#1 (15 pts)

Let

$\vec{A} = 3\vec{i} - 4\vec{j}$

$\vec{B} = \vec{i} - 4\vec{j}$

(a) Find the vectors $\vec{M}$ and $\vec{N}$ such that $\vec{A} = \vec{M} + \vec{N}$, where

$\vec{M}$ is parallel to $\vec{B}$, and $\vec{N}$ is perpendicular to $\vec{B}$.

Verify that $\vec{N}$ is perpendicular to $\vec{B}$.

SOLUTION :

$$\vec{M} = \overrightarrow{\text{proj}_{\vec{B}}}\,\vec{A} = \frac{\vec{A}\cdot\vec{B}}{\vec{B}\cdot\vec{B}}\,\vec{B} = \ = \frac{19}{17}\langle 1\,,\,-4\rangle = \left\langle \frac{19}{17}\,,\,\frac{-76}{17}\right\rangle$$

$$\vec{N} = \vec{A} - \vec{M} = \langle 3\,,\,-4\rangle - \left\langle \frac{19}{17}\,,\,\frac{-76}{17}\right\rangle = \left\langle \frac{32}{17}\,,\,\frac{8}{17}\right\rangle$$

Check :

$$\vec{M} + \vec{N} = \left\langle \frac{19}{17}\,,\,\frac{-76}{17}\right\rangle + \left\langle \frac{32}{17}\,,\,\frac{8}{17}\right\rangle = \left\langle \frac{51}{17}\,,\,\frac{-68}{17}\right\rangle = \langle 3\,,\,-4\rangle = \vec{A};$$

$$\vec{M} = \frac{19}{17}\langle 1\,,\,-4\rangle = \frac{19}{17}\,\vec{B}\,; \quad \text{and}$$

$$\vec{N}\cdot\vec{B} = \left\langle \frac{32}{17}\,,\,\frac{8}{17}\right\rangle \cdot \langle 1\,,\,-4\rangle = \frac{32}{17} - \frac{32}{17} = 0$$

(b) Find the unit vector with the same direction as $\vec{A}$.

$$\frac{\vec{A}}{|\,\vec{A}\,|} = \frac{\langle 3\,,\,-4\rangle}{5} = \left\langle \frac{3}{5}\,,\,\frac{-4}{5}\right\rangle$$

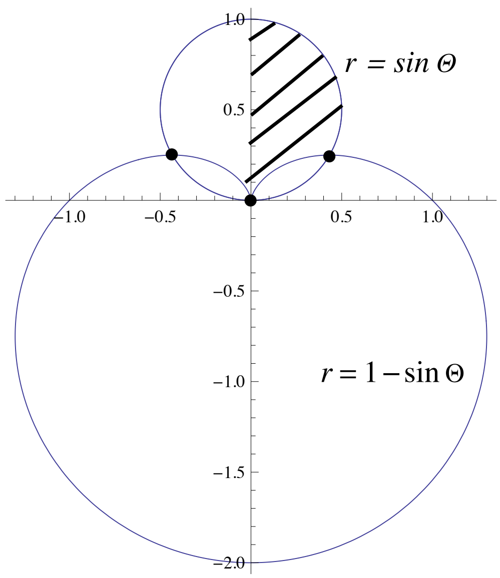
Midterm 3
Form A, Page 3

#2 (20 pts) Consider the polar curves $r = \sin\Theta$ (a circle)

and $r = 1 - \sin\Theta$ (a cardioid).

(a) (8 pts) Find all points of intersection of the two curves.

SOLUTION :



$$\sin\Theta = 1 - \sin\Theta$$
$$2\sin\Theta = 1$$
$$\sin\Theta = \frac{1}{2}$$

$$\Theta = \frac{\pi}{6},\ \frac{5\pi}{6}$$

Since $r = \sin\Theta$, $r = \frac{1}{2}$

POINTS OF INTERSECTION : $\left(\frac{1}{2}, \frac{\pi}{6}\right)$, $\left(\frac{1}{2}, \frac{5\pi}{6}\right)$, **and (0, 0) (GRAPH)**

(b) (12 pts) Find the area of the region that lies <u>inside</u> the circle and <u>outside</u> the cardioid.

$$\text{AREA} = 2 \text{ x SHADED AREA} = 2\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{1}{2}\left([\sin\Theta]^2 - [1 - \sin\Theta]^2\right) d\Theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left([\sin\Theta]^2 - 1 + 2\sin\Theta - [\sin\Theta]^2\right) d\Theta =$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (2\sin\Theta - 1)\, d\Theta = \left(-2\cos\Theta - \Theta\right)\Big|_{\frac{\pi}{6}}^{\frac{\pi}{2}} =$$

$$= -\frac{\pi}{2} + 2\,\frac{\sqrt{3}}{2} + \frac{\pi}{6} = \sqrt{3} - \frac{\pi}{3}$$

#3 (23 pts)

Let

$$\vec{r}(t) = \left\langle \sin\left(t^2\right),\ \cos\left(t^2\right),\ 4 \right\rangle,\ t \geq 0.$$

be a position vector of a moving particle.

(a) (9 pts) Compute the velocity, speed, and acceleration.

SOLUTION :

$$\vec{v}(t) = \vec{r}\,'(t) = \left\langle \cos\left(t^2\right) 2t,\ -\sin\left(t^2\right) 2t,\ 0 \right\rangle$$

$$\left|\vec{v}(t)\right| = \sqrt{\cos^2\left(t^2\right) 4t^2 + \sin^2\left(t^2\right) 4t^2} =$$

$$= \sqrt{4t^2\left(\cos^2\left(t^2\right) + \sin^2\left(t^2\right)\right)} =$$

$$= \sqrt{4t^2} = 2t$$

$$\vec{a}(t) = \vec{v}\,'(t) = \vec{r}\,''(t) =$$

$$= \left\langle -\sin\left(t^2\right) 4t^2 + \cos\left(t^2\right) 2,\ -\cos\left(t^2\right) 4t^2 - \sin\left(t^2\right) 2,\ 0 \right\rangle$$

(b) (2 pts) Find the unit tangent vector $\vec{T}(t)$.

SOLUTION :

$$\vec{T}(t) = \frac{\vec{v}(t)}{\left|\vec{v}(t)\right|} = \frac{\left\langle \cos\left(t^2\right) 2t,\ -\sin\left(t^2\right) 2t,\ 0 \right\rangle}{2t} =$$

$$= \left\langle \cos\left(t^2\right),\ -\sin\left(t^2\right),\ 0 \right\rangle$$

(c) (6 pts) Find the arc length function $s(t)$.

SOLUTION :

$$s(t) = \int_0^t \left|\vec{v}(u)\right| du = \int_0^t 2u\, du = u^2 \Big|_0^t = t^2$$

(d) (2 pts) Find the description of the curve that uses arc length as a parameter.

SOLUTION :

$$s = t^2 \Longrightarrow t = \sqrt{s} \Longrightarrow \vec{r}(s) = \langle \sin(s), \cos(s), 4 \rangle, \ s \geq 0.$$

(e) (4 pts) Find the angle between the vectors $\vec{r}(t)$ and $\vec{r}\,'(t)$.

SOLUTION :

Since

$$\vec{r}(t) \cdot \vec{r}\,'(t) = \left\langle \sin\left(t^2\right), \cos\left(t^2\right), 4 \right\rangle \cdot \left\langle \cos\left(t^2\right) 2t, -\sin\left(t^2\right) 2t, 0 \right\rangle =$$

$$= 2t \sin\left(t^2\right) \cos\left(t^2\right) - 2t \sin\left(t^2\right) \cos\left(t^2\right) + 0 = 0,$$

it follows that the angle between the vectors $\vec{r}(t)$ and $\vec{r}\,'(t)$ is $\frac{\pi}{2}$.

#4 (26 pts)

(a) Find the equation of the plane through the point (1, 2, 2) that contains the line $\vec{r}(t) = \langle 1 - t,\ 5t,\ 3 + t \rangle$.

SOLUTION :

We were given a point in the plane $P_0 = (1,\ 2,\ 2)$,

so let's choose two points on the line - say - corresponding to $t = 0$ and $t = 1$:

$P_1 = (1,\ 0,\ 3)$ and $P_2 = (0,\ 5,\ 4)$.

Let's form two vectors in the plane :

$$\boxed{\overrightarrow{P_0 P_1} = \langle 0,\ -1,\ 1 \rangle \text{ and } \overrightarrow{P_0 P_2} = \langle -1,\ 3,\ 2 \rangle}.$$

We can now find a normal vector :

$$\vec{N} = \overrightarrow{P_0 P_1} \times \overrightarrow{P_0 P_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & -2 & 1 \\ -1 & 3 & 2 \end{vmatrix} = \boxed{-7\,\vec{i} - 1\,\vec{j} - 2\,\vec{k}}$$

Equation : $\boxed{-7\,(x - 1) - (y - 2) - 2\,(z - 2) = 0}$

(b) Find <u>parametric</u> equations of the line through the point (4, -7, 0) that is parallel to the line $\vec{r}(t) = \langle 1 - t,\ 5t,\ 3 + t \rangle$.

SOLUTION :

We were given a point $P_0 = (4,\ -7,\ 0)$ and the direction vector is the same as for the given line : $\vec{v} = \langle -1,\ 5,\ 1 \rangle$.

Parametric equations :
$$x = 4 - t$$
$$y = -7 + 5t$$
$$z = t$$

#5 (16 pts) Let $f(x, y) = \dfrac{1}{x^2 - y^2}$.

(I) State the domain and the range of the function f.

DOMAIN :

$x^2 - y^2 \neq 0$

$y^2 \neq x^2$

$y \neq \pm x$

All points in the plane except the two lines : $y = x$ and $y = -x$.

RANGE : $(-\infty, 0) \cup (0, +\infty)$

(II) Sketch the following level curves :

(a) $f(x, y) = 1$; (b) $f(x, y) = -\dfrac{1}{4}$.

Label the level curves with their z - values.

SOLUTION :

(a) $\dfrac{1}{x^2 - y^2} = 1$

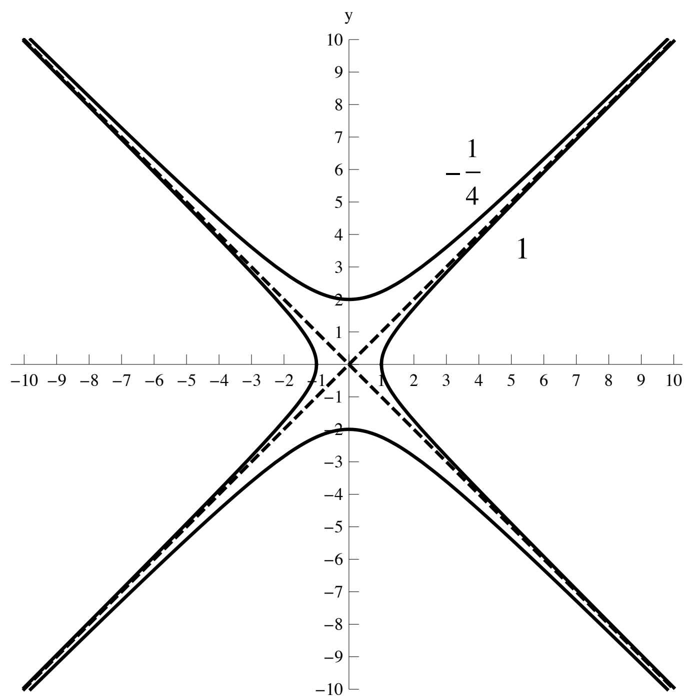
$x^2 - y^2 = 1$ hyperbola

(b) $\dfrac{1}{x^2 - y^2} = -\dfrac{1}{4}$

$x^2 - y^2 = -4$

$-\dfrac{x^2}{4} + \dfrac{y^2}{4} = 1$

(up - down)


y
10
9
8
7
6
5
4
3
2
1
-1
-2
-3
-4
-5
-6
-7
-8
-9
-10
-10 -9 -8 -7 -6 -5 -4 -3 -2 -1 1 2 3 4 5 6 7 8 9 10
$-\frac{1}{4}$
1