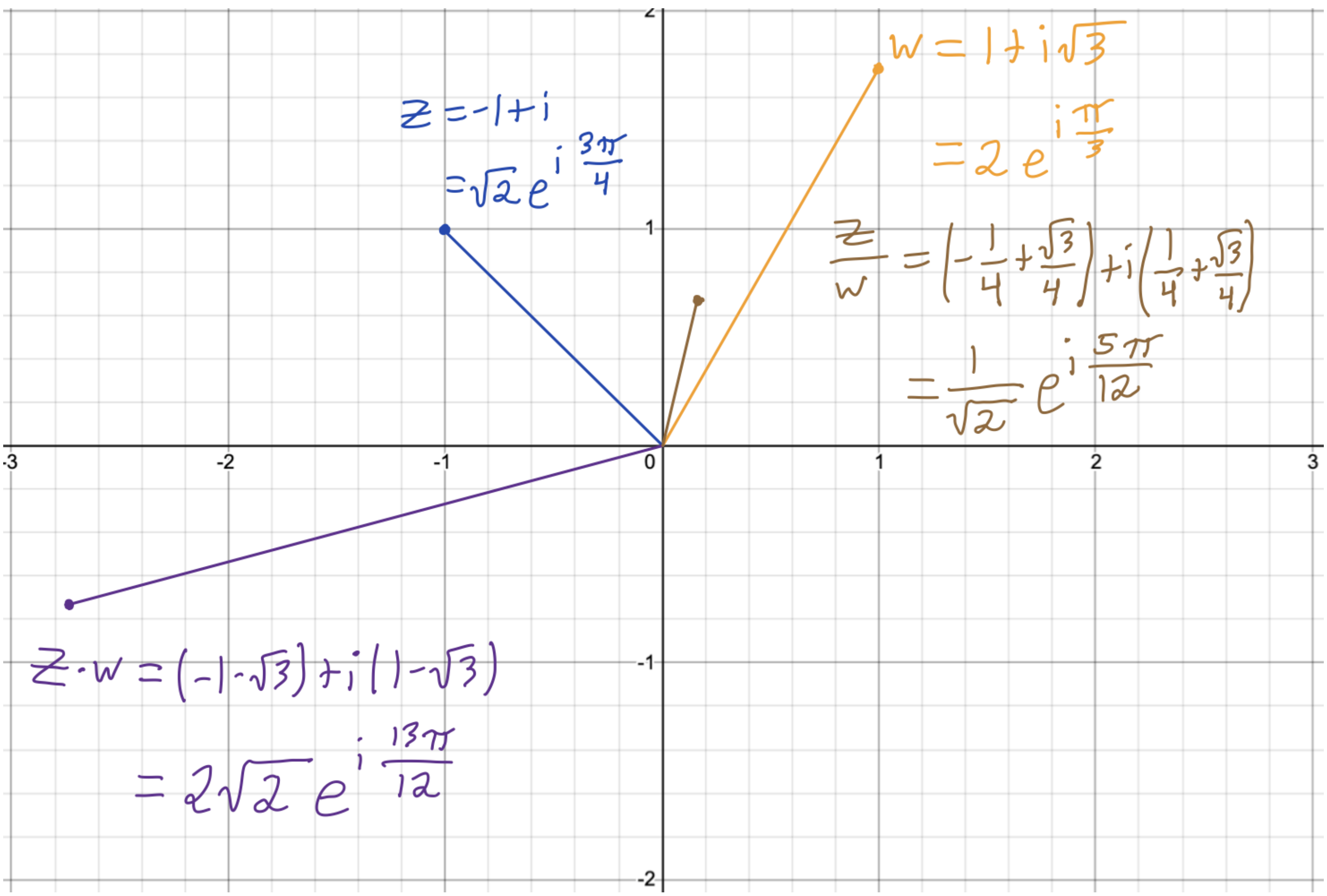

Figure 4. A grid to help you plot $w, z, z\cdot w$, and $\frac{z}{w}$.

Problem 2 (8 points): (Appendix C. 29, 30)
(1) Equate the real and imaginary parts of both sides of the identity

$$e^{i(a-b)} = e^{ia}e^{-ib}$$

to prove that

$$\cos(a-b) = \cos(a)\cos(b) + \sin(a)\sin(b);$$
$$\sin(a-b) = \sin(a)\cos(b) - \cos(a)\sin(b).$$

(2) Equate the real and imaginary parts of both sides of the identity

$$e^{i2\theta} = e^{i\theta} \cdot e^{i\theta}$$

to prove that

$$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta), \text{ and } \sin(2\theta) = 2\sin(\theta)\cos(\theta)$$

Solution to Part (1): Using Euler's formula, we see that

(19) $e^{i(a-b)} = \cos(a-b) + i\sin(a-b), e^{ia} = \cos(a) + i\sin(a)$, and

(20) $e^{-ib} = e^{i(-b)} = \cos(-b) + i\sin(-b) = \cos(b) - i\sin(b)$, so

(21) $$\cos(a-b) + i\sin(a-b) = e^{i(a-b)} = e^{ia}e^{-ib}$$

(22) $$= (\cos(a) + i\sin(a))(\cos(b) - i\sin(b))$$

(23) $$= \cos(a)\cos(b) - i\cos(a)\sin(b) + i\sin(a)\cos(b) - \underbrace{i^2}_{i^2=-1}\sin(a)\sin(b)$$

(24) $$= \cos(a)\cos(b) + \sin(a)\sin(b) + i(\sin(a)\cos(b) - \cos(a)\sin(b))$$

. . . .

$$\rightarrow \begin{array}{rcl} \cos(a-b) & = & \cos(a)\cos(b)+\sin(a)\sin(b) \\ \sin(a-b) & = & \sin(a)\cos(b)-\cos(a)\sin(b) \end{array}. \tag{25}$$

Solution to Part (2): We could deduce the result in part (2) by letting $a = -b = \theta$, but we will instead prove the result as a corollary to Euler's formula once again just for the extra practice. We once again begin the problem by using Euler's formula to see that

$$e^{i2\theta} = \cos(2\theta) + i\sin(2\theta) \text{ and } e^{i\theta} = \cos(\theta) + i\sin(\theta), \text{ so} \tag{26}$$

. . . .

$$e^{i2\theta} = e^{2i\theta} = (e^{i\theta})^2 = (\cos(\theta) + i\sin(\theta))^2 \tag{27}$$

. . . .

$$= \cos^2(\theta) + 2i\cos(\theta)\sin(\theta) + \underbrace{i^2}_{i^2=-1}\sin^2(\theta) \tag{28}$$

. . . .

$$= \cos^2(\theta) - \sin^2(\theta) + i2\cos(\theta)\sin(\theta) \tag{29}$$

. . . .

$$\rightarrow \begin{array}{rcl} \cos(2\theta) & = & \cos^2(\theta)-\sin^2(\theta) \\ \sin(2\theta) & = & 2\sin(\theta)\cos(\theta) \end{array}. \tag{30}$$

Problem 16.2.17 (10 points): Solve the following initial value problem.

$$y'' - 3y' - 18y = 0; \quad y(0) = 0, y'(0) = 4. \tag{31}$$

Draw the graph of the solution. (You may seek help from graphing website/software. Think about why the graph behave in that way and how is that related to the solution function.)

Solution: We see that the characteristic polynomial of equation (31) is

$$0 = r^2 - 3r - 18 = (r-6)(r+3), \tag{32}$$

which has roots $r = -3, 6$. It follows that the general solutions to equation (31) is

$$y(t) = c_1 e^{-3t} + c_2 e^{6t}. \tag{33}$$

Using the initial conditions, we see that

$$\begin{array}{rcl} 0 = y(0) & = & c_1 e^{-3\cdot 0} + c_2 e^{6\cdot 0} = c_1 + c_2 \\ 4 = y'(0) & = & -3c_1 e^{3\cdot 0} + 6c_2 e^{6\cdot 0} = -3c_1 + 6c_2 \end{array} \tag{34}$$

$$\rightarrow \begin{array}{rcrcl} c_1 & + & c_2 & = & 0 \\ -3c_1 & + & 6c_2 & = & 4 \end{array} \overset{R_2+3R_1}{\rightarrow} \begin{array}{rcrcl} c_1 & + & c_2 & = & 0 \\ & & 9c_2 & = & 4 \end{array} \tag{35}$$

$$\overset{\frac{1}{9}R_2}{\rightarrow} \begin{array}{rcrcl} c_1 & + & c_2 & = & 0 \\ & & c_2 & = & \frac{4}{9} \end{array} \overset{R_1-R_2}{\rightarrow} \begin{array}{rcl} c_1 & = & -\frac{4}{9} \\ c_2 & = & \frac{4}{9} \end{array} \tag{36}$$

$$\rightarrow \boxed{y(t) = -\frac{4}{9}e^{-3t} + \frac{4}{9}e^{6t}}. \tag{37}$$

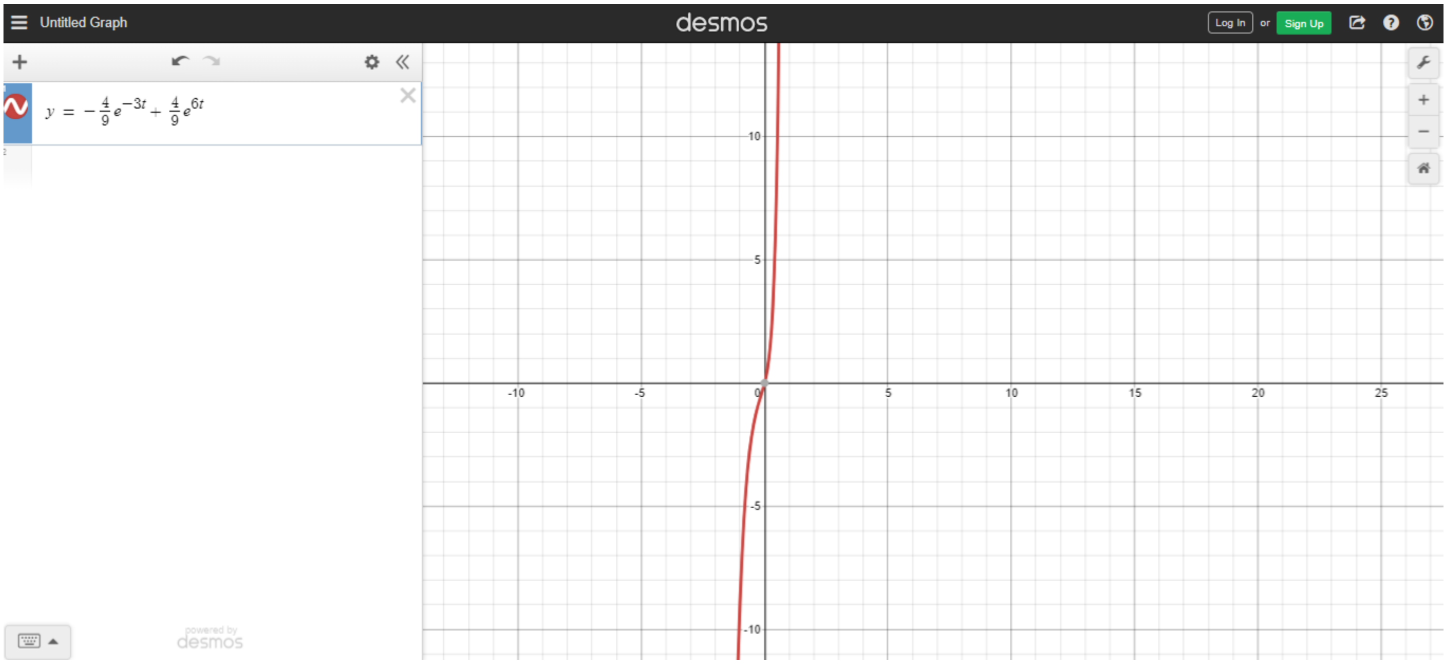

FIGURE 5. A graph of the solution to the initial value problem in (31).

Problem 16.2.23 (10 points): Solve the following initial value problem.

$$y'' - y' + \frac{1}{4}y = 0; \quad y(0) = 1, y'(0) = 2. \tag{38}$$

Draw the graph of the solution. (You may seek help from graphing website/software. Think about why the graph behave in that way and how is that related to the solution function.)

Solution: We see that the characteristic polynomial of equation (38) is

$$0 = r^2 - r + \frac{1}{4} = (r - \frac{1}{2})^2, \tag{39}$$

which has $r = \frac{1}{2}$ as a double root. It follows that the general solutions to equation (38) is

$$y(t) = c_1 e^{\frac{t}{2}} + c_2 t e^{\frac{t}{2}}. \tag{40}$$

Noting that

$$y'(t) = \frac{1}{2}c_1 e^{\frac{t}{2}} + c_2 e^{\frac{t}{2}} + \frac{1}{2}c_2 t e^{\frac{t}{2}} = (\frac{1}{2}c_1 + c_2)e^{\frac{t}{2}} + \frac{1}{2}c_2 t e^{\frac{t}{2}}, \tag{41}$$

we can use the initial conditions, to see that

$$\begin{array}{rcccccl} 1 & = & y(0) & = & c_1 e^{\frac{0}{2}} + c_2 \cdot 0 \cdot e^{\frac{0}{2}} & = & c_1 \\ 2 & = & y'(0) & = & (\frac{1}{2}c_1 + c_2)e^{\frac{0}{2}} + \frac{1}{2}c_2 \cdot 0 \cdot e^{\frac{0}{2}} & = & \frac{1}{2}c_1 + c_2 \end{array} \tag{42}$$

....

$$\to \begin{array}{ccccc} c_1 & & & = & 1 \\ \frac{1}{2}c_1 & + & c_2 & = & 2 \end{array} \to \begin{array}{ccccc} c_1 & = & & & 1 \\ c_2 & = & 2 - \frac{1}{2} \cdot 1 & = & \frac{3}{2} \end{array} \tag{43}$$

....

$$\to \boxed{y(t) = e^{\frac{t}{2}} + \frac{3}{2}te^{\frac{t}{2}}}. \tag{44}$$

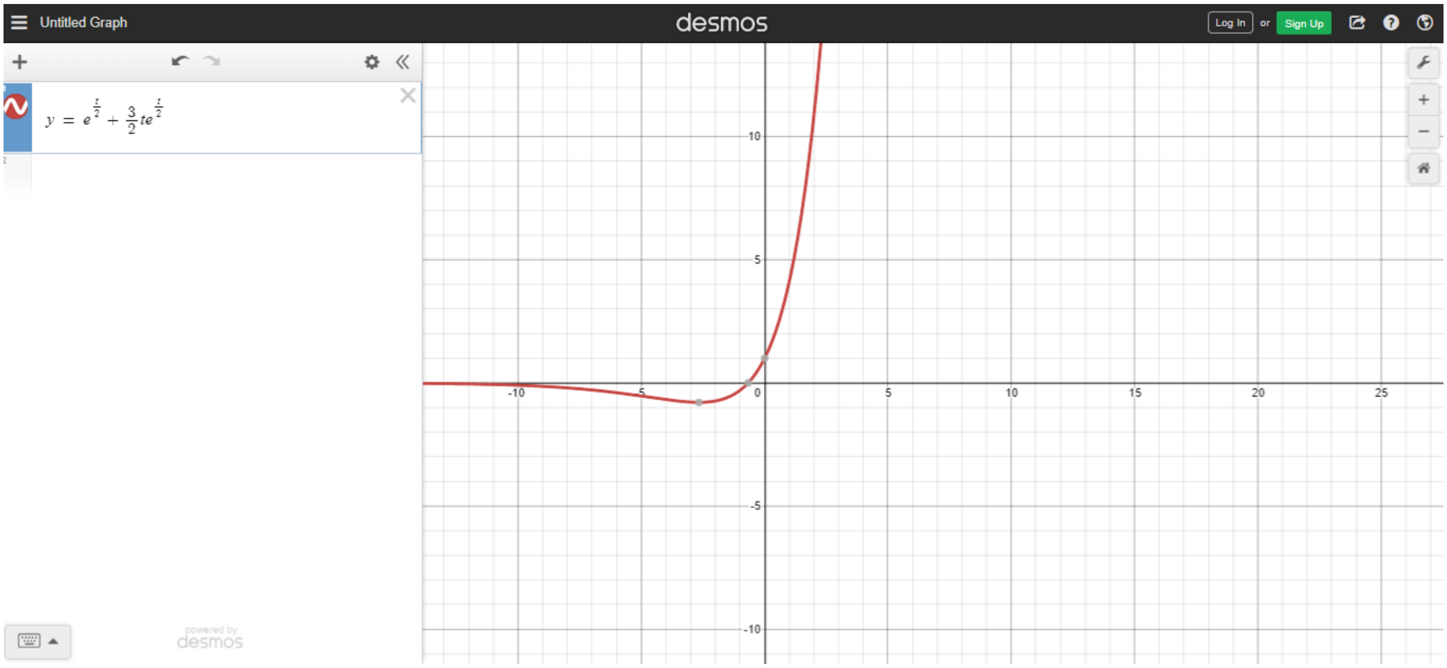

Figure 6. A graph of the solution to the initial value problem in (38).

Problem 16.2.31 (10 points): Solve the following initial value problem.

$$y'' + 6y' + 10y = 0; \quad y(0) = 0, y'(0) = 6. \tag{45}$$

Draw the graph of the solution. (You may seek help from graphing website/software. Think about why the graph behave in that way and how is that related to the solution function.)

Solution: We see that the characteristic polynomial of equation (45) is

$$0 = r^2 + 6r + 10 \to r = \frac{-6 \pm \sqrt{6^2 - 4 \cdot 1 \cdot 10}}{2 \cdot 1} = \frac{-6 \pm \sqrt{-4}}{2} = -3 \pm i, \tag{46}$$

It follows that the general solutions to equation (45) is

$$y(t) = c_1' e^{(-3+i)t} + c_2' e^{(-3-i)t} = c_1 \sin(t) e^{-3t} + c_2 \cos(t) e^{-3t}. \tag{47}$$

Noting that

$$y'(t) = c_1 \cos(t) e^{-3t} - 3c_1 \sin(t) e^{-3t} - c_2 \sin(t) e^{-3t} - 3c_2 \cos(t) e^{-3t} \tag{48}$$

$$= (-3c_1 - c_2) \sin(t) e^{-3t} + (c_1 - 3c_2) \cos(t) e^{-3t}, \tag{49}$$

we can use the initial conditions to see that

$$\begin{array}{rcl} 0 = y(0) & = & c_1 \sin(0) e^{-3\cdot 0} + c_2 \cos(0) e^{-3\cdot 0} \\ 6 = y'(0) & = & (-3c_1 - c_2) \sin(0) e^{-3\cdot 0} + (c_1 - 3c_2) \cos(0) e^{-3\cdot 0} \end{array} \tag{50}$$

. . . .

$$\to \begin{array}{rcl} 0 & = & c_2 \\ 6 & = & c_1 - 3c_2 \end{array} \to \begin{array}{rcl} c_2 & = & 0 \\ c_1 & = & 6 + 3c_2 = 6 \end{array} \tag{51}$$

. . . .

$$\to \boxed{y(t) = 6 \sin(t) e^{-3t}}. \tag{52}$$

FIGURE 7. A graph of the solution to the initial value problem in (45).

Problem 16.2.37 (10 points): Solve the following initial value problem.

$$t^2y'' + 6ty' + 6y = 0;\ y(1) = 0, y'(1) = -4. \tag{53}$$

Draw the graph of the solution. (You may seek help from graphing website/software. Think about why the graph behave in that way and how is that related to the solution function.)

Remark: The solution below is a detailed solutions 'from first principles' that also demonstrates how to perform a change of variables in a differential equation. For a homework or an exam, you are permitted to use the fact that the characteristic equation of $t^2y''+aty'+by = 0$ is given by $r^2+(a-1)r+b = 0$, and that the general solution is of the form $y(t) = c_1t^{r_1} + c_2t^{r_2}$ if the roots are distinct[1], or $y(t) = c_1t^r + c_2t^r \ln(t)$ if $r_1 = r_2 = r$.

Solution: We perform a substitution (or a change of variables) in order to convert equation (53) into a constant coefficient differential equation, which will then be straight-forward to solve. Letting $x = \ln(t)$, we see that $t = e^x$, and we may define $h(x) = y(e^x) = y(t)$. We see that

$$h'(x) = \frac{d}{dx}h(x) = \frac{d}{dx}y(e^x) = y'(e^x) \cdot \frac{d}{dx}e^x = y'(e^x) \cdot e^x = ty'(t), \text{ and} \tag{54}$$

$$h''(x) = \frac{d}{dx}h'(x) = \frac{d}{dx}(e^xy'(e^x)) = \frac{d}{dx}(e^x) \cdot y'(e^x) + e^x \cdot \frac{d}{dx}y'(e^x) \tag{55}$$

$$= e^xy'(e^x) + e^x \cdot e^xy''(e^x) = e^xy'(e^x) + e^{2x}y''(e^x) = ty'(t) + t^2y''(t). \tag{56}$$

We now see that

$$0 = t^2y'' + 6ty' + 6y = (t^2y'' + ty') + 5ty' + 6y \tag{57}$$

$$= (t^2y''(t) + ty'(t)) + 5ty'(t) + 6y(t) \tag{58}$$

[1] This general form is still correct if r_1 and r_2 are distinct complex numbers, but it is usually not the preferred form of the general solution. If $r\pm si$ are the distinct complex roots of the characteristic equation, then the preferred form the general solution is $y(t) = c_1t^r \cos(s\ln(t))+c_2t^r \sin(s\ln(t))$.

. . . .

$$= h''(x) + 5h'(x) + 6h(x) = h'' + 5h' + 6h. \tag{59}$$

We see that the characteristic equation of our converted equation is

$$0 = r^2 + 5r + 6 = (r+2)(r+3), \tag{60}$$

and has solutions $r = -3, -2$. It follows that the general solution to our converted equation is

$$h(x) = c_1e^{-2x} + c_2e^{-3x}. \tag{61}$$

Recalling that $x = \ln(t)$, we see that the general solution to equation (53) is

$$y(t) = h(x) = c_1e^{-2x} + c_2e^{-3x} = c_1e^{-2\ln(t)} + c_2e^{-3\ln(t)} = c_1t^{-2} + c_2t^{-3}. \tag{62}$$

Making use of the initial conditions, we see that

$$\begin{array}{rcrcrcr} 0 & = & y(1) & = & c_1 \cdot 1^{-2} + c_2 \cdot 1^{-3} & = & c_1 + c_2 \\ -4 & = & y'(1) & = & -2c_1 \cdot 1^{-3} - 3c_2 \cdot 1^{-4} & = & -2c_1 - 3c_2 \end{array} \tag{63}$$

. . . .

$$\to \begin{array}{rcrcr} c_1 & + & c_2 & = & 0 \\ -2c_1 & - & 3c_2 & = & -4 \end{array} \xrightarrow{R_2+2R_1} \begin{array}{rcrcr} c_1 & + & c_2 & = & 0 \\ & & -c_2 & = & -4 \end{array} \tag{64}$$

. . . .

$$\xrightarrow{\frac{1}{5}R_2} \begin{array}{rcrcr} c_1 & + & c_2 & = & 0 \\ & & c_2 & = & 4 \end{array} \xrightarrow{R_1-R_2} \begin{array}{rcr} c_1 & = & -4 \\ c_2 & = & 4 \end{array} \tag{65}$$

. . . .

$$\to \boxed{y(t) = -4t^{-2} + 4t^{-3}}. \tag{66}$$

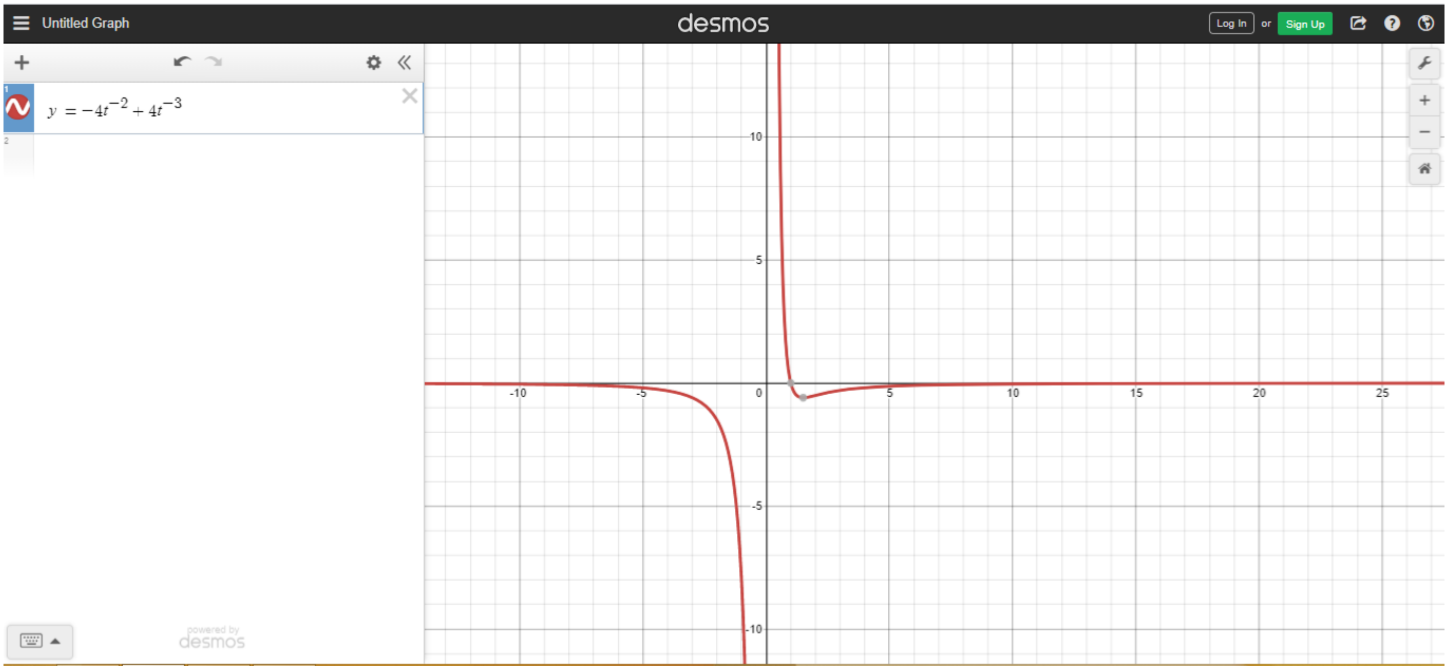

FIGURE 8. A graph of the solution to the initial value problem in (53).