Math 1172

Midterm 3

Spring 2014

Form B

Name: Yanli Wang

OSU username (name.nn): ______________________

Lecturer: ______________________

Recitation Instructor: ______________________

Recitation Time: ______________________

## Instructions

•) SHOW ALL WORK!!! Incorrect answers with work shown may receive partial credit, but unsubstantiated answers may receive NO credit.

•) Give EXACT answers unless asked to do otherwise.

•) You do not need to simplify numerical answers such as $\frac{5}{\sqrt{8}} - \frac{5}{\sqrt{12}}$.

•) Calculators are permitted EXCEPT those calculators that have computer algebra systems (CAS) or ability to communicate with others. Furthermore, all memory must be cleared and all apps must be removed. PDA's, laptops, and cell phones are prohibited. Do not have these devices out!

•) The exam duration is 55 minutes.

•) The exam consists of 5 problems starting on page 2 and ending on page 6. Make sure your exam is not missing any pages before you start. Page 7 contains formulas. Page 8 may be used for extra work space.

| Problem Number | Maximum Point Value | Score |
|---|---|---|
| 1 | 18 | |
| 2 | 20 | |
| 3 | 22 | |
| 4 | 24 | |
| 5 | 16 | |
| Total | 100 | |

**Problem 1**

[18 pts] True or False. You do not need to show work for problems on this page.

a) [2 pts] $\mathbf{u} = < \sin(4), \cos(4) >$ is a unit vector.

True. Since $|\vec{u}| = \sqrt{\sin(4)^2 + \cos(4)^2} = \sqrt{1} = 1$

b) [2 pts] The planes $x + 2y + 3z = 2$ and $4x + z = -1$ intersect along the line $\mathbf{r}(t) = < -1 + 2t, -3 + 11t, 3 - 8t >$.

True: If $\vec{r}(t)$ is the intersection of two planes, then the direction of the line is orthogonal to the normal vectors of both planes. Also, the line contains at least one point from each plane.

Check: Direction: $<2, 11, -8>$. Normal vectors $<1, 2, 3>$ $<4, 0, 1>$. It's easy to verify $<2,11,-8> \perp <1,2,3>$, $<2,11,-8> \perp <4,0,1>$. And $(-1,-3,3)$ is on both planes

c) [2 pts] The parametric curve given by $x = 3 + 2t^2$ and $y = t^2$ for $1 \le t \le 4$ is the line segment from the point $(5, 1)$ to the point $(35, 16)$.

True

$x = 3 + 2t^2$ $y = t^2$ $\Rightarrow$ $x = 3 + 2y$. It's a line.

when $t = 1$, $x = 3 + 2 = 5$, $y = 1$

when $t = 4$, $x = 3 + 2 \cdot 16 = 35$, $y = 16$

line segment from $(5, 1)$ to $(35, 16)$

d) [2 pts] If the speed $|\mathbf{v}(t)|$ is a constant then the curve $\mathbf{r}(t)$ is a line.

False

Not necessary. Counter example: $V(t) = < \sin t, \cos t >$ $|V(t)| \equiv 1$ but the trajectory is a circle!

e) [2 pts] $\displaystyle\lim_{(x,y)\to(-2,2)} \frac{x^3y - xy^3}{x^2 - y^2} = -4$

True.

$= \displaystyle\lim_{(x,y)\to(-2,2)} \frac{xy\cancel{(x^2-y^2)}}{\cancel{x^2-y^2}} = \lim_{(x,y)\to(-2,2)} xy = -4$

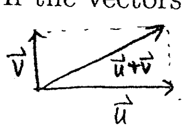
f) [2 pts] For any vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^3$, $\mathbf{u} \times (\mathbf{u} \times \mathbf{v}) = 0$.

False

$\vec{u} \perp \vec{u} \times \vec{v}$ So $\vec{u} \cdot (\vec{u} \times \vec{v}) = 0$

g) [2 pts] If the vectors $\mathbf{u}$ and $\mathbf{v}$ are perpendicular, then $|\mathbf{u} + \mathbf{v}| = |\mathbf{u}| + |\mathbf{v}|$.

False.




$\leftarrow$ It's easy to see $|\vec{u}+\vec{v}| \ne |\vec{u}| + |\vec{v}|$

h) [2 pts] For any plane parallel to the $yz$-plane, the trace of the surface $x - 3y^2 - 4z^2 = 1$ in that plane is a parabola.

False

Trace for $x = x_0$ is $x_0 - 3y^2 - 4z^2 = 1$ $\quad 3y^2 + 4z^2 = x_0 - 1$ $\leftarrow$ ellipse.

i) [2 pts] Given functions $f(t)$ and $g(t)$ with $f'(t)$ and $g'(t)$ continuous, the curves defined by $\mathbf{r}(t) = < f(t), g(t) >$ and $\mathbf{R}(t) = < g(t), f(t) >$ have the same length over the interval $[a, b]$.

True.

The arc length for both curves is $s = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2}\, dt$.

**Problem 2**

[20 pts] Consider the polar curves $r = \cos\theta$ (a circle) and $r = 1 - \cos\theta$ (a cardioid).

a) [6 pts] Find the slope of the line that is tangent to the graph of the cardioid $r = 1 - \cos\theta$ at the point $\left(\frac{3}{2}, \frac{2\pi}{3}\right)$. Give the **exact** value.

$$x_0 = r_0 \cos\theta_0 = \frac{3}{2}\cos\frac{2\pi}{3} = -\frac{3}{2}\cdot\frac{1}{2} = -\frac{3}{4}$$

$$y_0 = r_0 \sin\theta_0 = \frac{3}{2}\sin\frac{2\pi}{3} = +\frac{3}{2}\cdot\frac{\sqrt{3}}{2} = +\frac{3\sqrt{3}}{4}$$

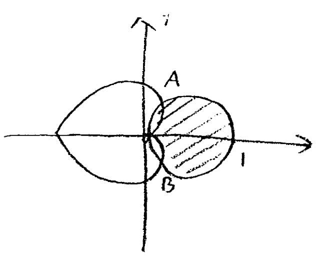
$$k = \frac{f'(\theta_0)\sin\theta_0 - f(\theta_0)\cos\theta_0}{f'(\theta_0)\cos\theta_0 - f(\theta_0)\sin\theta_0} = \frac{\sin\frac{2\pi}{3}\sin\frac{2\pi}{3} + (1-\cos\frac{2\pi}{3})\cos\frac{2\pi}{3}}{\sin\frac{2\pi}{3}\cos\frac{2\pi}{3} - (1-\cos\frac{2\pi}{3})\sin\frac{2\pi}{3}} = \frac{(\frac{\sqrt{3}}{2})^2 - (1+\frac{1}{2})\frac{1}{2}}{\frac{\sqrt{3}}{2}\cdot(-\frac{1}{2}) - \frac{3}{2}\cdot\frac{\sqrt{3}}{2}} = 0.$$

where $f(\theta) = 1 - \cos\theta$

$f'(\theta) = \sin\theta$

The tangent line is $y - \frac{3\sqrt{3}}{4} = 0\cdot(x - -\frac{3}{4}) \Rightarrow y = \frac{3\sqrt{3}}{4}$

b) [8 pts] Find all points of intersection of the two curves. Give **exact** values for the coordinates in $(r, \theta)$.




Three intersections

① The origin

②
$\cos\theta = 1 - \cos\theta$

$2\cos\theta = 1$

$\cos\theta = \frac{1}{2}$.

$\theta = \frac{\pi}{3}$ or $\theta = -\frac{\pi}{3}$.

$r = \cos\frac{\pi}{3} = \frac{1}{2}$

So. $A(\frac{1}{2}, \frac{\pi}{3})$ $B(\frac{1}{2}, -\frac{\pi}{3})$

c) [6 pts] $\boxed{\text{Set up}}$ an integral that represents the area of the region that lies inside the circle and outside the cardioid. DO NOT EVALUATE THE INTEGRAL.

$$A = \frac{1}{2}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} (\cos\theta)^2 - (1-\cos\theta)^2 \, d\theta$$

**Problem 3**

[22 pts] Consider the vectors $\mathbf{v} = 2\mathbf{i} - 4\mathbf{j}$ and $\mathbf{w} = 4\mathbf{i} + 4\mathbf{j} - \mathbf{k}$.

$\vec{V} = <2, -4, 0>$ $\quad \vec{w} = <4, 4, -1>$

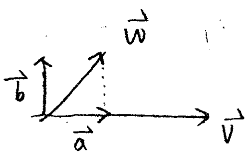
a) [4 pts] Compute $\mathbf{v} \cdot \mathbf{w}$.

$$\vec{V} \cdot \vec{W} = <2, -4, 0> \cdot <4, 4, -1>$$
$$= 2 \cdot 4 - 4 \cdot 4 + 0$$
$$= -8$$

b) [4 pts] Find $\text{proj}_{\mathbf{v}}\mathbf{w}$.

$$\text{Proj}_V W = \frac{V \cdot W}{V \cdot V} \cdot V$$
$$= \frac{-8}{4+16} \cdot <2, -4, 0>$$
$$= -\frac{2}{5} <2, -4, 0>$$
$$= <-\frac{4}{5}, \frac{8}{5}, 0>$$

c) [5 pts] Find vectors $\mathbf{a}$ and $\mathbf{b}$ such that $\mathbf{w} = \mathbf{a} + \mathbf{b}$, where $\mathbf{a}$ is parallel to $\mathbf{v}$ and $\mathbf{b}$ is perpendicular to $\mathbf{v}$.




$$\vec{a} = \text{Proj}_{\vec{V}} \vec{W} = <-\frac{4}{5}, \frac{8}{5}, 0>$$
$$\vec{b} = \vec{W} - \vec{a} = <4, 4, -1> - <-\frac{4}{5}, \frac{8}{5}, 0>$$
$$= <4\tfrac{4}{5}, 2\tfrac{2}{5}, -1>$$

d) [4 pts] Find an equation for the line that passes through the point $(-2, 1, 5)$ and is parallel to the line $\mathbf{r}(t) =< 3, 3, -5 > + t\mathbf{v}$.

One point $(: -2, 1, 5)$

Direction: $\vec{V} = <2, -4, 0>$

The equation for the line is $\vec{R}(t) = (-2, 1, 5) + <2, -4, 0> t$

$\vec{R}(t) = <2t-2, -4t+1, 5>$ or $\begin{cases} X(t) = 2t - 2 \\ Y(t) = -4t + 1 \\ Z(t) = 5 \end{cases}$

e) [5 pts] Find vectors $\mathbf{a}$ and $\mathbf{b}$ such that $\mathbf{a} + \mathbf{b} = \mathbf{v}$ and $\mathbf{a} - \mathbf{b} = \mathbf{w}$.

$\begin{cases} a + b = V \quad ① \\ a - b = W \quad ② \end{cases}$ solve for $a$ and $b$.

$$① + ② \Rightarrow 2a = V + W \Rightarrow a = \tfrac{1}{2}(V + W) = \tfrac{1}{2}(<2, -4, 0> + <4, 4, -1>)$$
$$= \tfrac{1}{2} <6, 0, -1> = <3, 0, -\tfrac{1}{2}>$$

$$b = V - a = <2, -4, 0> - <3, 0, -\tfrac{1}{2}>$$
$$= <-1, -4, \tfrac{1}{2}>$$

**Problem 4**

[24 pts] Consider a particle that has initial position $\mathbf{r}(0) =< 2, 6, 0 >$ and initial velocity $\mathbf{v}(0) =< 3, -2, -1 >$. Moreover, the particle has acceleration given by $\mathbf{a}(t) =< 0, 2, 6t >$ for $t \geq 0$.

a) [6 pts] Find the velocity, $\mathbf{v}(t)$.

$$\vec{v}(t) = \int a(t)\,dt + \vec{c}$$
$$= < \int 0\,dt, \int 2\,dt, \int 6t\,dt > + \vec{c}$$
$$= < 0, 2t, 3t^2 > + \vec{c}$$
$$\vec{v}(0) = < 0, 2t, 3t^2 > \Big|_{t=0} + \vec{c} = < 3, -2, -1 >$$

So $\vec{c} = < 3, -2, -1 >$

then $\vec{v}(t) = < 0, 2t, 3t^2 > + < 3, -2, -1 >$
$$= < 3, 2t-2, 3t^2-1 >$$

b) [4 pts] Find the speed of the particle at time $t = 2$.

$$|\vec{v}(t)| = \sqrt{3^2 + (2t-2)^2 + (3t^2-1)^2}$$
$$|\vec{v}(2)| = \sqrt{3^2 + 2^2 + 11^2}$$
$$= \sqrt{134}$$

c) [6 pts] Find the position function $\mathbf{r}(t)$.

$$\vec{r}(t) = \int \vec{v}(t)\,dt + \vec{c}$$
$$= < \int 3\,dt, \int 2t-2\,dt, \int 3t^2-1\,dt > + \vec{c}$$
$$= < 3t, t^2-2t, t^3-t > + \vec{c}$$
$$\vec{r}(0) = < 0, 0, 0 > + \vec{c} = < 2, 6, 0 >$$
$$\Rightarrow \quad \vec{c} = < 2, 6, 0 >$$

$$\vec{r}(t) = < 3t, t^2-2t, t^3-t > + < 2, 6, 0 >$$
$$= < 3t+2, t^2-2t+6, t^3-t >$$

d) [4 pts] Find the points (if any) where the curve $\mathbf{r}(t)$ intersects the plane $y = 5$.

$$\vec{r}(t) = < 3t+2, t^2-2t+6, t^3-t >$$
$$t^2-2t+6 = 5$$
$$t^2-2t+1 = 0$$
$$(t-1)^2 = 0$$
$$t = 1$$

Only one intersection when $t=1$

The intersection is

$(5, 5, 0)$

e) [4 pts] Set up an integral that represents the distance traveled by the particle along the curve over the interval $0 \leq t \leq 2$. DO NOT EVALUATE THE INTEGRAL.

$$S = \int_0^t |\vec{r}'(t)|\,dt = \int_0^t |v(t)|\,dt = \int_0^t \sqrt{3^2 + (2t-2)^2 + (3t^2-1)^2}\,dt$$

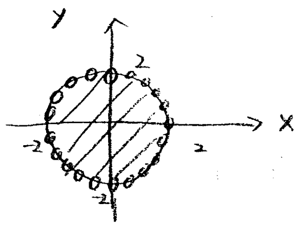
**Problem 5**

[16 pts] Consider the function $f(x,y) = \dfrac{1}{\sqrt{4 - x^2 - y^2}}$.

a) [4 pts] Find the domain of $f$. Sketch the domain in the $xy$-plane.

Domain: $\{(x,y) \mid 4 - x^2 - y^2 > 0\}$

Graph:




Step 1: Boundary:

$4 - x^2 - y^2 = 0 \iff x^2 + y^2 = 4$ circle

Step 2: choose side:

Do not include the boundary!

b) [4 pts] At what points of $\mathbb{R}^2$ is the function continuous?

$f(x,y)$ is continuous in its domain $\{(x,y) \mid 4 - x^2 - y^2 > 0\}$

c) [6 pts] Sketch the the level curves (in the $xy$-plane) corresponding to $z_0 = 1$, $z_0 = 2$, and $z_0 = 3$.

Level curve $z_0 = 1$

$1 = \dfrac{1}{\sqrt{4-x^2-y^2}}$

$\sqrt{4-x^2-y^2} = 1$

$4 - x^2 - y^2 = 1$

$x^2 + y^2 = 3$

$z_0 = 2$

$2 = \dfrac{1}{\sqrt{4-x^2-y^2}}$

$2\sqrt{4-x^2-y^2} = 1$

$\sqrt{4-x^2-y^2} = \frac{1}{2}$

$4 - x^2 - y^2 = \frac{1}{4}$

$x^2 + y^2 = 3\frac{3}{4}$

$z_0 = 3$

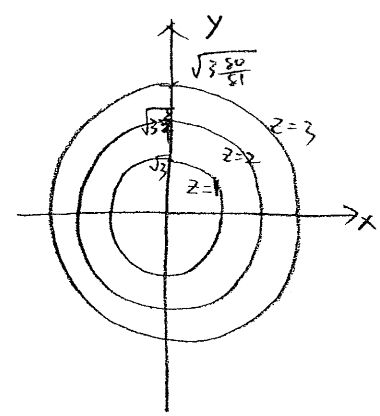
$3 = \dfrac{1}{\sqrt{4-x^2-y^2}}$

$9\sqrt{4-x^2-y^2} = 1$

$\sqrt{4-x^2-y^2} = \frac{1}{9}$

$4 - x^2 - y^2 = \frac{1}{81}$

$x^2 + y^2 = 3\frac{80}{81}$




d) [2 pts] Evaluate the limit. $\lim_{(x,y)\to(-1,1)} f(x,y)$

$\lim_{(x,y)\to(-1,1)} \dfrac{1}{\sqrt{4-x^2-y^2}} = \dfrac{1}{\sqrt{4-1-1}} = \dfrac{1}{\sqrt{2}}$

## A Few Trigonometric Identities

1) $\sin^2\theta = \dfrac{1-\cos(2\theta)}{2}$

2) $\cos^2\theta = \dfrac{1+\cos(2\theta)}{2}$

3) $\cos^2\theta + \sin^2\theta = 1$

4) $\sec^2\theta - \tan^2\theta = 1$

5) $\csc^2\theta - \cot^2\theta = 1$

## A Few Reduction Formulas

Assume $n$ is a positive integer.

1) $\displaystyle\int \sin^n x\,dx = -\frac{\sin^{n-1}x\cos x}{n} + \frac{n-1}{n}\int \sin^{n-2}x\,dx$

2) $\displaystyle\int \cos^n x\,dx = \frac{\cos^{n-1}x\sin x}{n} + \frac{n-1}{n}\int \cos^{n-2}x\,dx$

3) $\displaystyle\int \tan^n x\,dx = \frac{\tan^{n-1}x}{n-1} - \int \tan^{n-2}x\,dx, \quad n\neq 1$

4) $\displaystyle\int \sec^n x\,dx = \frac{\sec^{n-2}x\tan x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2}x\,dx, \quad n\neq 1$