Math 1172

Midterm 3

Autumn 2014

Form A

Name: ________________

OSU username (name.nn): ________________

Lecturer: ________________

Recitation Instructor: ________________

Recitation Time: ________________

Instructions

•) SHOW ALL WORK!!! Incorrect answers with work shown may receive partial credit, but unsubstantiated answers may receive NO credit.

•) Give EXACT answers unless asked to do otherwise.

•) You do not need to simplify numerical answers such as $\frac{5}{\sqrt{8}} - \frac{5}{\sqrt{12}}$.

•) Calculators are permitted EXCEPT those calculators that have computer algebra systems (CAS) or ability to communicate with others. Furthermore, all memory must be cleared and all apps must be removed. PDA's, laptops, and cell phones are prohibited. Do not have these devices out!

•) The exam duration is 55 minutes.

•) The exam consists of 5 problems starting on page 2 and ending on page 6. Make sure your exam is not missing any pages before you start. Page 7 contains formulas. Page 8 may be used for extra work space.

Problem Number	Maximum Point Value	Score
1	18	
2	16	
3	22	
4	24	
5	20	
Total	100	

Problem 1

[18 pts] True or False. You do not need to show work for problems on this page.

a) [2 pts] The level curves of the function $z = 3x^2 + y^2$ for $z > 0$ are ellipses.

True. At level $z = z_0 > 0$, we have level curve
$z_0 = 3x^2 + y^2$, or $\dfrac{x^2}{\left(\frac{z_0}{3}\right)} + \dfrac{y^2}{z_0} = 1$, which is an ellipse.

b) [2 pts] If the vectors $\mathbf{u}$ and $\mathbf{v}$ are parallel, then $|\mathbf{u} + \mathbf{v}| = |\mathbf{u}| + |\mathbf{v}|$.

False. Counter example: $\vec{u} = -\vec{v}$, $\vec{u} \neq \vec{0}$. Then
$|\vec{u} + \vec{v}| = |\vec{0}| = 0$, but $|\vec{u}| + |\vec{v}| = 2|\vec{u}| \neq 0$.

c) [2 pts] For any plane parallel to the yz-plane, the trace of the surface $x^2 - y - 4z^2 = 1$ in that plane is a parabola.
is of the form $x = c$ for some constant c.

True. the trace of $x^2 - y - 4z^2 = 1$ in $x = c$ gives $c^2 - y - 4z^2 = 1$,
i.e. $y = -4z^2 + c^2 - 1$, so we get a parabola.

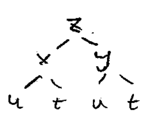
d) [2 pts] Given functions $f(t)$ and $g(t)$ with $f'(t)$ and $g'(t)$ continuous, the curves defined by $\mathbf{r}(t) = < f(t), g(t) >$ and $\mathbf{R}(t) = < g(t), f(t) >$ have the same length over the interval $[a, b]$.

True. $L_{\vec{r}} = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2}\, dt = \int_a^b \sqrt{(g'(t))^2 + (f'(t))^2}\, dt = L_{\vec{R}}$.

e) [2 pts] Assume that a constant force $\mathbf{F} = \langle 3, -1, 2 \rangle$ (in newtons) is used to move an object from $(2, 0, 2)$ to $(3, 3, 3)$ (distance measured in meters.) Then, the work done by the force is 12 Joules.

False. Work in question $= \vec{F} \cdot \langle 3-2, 3-0, 3-2 \rangle$
$= \langle 3, -1, 2 \rangle \cdot \langle 1, 3, 1 \rangle = 3 \cdot 1 + (-1) \cdot 3 + 2 \cdot 1 = 2$ J

f) [2 pts] If $z = f(x, y)$ is differentiable and $x = u + t$ and $y = u - t$, then $\dfrac{\partial z}{\partial t} = \dfrac{\partial z}{\partial x} - \dfrac{\partial z}{\partial y}$.

True. 
$\dfrac{\partial z}{\partial t} = \dfrac{\partial z}{\partial x} \cdot \dfrac{\partial x}{\partial t} + \dfrac{\partial z}{\partial y} \cdot \dfrac{\partial y}{\partial t} = \dfrac{\partial z}{\partial x} - \dfrac{\partial z}{\partial y}$ b/c $\dfrac{\partial x}{\partial t} = 1$, $\dfrac{\partial y}{\partial t} = -1$.

g) [2 pts] The curve $\mathbf{r}(t) = \langle \cos t, t, \sin t \rangle$ is parameterized by arc length.

False. $\vec{r}'(t) = \langle -\sin t, 1, \cos t \rangle$, $|\vec{r}'(t)| = \sqrt{(-\sin t)^2 + 1 + (\cos t)^2} = \sqrt{2}$
$\int_0^t |\vec{r}'(t)|\, dt = \int_0^t \sqrt{2}\, dt = \sqrt{2}\, t \neq t$. So NOT parameterized by arc length.

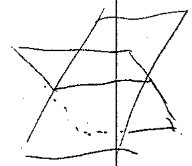
h) [2 pts] If the speed $|\mathbf{v}(t)|$ is a constant then the curve $\mathbf{r}(t)$ is a line.

False. Counterexample. The curve in (g) is not a line but
$|\vec{v}(t)| = |\vec{r}'(t)| = \sqrt{2}$.

i) [2 pts] $\displaystyle\lim_{(x,y)\to(-2,2)} \frac{x^3 y - x y^3}{2(x^2 - y^2)} = -2$

True. $\displaystyle = \lim_{(x,y)\to(-2,2)} \frac{xy\cancel{(x^2 - y^2)}}{2\cancel{(x^2 - y^2)}} = \lim_{(x,y)\to(-2,2)} \frac{xy}{2} = \frac{-2 \cdot 2}{2} = -2$

Problem 2



[16 pts] Lines and Planes.

a) [8 pts] Find an equation of the line of intersection of the planes $2x - z = 3$ and $x + y + 3z = 2$.

The normal vectors are $\vec{n}_1 = \langle 2, 0, -1 \rangle$ and $\vec{n}_2 = \langle 1, 1, 3 \rangle$.

Then the line of intersection has direction vector

$$\vec{v} = \vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 0 & -1 \\ 1 & 1 & 3 \end{vmatrix} = \begin{vmatrix} 0 & -1 \\ 1 & 3 \end{vmatrix} \vec{i} - \begin{vmatrix} 2 & -1 \\ 1 & 3 \end{vmatrix} \vec{j} + \begin{vmatrix} 2 & 0 \\ 1 & 1 \end{vmatrix} \vec{k}$$

$$= \vec{i} - 7\vec{j} + 2\vec{k} = \langle 1, -7, 2 \rangle.$$

Next we need a point on the line.

$$\begin{cases} 2x - z = 3 \cdots (1) \\ x + y + 3z = 2 \cdots (2) \end{cases}$$

$(1) \Rightarrow z = 2x - 3$.

plug it in (2) $\Rightarrow x + y + 3(2x-3) = 7x + y - 9 = 2$

$\Leftrightarrow 7x + y = 11$.

Hence if $x = 1$, then $y = 4$. and $z = -1$.

So $(1, 4, -1)$ is a point on the line, and the equation is

$$\vec{r}(t) = \langle 1, 4, -1 \rangle + t\vec{v} = \langle 1, 4, -1 \rangle + \langle t, -7t, 2t \rangle$$

$$= \langle 1+t, 4-7t, -1+2t \rangle.$$

b) [8 pts] Find an equation of the plane that passes through the points $(2, 0, 3)$, $(-1, 1, 2)$, and $(0, 2, 0)$.

Vectors in plane: e.g. $\langle -1-2, 1-0, 2-3 \rangle = \langle -3, 1, -1 \rangle$

and $\langle 0-2, 2-0, 0-3 \rangle = \langle -2, 2, -3 \rangle$.

So a normal vector of the plane is

$$\langle -3, 1, -1 \rangle \times \langle -2, 2, -3 \rangle = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -3 & 1 & -1 \\ -2 & 2 & -3 \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 2 & -3 \end{vmatrix} \vec{i} - \begin{vmatrix} -3 & -1 \\ -2 & -3 \end{vmatrix} \vec{j} + \begin{vmatrix} -3 & 1 \\ -2 & 2 \end{vmatrix} \vec{k}$$

$$= -\vec{i} - 7\vec{j} - 4\vec{k} = \langle -1, -7, -4 \rangle$$

a point on the plane is $(2, 0, 3)$, for example.

So equation of the plane is

$$\langle x-2, y-0, z-3 \rangle \cdot \langle -1, -7, -4 \rangle = 0$$

i.e.

$$-(x-2) - 7y - 4(z-3) = 0$$

$$\Leftrightarrow \quad x + 7y + 4z = 14.$$

Problem 3

[22 pts] Polar Curves.

a) [6 pts] Given the polar curve $r = 2 + 2\sin\theta$, find $\frac{dy}{dx}$ in terms of r and θ.

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \qquad \frac{dr}{d\theta} = 2\cos\theta$$

$$\frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta} = \frac{2\cos\theta\sin\theta + (2+2\sin\theta)\cos\theta}{2\cos\theta\cos\theta - (2+2\sin\theta)\sin\theta}.$$

b) [6 pts] Find all points, (r, θ), on the polar curve $r = 2 + 2\sin\theta$ where the tangent line is horizontal.

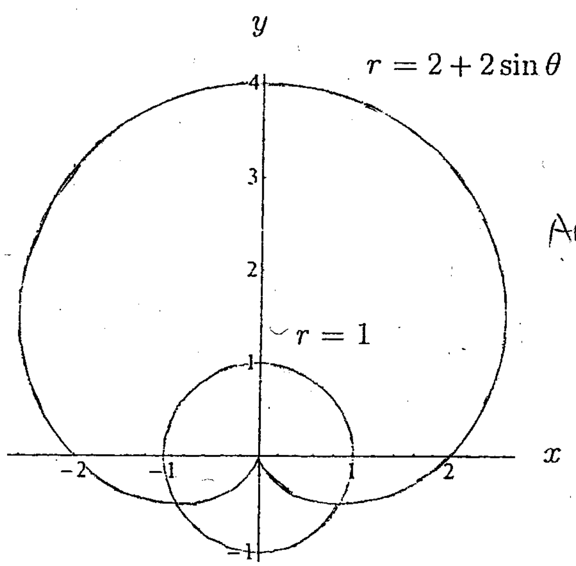
$$\frac{dy}{dx} = 0 \iff 2\cos\theta\sin\theta + (2+2\sin\theta)\cos\theta = 0.$$
$$\iff 2\cos\theta(\sin\theta + 1 + \sin\theta) = 0$$
$$\iff 2\cos\theta(2\sin\theta + 1) = 0$$
$$\iff \cos\theta = 0 \text{ or } \sin\theta = -\frac{1}{2}$$

$\theta = -\frac{\pi}{2}$ not ok b/c the bottom of $\frac{dy}{dx}$ is also 0 then.

so $\theta = \frac{\pi}{2}$, $\theta = -\frac{\pi}{6}, \frac{7\pi}{6}$. Find corresponding r values to be $4, 1, 1$.

so the points are $(4, \frac{\pi}{2})$, $(1, -\frac{\pi}{6})$, $(1, \frac{7\pi}{6})$

c) [10 pts] Set up an integral that represents the area of the region that lies inside the curve $r = 2 + 2\sin\theta$ and outside the circle $r = 1$. DO NOT EVALUATE THE INTEGRAL.



Set $2 + 2\sin\theta = 1$

$$\iff \sin\theta = -\frac{1}{2} \iff \theta = -\frac{\pi}{6}, \frac{7\pi}{6}.$$

So points of intersection are $(1, -\frac{\pi}{6})$, $(1, \frac{7\pi}{6})$.

$$\text{Area} = \int \frac{1}{2}(\text{outer}^2 - \text{inner}^2)\, d\theta$$
$$= \int_{-\frac{\pi}{6}}^{\frac{7\pi}{6}} \frac{1}{2}\left[(2+2\sin\theta)^2 - 1^2\right] d\theta.$$

Problem 4

[24 pts] Consider a particle that at time $t = 0$ has position $\mathbf{r}(0) = \langle 1, 4, -1 \rangle$ and velocity $\mathbf{v}(0) = \langle 0, 0, 2 \rangle$. Moreover, the particle has acceleration given by $\mathbf{a}(t) = \langle 6t, -2, 0 \rangle$ for $-\infty < t < \infty$.

a) [6 pts] Find the velocity, $\mathbf{v}(t)$.

$$\vec{v}(t) = \int \vec{a}(t)\,dt = \int \langle 6t, -2, 0 \rangle\, dt = \langle \int 6t\,dt, \int -2\,dt, \int 0\,dt \rangle$$
$$= \langle 3t^2 + C_1, -2t + C_2, C_3 \rangle.$$

By $\vec{v}(0) = \langle 0, 0, 2 \rangle$, we plug $t = 0$ in the above to get

$$\langle 0, 0, 2 \rangle = \langle C_1, C_2, C_3 \rangle \Rightarrow C_1 = 0, C_2 = 0, C_3 = 2,$$ so that

$$\vec{v}(t) = \langle 3t^2, -2t, 2 \rangle.$$

b) [4 pts] Find the unit tangent vector at time $t = 1$.

The unit tangent vector is $\vec{T}(t) = \dfrac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = \dfrac{\vec{v}(t)}{|\vec{v}(t)|} = \dfrac{\langle 3t^2, -2t, 2 \rangle}{\sqrt{(3t^2)^2 + (-2t)^2 + 2^2}}$

Hence at 1, we have

$$\vec{T}(t) = \frac{\vec{v}(1)}{|\vec{v}(1)|} = \frac{\langle 3, -2, 2 \rangle}{\sqrt{3^2 + (-2)^2 + 2^2}} = \frac{\langle 3, -2, 2 \rangle}{\sqrt{17}} = \langle \frac{3}{\sqrt{17}}, \frac{-2}{\sqrt{17}}, \frac{2}{\sqrt{17}} \rangle.$$

c) [6 pts] Find the position function $\mathbf{r}(t)$.

9 pts

$$\vec{r}(t) = \int \vec{v}(t)\,dt = \int \langle 3t^2, -2t, 2 \rangle\, dt = \langle \int 3t^2\,dt, \int -2t\,dt, \int 2\,dt \rangle$$
$$= \langle t^3 + D_1, -t^2 + D_2, 2t + D_3 \rangle$$

But $\vec{r}(0) = \langle D_1, D_2, D_3 \rangle = \langle 1, 4, -1 \rangle \Rightarrow D_1 = 1, D_2 = 4, D_3 = -1$, so we have

$$\vec{r}(t) = \langle t^3 + 1, -t^2 + 4, 2t - 1 \rangle$$

d) [4 pts] Find the points (if any) where the curve $\mathbf{r}(t)$ intersects the plane $y = 3$.

$\vec{r}(t) = \langle t^3 + 1, -t^2 + 4, 2t - 1 \rangle$ intersects the plane $y = 3$

$$\Leftrightarrow -t^2 + 4 = 3 \Leftrightarrow t^2 = 1 \Leftrightarrow t = \pm 1.$$

$$\vec{r}(-1) = \langle 0, 3, -3 \rangle, \quad \vec{r}(1) = \langle 2, 3, 1 \rangle$$

So points of intersection are $(0, 3, -3)$ and $(2, 3, 1)$.

e) [4 pts] Set up an integral that represents the distance traveled by the particle along the curve over the interval $0 \le t \le 7$. DO NOT EVALUATE THE INTEGRAL.

$$D = \int_0^7 |\vec{r}\,'(t)|\,dt = \int_0^7 |\vec{v}(t)|\,dt = \int_0^7 \sqrt{(3t^2)^2 + (-2t)^2 + 2^2}\,dt$$

Problem 5

[20 pts] Multivariable Functions.

a) [8 pts] If $f(x, y) = e^{x^2y} \sin x$, find f_x and f_y.

$$f_x(x,y) = \frac{\partial}{\partial x}\left(e^{x^2y}\sin x\right) = \left(\frac{\partial}{\partial x}e^{x^2y}\right)\sin x + e^{x^2y}(\sin x)'$$

$$= 2xye^{x^2y}\sin x + e^{x^2y}\cos x = e^{x^2y}(2xy\sin x + \cos x)$$

$$f_y(x,y) = \frac{\partial}{\partial y}\left(e^{x^2y}\sin x\right) = \left(\frac{\partial}{\partial y}e^{x^2y}\right)\sin x = x^2e^{x^2y}\sin x.$$

b) [6 pts] State the domain and range of the function $g(x, y) = \dfrac{1}{x^2 + y^2}$.

Domain of g is the set $\{(x,y) \mid x \neq 0, y \neq 0\}$.

Range of g is $(0, \infty)$.

c) [6 pts] Use the Two-Path Test to prove that $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y}{x^3 - y^3}$ does not exist.

① use path $y = -x$ and $x \to 0$, we have

$$\lim_{x\to 0} \frac{x^2\cdot(-x)}{x^3 - (-x)^3} = \lim_{x\to 0}\frac{-x^3}{2x^3} = -\frac{1}{2}$$

② use path $y = 0$ and $x \to 0$, we have

$$\lim_{x\to 0}\frac{x^2\cdot 0}{x^3 - 0} = \lim_{x\to 0}\frac{0}{x^3} = 0$$

$-\frac{1}{2} \neq 0$, so by the Two-Path Test, $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y}{x^3 - y^3}$ DNE.

A Few Trigonometric Identities

1) $\sin^2\theta = \dfrac{1-\cos(2\theta)}{2}$

2) $\cos^2\theta = \dfrac{1+\cos(2\theta)}{2}$

3) $\cos^2\theta + \sin^2\theta = 1$

4) $\sec^2\theta - \tan^2\theta = 1$

5) $\csc^2\theta - \cot^2\theta = 1$

A Few Reduction Formulas

Assume n is a positive integer.

1) $\displaystyle\int \sin^n x\,dx = -\frac{\sin^{n-1} x\cos x}{n} + \frac{n-1}{n}\int \sin^{n-2} x\,dx$

2) $\displaystyle\int \cos^n x\,dx = \frac{\cos^{n-1} x\sin x}{n} + \frac{n-1}{n}\int \cos^{n-2} x\,dx$

3) $\displaystyle\int \tan^n x\,dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x\,dx, \quad n \neq 1$

4) $\displaystyle\int \sec^n x\,dx = \frac{\sec^{n-2} x\tan x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2} x\,dx, \quad n \neq 1$