# Galois Groups and Branched Covers of Riemann Surfaces

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The goal of this paper is to explain, using the theory of Riemann surfaces, one aspect of the relationship between covering spaces and Galois theory. We mostly follow Tamás Szamuely's treatment in [8]. We assume some knowledge of Galois theory and covering spaces, but develop Riemann surfaces mostly from scratch. However, we refer the reader to [8] for some of the proofs. To learn the basics of Galois theory, see [1], along with the first chapter of [8]. For an introduction to covering spaces, see chapter 2 of [8]. For an even more informal summary of some of the same ideas, also based on [8], see [5].

Although we won't emphasize it much, it's worth noting that the results we discuss here parallel similar ideas in the study of algebraic curves. For instance, Theorem 4.7, discussed below, establishes a connection between finite extensions of the field of functions on a Riemann surface and maps between Riemann surfaces. Almost the same connection exists in the world of algebraic curves: if $X, Y$ are nonsingular and projective, with function fields $k(X), k(Y)$, then nonconstant maps $f\colon X \to Y$ correspond to maps $\tilde{f}\colon k(Y) \to k(X)$, which exhibit $k(X)$ as a finite field extensin of $k(Y)$. In fact, as we will briefly discuss later, every compact Riemann surface can be understood as a smooth projective curve over $\mathbb{C}$, so these parallels are not so surprising. For background on the general theory of algebraic curves, see [2].

## 1 Introduction & Background

Our starting point is the fundamental theorem of Galois theory:

**Theorem 1.1.** *Let $K/F$ be a finite Galois field extension, and let $Gal(K/F)$ be the group of automorphisms of $K$ fixing $F$. There is a bijective correspondence between subgroups $G \subset Gal(K/F)$ and intermediate fields $L$ where $F \subset K \subset L$, given by mapping $G$ to the fixed field $F^G = \{x \in K : \sigma(x) = x \forall \sigma \in G\}$, and mapping the intermediate field $L$ to $Gal(K/L) \subset Gal(K/F)$. In particular $L$ is normal if and only if $Gal(K/L) \trianglelefteq Gal(K/F)$, in which case $Gal(L/F) \cong Gal(K/F)/Gal(K/L)$.*

By giving some definitions, we can state a strikingly similar result for covering spaces in topology. Suppose $p\colon Y \to X$ is a cover. Let $G = \mathrm{Aut}(Y/X)$ be the group of topological automorphisms of $Y$ which are compatible with $p$. There is an obvious projection map $\pi\colon Y \to G \setminus Y$, where $G \setminus Y$ denotes the quotient of $Y$ by the action of $G$. Furthermore, since $G$ acts compatibly with $p$, the map $p$ factors through the quotient, so we have $p = \overline{p} \circ \pi$, where $\overline{p}$ is the induced map from $G \setminus Y$ to $X$. Call the cover $Y$ *Galois* if $Y$ is connected the map $\overline{p}\colon G \setminus Y \to X$ is a homeomorphism. Working from this definition, we obtain the following result:

**Theorem 1.2** ([8], Theorem 2.2.10)**.** *Suppose $p\colon Y \to X$ is a Galois cover. For any $H \subset G = Aut(Y/X)$, $p$ induces a map $\overline{p_H}\colon H \setminus Y \to X$ which turns $H \setminus Y$ into a covering of $X$. Conversely, suppose $q\colon Z \to X$ is a connected cover for which the following diagram commutes:*

$$\begin{array}{ccc} Y & \xrightarrow{f} & Z \\ & \searrow^{p} & \downarrow{\scriptstyle q} \\ & & X \end{array}$$

*Then $f\colon Y \to Z$ is a Galois cover, and $Z \cong Aut(Y/Z) \setminus Y$. These maps induce a bijection between subgroups of $G$ and intermediate covers $Z$. The cover $q\colon Z \to X$ is Galois if and only if $Aut(Y/Z) \trianglelefteq G$, and if this condition holds, then $Aut(Z/X) \cong G/H$.*

Our goal in the rest of the paper is to show that this correspondence is not an accident. We will focus on fields $F$ which arise as the function field of a Riemann surface. In this case, we find that finite Galois extensions of $F$ correspond almost exactly to certain kinds of topological coverings, and in particular, automorphisms of an extension over the base field are almost the same thing as automorphisms of topological covers.

# 2 Covering Spaces, More Carefully

Before we study Riemann surfaces, we need some additional facts about covers. First, we develop an alternate formulation of covers, based on the idea of a sheaf.

**Definition 2.1.** Let $X$ be a topological space. Define $X_{Top}$ to be the category whose elements are the open subsets of $X$ and whose morphisms are the inclusion maps $i_{VU}\colon V \to U$ for $V \subset U$. A *presheaf of sets* on $X$ is a contravariant functor $\mathcal{F}\colon X_{\text{Top}} \to \text{Set}$. If $s \in \mathcal{F}(U)$, we write $s|_V$ for $\mathcal{F}(i_{VU})(s)$, and we think of $s|_V$ as the "restriction" of $s$ to $V$. Elements $s \in \mathcal{F}(U)$ are called sections of $\mathcal{F}$ over $U$.

**Definition 2.2.** A *sheaf* on $X$ is a presheaf, satisfying the following additional condition: For any open $U \subset X$, and any open covering $\cup U_i$ of $U$, every collection of elements $s_i$ such that $s_i \in \mathcal{F}(U_i)$ and $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ comes from the restriction of a unique $s \in \mathcal{F}(U)$. In other words, elements with compatible restrictions can be glued together, and an element is completely determined by its restriction to each element of an open cover.

A motivating example to keep in mind is the sheaf $\mathcal{F}$ of continuous real-valued functions on a space $X$: Given any open $U \subset X$, $\mathcal{F}(U)$ is defined to be the set of continuous functions $f\colon U \to \mathbb{R}$ and the restriction maps are given by restriction of functions.

For a fixed space $X$, we define a morphism of sheaves (implicitly, sheaves of sets) to be a morphism of the corresponding functors, namely a natural transformation. This turns sheaves over $X$ into a category. We will focus on sheaves with a specific property that relates them to covers:

**Definition 2.3.** Let $X$ be a space. A constant sheaf is the sheaf of continuous functions into a discrete space. A sheaf $\mathcal{F}$ is called locally constant if, for every $x \in X$, there exists some $V \ni x$ such that the restriction of $\mathcal{F}$ to $V$, which also forms a sheaf, is isomorphic to a constant sheaf on $V$.

The relationship between locally constant sheaves and covers is the following: let $p\colon Y \to X$ be a cover. For any open $U \subset X$, $s\colon U \to Y$ is called a *section* of $p$ over $U$ if $p \circ s$ is the identity on $U$. The covers of $X$ form a category, with morphisms given by continuous maps compatible with the projection maps. One can verify that letting $\mathcal{F}(U)$ be the sections of $p$ over $U$ and letting the restriction maps be restriction of functions gives a functor from covers of $X$, to locally constant sheaves on $X$.

**Theorem 2.4** ([8], Theorem 2.5.9)**.** *The functor $\mathcal{F}$ gives an equivalence of categories between covers of $X$ and locally constant sheaves on $X$.*

The construction of a functor in the other direction is more involved: we will describe it, but not prove that it works. To do so, we need one more definition:

**Definition 2.5.** Let $\mathcal{F}$ be a presheaf on $X$, $x \in X$. The *stalk* $\mathcal{F}_x$ of $\mathcal{F}$ at $X$ is the disjoint union of the sets $\mathcal{F}(U)$ for $U \ni x$, subject to the following relation: $s \in \mathcal{F}(U)$ and $t \in \mathcal{F}(V)$ are equivalent if their restrictions agree on some sufficiently small open neighborhood. [1]

Now, given a locally constant sheaf $\mathcal{F}$ on $X$, we define $X_{\mathcal{F}}$, as a set, to be the disjoint union of the stalks $\mathcal{F}_x$ for every $x \in X$. This has an obvious projection map $p_{\mathcal{F}}$, sending every element of the stalk $\mathcal{F}_x$ to the element $x$. Also, for every $U$ and every section $s \in \mathcal{F}(U)$, there is a map $i_s\colon U \to X_{\mathcal{F}}$, which sends every element $x \in U$ to the image of $s$ in the stalk $\mathcal{F}_x$. We want the maps $i_s$ to be sections, in the literal sense, of the covering map, so they should in particular be open maps. Accordingly, we define the topology to be the coarsest topology in which every $i_s(U)$ is open. It is not too hard to check that the resulting construction has the desired properties.

The next idea we need to consider is that of a proper map. A continuous map $f\colon A \to B$ is called *proper* if for any compact $K \subset X$, $f^{-1}(K)$ is compact. Intuitively, this indicates that the fibers of $f$ are "not too big:" For instance, if $f$ maps $\mathbb{R}$ into itself, $f$ is proper if and only if the pre-image of a bounded set is bounded, as a result of the Heine-Borel theorem.

We will need a few facts later: First, if $A$ is compact, and assuming all spaces are Hausdorff $f$ is automatically proper. This follows because if $K \subset B$ is compact, $f^{-1}(K)$ is closed in $A$, and hence compact. Next, suppose $p\colon Y \to X$ is a covering map. Then $p$ will have finite fibers if, and only if, $p$ is proper (in this case, we call $Y$ a finite cover of $X$). Indeed, if $p$ is proper, then $p^{-1}(x)$ is compact for any $x$. However, since $p$ is a covering map, it must also be discrete, and any discrete compact set is finite.

In the other direction, suppose all fibers are finite, and let $Z \subset X$ be compact. Suppose $\cup U_i$ is an open cover of $p^{-1}(Z)$. We may assume that each $U_i$ is small enough that $p^{-1}(p(U_i))$ is isomorphic to the disjoint union of copies of $p(U_i)$: indeed, the definition of a covering space ensures some small enough open set exists around each point of $X$, and if we can find a finite subcover for a refinement of the given cover, we can find a finite subcover of the given cover as well. Note that $p$ is an open map, since covering maps are local homeomorphisms. But then the sets $p(U_i)$ are an open cover of $Z$ by surjectivity, and hence have a finite subcover. Finally, note that since each $p(U_i)$ is trivially covered and $p$ has finite fibers, there are only finitely many $U_j$ such that $p(U_j)$ intersects $p(U_i)$. It follows that the finite subcover of $Z$ pulls back to a finite subcover of $p^{-1}(Z)$, so we're done.

Finally, suppose we have a proper map $f\colon A \to B$ and that all spaces are locally compact. Then $f$ is closed. Indeed, for any closed $C \subset A$, $f(C)$ is closed if and only if its intersection with every open set $U \subset B$ is closed in $U$. Since $B$ is locally compact, this will hold if and only if $f(C) \cap K$

[1] More categorically, the stalk is the direct limit of the sets $\mathcal{F}(U)$ along with the restriction maps.

is closed for every compact $K \subset B$. But $f^{-1}(K)$ is compact, so $f^{-1}(K) \cap C$ is compact, and its image will be compact, hence closed in $K$.

# 3 Riemann Surfaces

A *Riemann surface* is a one-dimensional complex manifold $X$, whose transition maps are holomorphic. In other words, around every point $x \in X$ there exists an open $U_x \ni x$ and a homeomorphism $\varphi_x \colon U_x \to V_x$ for $V_x \subset \mathbb{C}$ open, called a chart, such that whenever $U_x \cap U_y \neq \emptyset$, $\varphi_x \circ \varphi_y^{-1}$ is holomorphic. We will often assume Riemann surfaces are compact or connected, simply because it gives a nicer correspondence.

Given two Riemann surfaces $X$ and $Y$, a *holomorphic map* $\varphi \colon Y \to X$ is a continuous map such that if $U \subset X$, $V \subset Y$ are open, $\varphi(V) \subset U$ and $f \colon U \to \mathbb{C}$ and $g \colon V \to \mathbb{C}$ are charts, then $f \circ \varphi \circ g^{-1} \colon g(V) \to \mathbb{C}$ is holomorphic. We will always assume these maps are nonconstant.

A very basic example of a Riemann surface is $\mathbb{C}$ itself, and the maps $z \to z^k$ are holomorphic for every $k \geq 1$. Notice that each such map turns $\mathbb{C}^*$ into a finite cover of itself. However, these are not covering maps at the point 0, intuitively because multiple sheets of the covering "meet." The first surprise of Riemann surfaces is that this analysis extends to any holomorphic map, at any point:

**Theorem 3.1** ([8], Theorem 3.2.1)**.** *Let $\varphi \colon Y \to X$ be a holomorphic map, pick $y \in Y$ and write $\varphi(y) = x$. There exist open $V_y \ni y, U_x \ni x$ such that $\varphi(V_y) \subset U_x$, and complex charts $g_y \colon V_y \to \mathbb{C}$, $f_x \colon U_x \to \mathbb{C}$ such that $f_x(x) = g_y(y) = 0$, and such that the following diagram commutes for some positive integer $e_y$:*

$$\begin{array}{ccc} V_y & \xrightarrow{\varphi} & U_x \\ \downarrow{\scriptstyle g_y} & & \downarrow{\scriptstyle f_x} \\ \mathbb{C} & \xrightarrow{z \mapsto z^{e_y}} & \mathbb{C} \end{array}$$

*The integer $e_y$ is independent of the choice of charts.*

We call the integer $e_y$ the ramification index of $\varphi$ at $y$, and call $y$ a branch point of $e_y > 1$. Let $S_\varphi$ be the set of branch points of $\varphi$.

As the example of $z \mapsto z^k$ shows, we can't expect all nonconstant holomorphic maps to be coverings. However, it turns out that branch points are more or less the only thing that can go wrong:

**Proposition 3.2** ([8], Proposition 3.2.6)**.** *Let $Y$ be a Riemann surface, $X$ a connected Riemann surface, $\varphi \colon Y \to X$ a proper holomorphic map. Then $\varphi$ is surjective with finite fibers, and its restriction to $Y \setminus \varphi^{-1}(\varphi(S_\varphi))$ is a finite cover of $X \setminus \varphi(S_\varphi)$.*

*Proof.* Since $\varphi$ is proper, the fiber of a point will be compact. Therefore, to show all fibers are finite it's enough to show all fibers are discrete. However, this follows from Theorem 3.1: since every holomorphic map behaves locally like $z^k$, every holomorphic map is locally finite-to-one. In particular, if $\varphi(y) = x$, then $y$ has a neighborhood only finitely many points of which are mapped to $x$, so $\varphi^{-1}(x)$ is discrete.

Next, we show $\varphi$ is surjective. Another corollary of Theorem 3.1 is that $\varphi$ must be open, since the maps $z^k$ are all open. On the other hand, since $\varphi$ is proper and $\mathbb{C}$ is locally compact, we showed above that $\varphi$ is closed. Therefore, $\varphi(Y)$ is both open and closed, and hence must be all of $X$ since $X$ is connected. It remains to show that $\varphi$ is a covering map away from branch points. But if $x$ is not the image of a branch point, then each element of its fiber maps homeomorphically onto a neighborhood of $x$ (since the bottom arrow in the commutative diagram for Theorem 3.1 will be the identity map). Since the fibers are finite, it's clear that $\varphi$ will be a covering map. $\square$

This motivates the following definition: a *branched cover* is a map which restricts to a covering map outside of a discrete closed subset of its domain. Let $\mathrm{Hol}_{X,S}$ be the category of Riemann surfaces $Y$ along with proper holomorphic maps $Y \to X$, by mean of which $Y$ is a branched covering of $X$ and such that all branch points are contained in $\varphi^{-1}(S)$. Morphisms in this category are holomorphic maps of Riemann surfaces compatible with the projection maps. The following result effectively says that studying branched covers of a connected Riemann surface $X$ is the same as studying branched covers by Riemann surfaces.

**Theorem 3.3.** *The functor which sends a Riemann surface $\varphi\colon Y \to X$ to the corresponding topological cover of $X \setminus S$ induces an equivalence of categories between the category of finite covers of $X$ and $Hol_{X,S}$.*

# 4 Galois Theory on Riemann Surfaces

It's time to connect our study of Riemann surfaces back to questions about fields. Let $X$ be a Riemann surface. A meromorphic function on $X$ is a function which is holomorphic on some domain $X \setminus S$ for some discrete closed $S \subset X$, such that for every chart $\varphi\colon U \to \mathbb{C}$, $f \circ \varphi^{-1}$ is meromorphic in the usual sense. The meromorphic functions form a ring $\mathcal{M}(X)$. Without additional hypotheses, they do not form a field. In particular, if $X = A \cup B$ has two connected components, then the function which is 0 on $A$ and 1 on $B$ is holomorphic and nonzero, but its inverse is not meromorphic, since it will not be holomorphic anywhere on $A$. However, this pathology goes away when we assume connectedness.

**Proposition 4.1.** *Suppose $X$ is a connected Riemann surface. Then $\mathcal{M}(X)$ is a field.*

*Proof.* We only need to show that nonzero elements of the ring have inverses. Suppose $f \in \mathcal{M}(X)$ is nonzero. Then $1/f$ will be meromorphic as long as the zeroes of $f$, corresponding to poles of $1/f$, form a discrete closed subset. If they did not, then they would have a limit point. Locally, this gives us a holomorphic function on an open, connected subset of $\mathbb{C}$ whose zeroes accumulate, by mapping through one of the charts. But by the identity theorem, any such function is identically zero. Consider the points $x$ such that $f$ vanishes identically on an open set around $x$. This is defined to be open, and must also be closed: indeed, a boundary point of this set, by the same argument given above, must see the function locally vanish. It follows that this set is clopen, and also nonempty by the argument we gave above. By connectedness $f$ must be identically zero on $X$, a contradiction. $\square$

For the rest of the paper, we will always assume both $X$ and $Y$ are connected. In this case, both $\mathcal{M}(X)$ and $\mathcal{M}(Y)$ are fields, since both surfaces are connected. Also, for any holomorphic $\varphi\colon Y \to X$, there exists a corresponding $\tilde{\varphi}\colon \mathcal{M}(X) \to \mathcal{M}(Y)$, given by $\tilde{\varphi}(f) = f \circ \varphi$. If $\varphi$ is nonconstant, then $\tilde{\varphi}$ is also nonconstant, hence injective. This allows us to realize $\mathcal{M}(X)$ as a subfield of $\mathcal{M}(y)$, identifying $\mathcal{M}(X)$ with its image under $\tilde{\varphi}$.

If $p\colon A \to B$ is a finite cover of connected spaces, the *degree* of $p$ is defined to be the cardinality of the fiber over any point. This is well-defined, basically because $B$ is connected: one can verify that the set on which the fiber has size $n$ is open for every $n$, which means only one such $n$ must occur or else $B$ would be disconnected. Likewise, given a branched cover, we define its degree to be the degree of the corresponding topological cover on a restriction.

It turns out that in the case of Riemann surfaces, degrees of branched coverings and of field extensions are closely related. As a first demonstration of this, we prove the following result:

**Lemma 4.2** ([8], Lemma 3.3.6). *Let $\varphi\colon Y \to X$ be a nonconstant map of compact, connected Riemann surfaces which has degree $d$ as a branched cover. Every meromorphic $f \in \mathcal{M}(Y)$ satisfies a polynomial of degree $d$, not necessarily irreducible, over $\mathcal{M}(X)$.*

*Proof.* Pick any $x \notin \varphi(S)$, where $S$ is the set of branch points. By the definition of degree of a cover, $\varphi^{-1}(x)$ has size $d$, so there exists some small open $U \ni x$ whose pre-image is the disjoint union of open sets $V_1, \ldots, V_d$ each homeomorphic to $U$. For each $i \le d$, there's a corresponding section $s_i\colon U \to V_i$. Let $f_i = f \circ s_i$. Then $f_i$ is meromorphic on $U$ for every $i$. Set:

$$F = \pi(t - f_i) = t^d + a_{d-1}t^{d-1} + \cdots + a_0$$

Currently, this is an element of $\mathcal{M}(U)$, but we will show it extends to all of $X \setminus \varphi(S)$. To see this, pick some other $x_1$ and some $U_1$, in the same way they were chosen above. On $U \cap U_1$, the sections of $\varphi$ are the same in both cases, so the same polynomials $f_i$ will occur, possibly in a different order. Since $F$ is symmetrical in the $f_i$, this means the coefficients of $F$ agree, as functions, with the coefficients of $F_1$, on the shared domain $U \cap U_1$. Since we can repeat this construction for every point, we can take the coefficients $a_i$ to be meromorphic functions on $X \setminus \varphi(S)$.

It remains to show that the $a_i$ are meromorphic on all of $X$. Pick some $x \in \varphi(S)$, the image of a branch point $y \in S$. Pick a coordinate chart $f_x\colon U_x \to \mathbb{C}$, where $U_x$ is a neighborhood of $x$ and $f_x(x) = 0$. Then by composition, $f_x \circ \varphi$ defines a holomorphic function in a neighborhood of every $y \in \varphi^{-1}(x)$, such that $f_x(\varphi(y)) = f_x(x) = 0$. Since $f$ is a meromorphic function on $Y$, its poles each have some finite order. By picking some sufficiently large $k$, and since $f_x \circ \varphi$ has a zero at each $y \in \varphi^{-1}(x)$, we find that the function $(f_x \circ \varphi)^k f$ is holomorphic for every $y \in \varphi^{-1}(x)$, and in particular, bounded on a (punctured) neighborhood around each $y$.

Note that $((f_x \circ \varphi)^k f) \circ s_i = (f_x \circ \varphi \circ s_i)^k f \circ s_i = f_x^k f_i$, applying the fact that $s_i$ is a section. Since $s_i$ is also holomorphic around each point, we find that $f_x^k f_i$ is bounded on $U_x \setminus \{x\}$. So is $f_x^{kd} a_j$, since each $a_j$ is a sum of products of at most $d$ functions whose product with $f_x^k$ is holomorphic Now, by Riemann's removable singularity theorem, this allows us to patch over $x$ and define a holomorphic function on all of $U_x$. But then, taking ratios makes it clear that $a_j$ is a meromorphic function everywhere. That is, $a_j \in \mathcal{M}(X)$, and in particular $F \in \mathcal{M}(X)[t]$.

It remains to show that $f$ satisfies the polynomial $F$. Recalling the identification between $\mathcal{M}(X)$ and $\tilde{\varphi}(\mathcal{M}(X))$, what we actually have to show is that $f$ satisfies $\tilde{\varphi}(F) = t^d + (a_{d-1} \circ \varphi)t^{d-1} + \cdots + a_0 \circ \varphi$. Note that for every $s_i$, $(\tilde{\varphi}(F) \circ s_i)(f \circ s_i) = F(f_i) = 0$, applying the fact that $s_i$ is a section. That is, $\tilde{\varphi}(F)(f) \circ s_i$ is identically 0 on $U$, so $\tilde{\varphi}(F)(f)$ is identically 0 on $V_i$. Since the same holds for every $x \notin \varphi(S)$ and every $U \ni x$, we conclude the function is zero everywhere except possibly at branch points, hence zero everywhere. Thus, $f$ satisfies the degree $d$ polynomial $F$. $\square$

In fact, the degree of this field extension over $\mathcal{M}(X)$ will be exactly $d$, as we now show. Following [8], we use the following result, which we will not prove:

**Theorem 4.3.** *Let $X$ be a compact Riemann surface, $x_1, \dots, x_n \in X$ a finite set of points, and $a_1, \dots, a_n$ any complex numbers. There exists a meromorphic function $f \in \mathcal{M}(X)$ such that $f$ is holomorphic at all $x_i$ and $f(x_i) = a_i$.*

As an aside, the same result plays a key part in the proof of the following:

**Theorem 4.4.** *Let $C$ be a compact Riemann surface. Then $C$ is isomorphic, as a complex manifold, to some smooth projective curve $C'$.*

For more details on the result and its proof, see [4] section 2.1, [6], the appendix to [3], and [7]. The basic idea is that by using 4.3, we can define meromorphic functions which separate points on a Riemann surface, and this gives an embedding into projective space. By a result known as Chow's Theorem (discussed in section 1.3 of [4]), any closed complex submanifold of projective space is an algebraic variety, so in particular the image of the Riemann surface is a curve.

Returning to our main point, we prove the following:

**Proposition 4.5.** *Let $\varphi\colon Y \to X$ be nonconstant, where $X, Y$ are compact, connected Riemann surfaces, and suppose $\varphi$ has degree $d$ as a branched cover. The field extension $\mathcal{M}(Y)/\tilde{\varphi}(\mathcal{M}(X))$ has degree $d$.*

*Proof.* First, we show the degree is at least $d$. Pick some $x \in X \setminus \varphi(S)$, where $S$ as usual is the set of branch points. Then $\varphi^{-1}(x) = y_1, \dots, y_d$, for some distinct $y_i$. By Theorem 4.3, there exists $f \in \mathcal{M}(Y)$ which is holomorphic at each $y_i$ and such that $f(y_i) \neq f(y_j)$ for $i \neq j$. Applying the result proved above, $f$ satisfies an irreducible polynomial $f^n + \tilde{\varphi}(a_{n-1})f^{n-1} + \cdots + \tilde{\varphi}(a_0) = 0$ over $\mathcal{M}(X)$, where $n \leq d$. If each coefficient $a_i$ is holomorphic at $x$, then by substitution we obtain a polynomial $t^n + \cdots + a_0(x)$. Every value $f(y_i)$ must be a root of this polynomial, so it has $d$ distinct roots, so $n = d$. Otherwise, suppose some $a_i$ has a pole at $x$. Then we can simply pick a sufficiently close point $x'$: $x'$ will still not be in $\varphi(S)$, the values $f$ attains on its inverse image points will still be distinct, and we may pick $x'$ so that all the coefficients are holomorphic. Then, repeat the same argument. This shows that some element has degree $d$, so the degree of the extension is exactly $d$.

Suppose $g \in \mathcal{M}(Y)$ is some other element. By the primitive element theorem, we have $\mathcal{M}(X)(f, g) = \mathcal{M}(X)(h)$ for some $h$. But by the result we proved above, $h$ also has degree at most $d$ over $\mathcal{M}(X)$, so $\mathcal{M}(X)(f) = \mathcal{M}(X)(h)$. In particular, $f$ already generates $\mathcal{M}(Y)$, so the extension has degree exactly $d$. □

We now have a functor $F$ from the category of compact, connected Riemann surfaces acting as proper holomorphic branched covers of $X$ to the category of finite field extensions of $\mathcal{M}(X)$. The functor $F$ is contravariant, since a map $\varphi\colon Y \to Z$ induces a map of fields in the opposite direction. It will turn out that the functor $F$ induces an equivalence of categories.

Recall that a functor induces an equivalence of categories if, and only if, it's fully faithful and essentially surjective. We will show that the functor $F$ is essentially surjective, or in other words that every finite field extension of $\mathcal{M}(X)$ occurs as $\mathcal{M}(Y)$ for some $Y$:

**Proposition 4.6** ([8], Proposition 3.3.8[2])**.** *Let $X$ be a connected compact Riemann surface, and let $L$ be a finite field extension of of $\mathcal{M}(X)$. There exists a compact, connected Riemann surface $Y$ mapping holomorphically onto $X$, such that $\mathcal{M}(Y) \cong L$ as a finite extension over $\mathcal{M}(X)$.*

[2]Note, however, that we restrict attention to connected covers, so we phrase the result in terms of field extensions rather than the product of multiple field extensions.

*Proof.* Let $\alpha$ be a primitive element for $L$, let $F \in \mathcal{M}(X)[t]$ be the minimal polynomial of $\alpha$, and let $d$ be its degree. Since $F$ is irreducible, $F$ is separable, and hence has no common factor with its derivative $F_t$ with respect to $t$. Since the polynomial ring over a field is Euclidean, $(F, F_t)$ is not a proper ideal. Thus, there exist $A, B \in \mathcal{M}(X)$ such that $AF + BF_t = 1$.

Note that for every $x$ where the coefficients of $F$ are holomorphic, we have a complex polynomial $F(x) \in \mathbb{C}[t]$. Suppose $A, B$ do not have a pole at $x$, and that all coefficients of $F$ are holomorphic at $x$. By evaluating, we obtain an equality $A(x)F(x) + B(x)F_t(x) = 1$, as functions of $t$. It follows that $F(x), F_t(x)$, as polynomials in $t$, cannot have a common root unless one of the coefficients has pole, or $A$ does, or $B$ does. Let $S$ be the set of points $X$ for which any of these functions has a pole: Then $S$ is discrete and closed, and for any $x \in X'$, $F(x)$ has no root shared with $F_t(x)$. In particular, $F(x)$ has no multiple roots, so it has $d$ distinct roots $a_1, \ldots, a_d$.

Now, we apply our results about sheaves from above. For any open $U \subset X'$, let $\mathcal{F}(U)$ be the set of holomorphic functions $f$ on $U$ such that $F(f) = 0$. This is a well-defined notion: since $F$ is a polynomial with coefficients holomorphic on $X'$, we can interpret $F \in \mathcal{M}(U)[t]$. Further, this forms a sheaf in the obvious way; in particular, given compatible holomorphic functions such that $F(f) = 0$, they "glue" to form a holomorphic function with the same property on their combined domain.

We will show, by invoking some complex analysis, that $\mathcal{F}$ is locally constant. Pick $x \in X'$ and some $a_i$ which is a root of $F(x)$. We can interpret $F$ in some neighborhood as a holomorphic function $G$ of two variables, by setting $G(x, t) = F(x)(t)$. Then $G(x, a_i) = 0$, but $D_t(x, a_i) \neq 0$, since $F(x)$ and $F_t(x)$ do not share any roots. The holomorphic implicit function theorem then tells us that there exists a function $f_i$ in a neighborhood of $x$ with the property that $f_i(x) = a_i$, $G(x, f_i(x)) = F(x)(f_i(x)) = 0$. Thus, we obtain $d$ distinct functions $f_1, \ldots, f_d$, all contained in $\mathcal{F}(V)$ for some sufficiently small neighborhood $V$ of $x$ which we may assume is connected. But this means that, interpreting $F$ as an element of $\mathcal{F}(V)[t]$, $F$ "locally factors" as $F(t) = \prod_{i \leq d}(t - f_i)$. This shows that these functions are locally the only sections of $\mathcal{F}$, so the sheaf is locally constant.

Therefore, applying Theorem 2.4, we obtain a cover $p_{\mathcal{F}} \colon X'_{\mathcal{F}} \to X'$. For each connected component $X'_j$ of $X'_{\mathcal{F}}$, Theorem 3.3 gives a Riemann surface $X_j$, such that $X_j$ maps onto $X$ via a proper holomorphic map, and such that its restriction is isomorphic as a cover of $X'$ to $X'_j$. Since $X$ is connected and the maps are proper, each component must be compact and thus the whole cover is. However, Theorem 2.4 does not guarantee that the cover is connected, so there may be multiple components.

To prove this is impossible, we need to use some details of the functor from locally constant sheaves to functors described earlier. For any $x \in X'$, we can construct the functions $f_1, \ldots, f_d$, as we did above. Since the sheaf $\mathcal{F}$ is locally constant, the stalk $\mathcal{F}_x$ bijects in an obvious way with $f_1, \ldots, f_d$, since these are the only sections in a neighborhood of $x$. In fact, there exists some neighborhood $V$ containing $x$ on which all the stalks have the same form. For each $f_i$, there exists a map $s_i$ sending $x \in V$ to the element of $\mathcal{F}_x$ corresponding to $f_i$; this is a local section, and by the definition of the topology on $X'_{\mathcal{F}}$, we know $s_i(V)$ is open. Define a function $f$ on $s_i(V)$, by setting:

$$f(f_i) = f_i(p_{\mathcal{F}}(f_i))$$

Note the equivocation: $f_i$, in its guise as an element of the stalks $\mathcal{F}_x$, can be the argument for $p_{\mathcal{F}}$, while $f_i$ as a function on $V$, can take $p_{\mathcal{F}}(f_i)) \in V$ as an argument. We claim that this actually defines a meromorphic function on all of $X_{\mathcal{F}}$. Indeed, the proof is nearly identical to the one we already gave for 4.2. Again, we can repeat the same construction locally around every point, giving functions which must be compatible everywhere except at branch points Likewise, we can use a similar argument via Riemann's removable singularity theorem to extend to all of $X_{\mathcal{F}}$.

In particular, we can regard the function $f$ as an element of $\mathcal{M}(X_j)$ for every connected component $X_j$. Let $d_j$ be the degree of the cover $X_j' \to X'$. Then by 4.2, the minimal polynomial, $G$, of $f$ over $\mathcal{M}(X)$ has degree at most $d_j \leq d$. Suppose we could show that $F(f) = 0$ (as usual, this should actually be taken to mean that $\widetilde{pi}(F)(f) = 0$, where $\pi X_j \to X$ is the holomorphic map). Then by unique factorization $G$ divides $F$, but since $F$ is irreducible, this means $F = G$. Thus, the degree of $G$ must be $d$, and $d_j = d$. This means that fibers of points in $X'$ already have $d$ elements from $X_j'$, so $X_j$ must actually be the only component, and $X_{\mathcal{F}}$ is connected. Also, since every element of $\mathcal{M}(X_{\mathcal{F}})$ has degree at most $d$, by the primitive element theorem, we have $\mathcal{M}(X_{\mathcal{F}}) = \mathcal{M}(X)(f) \cong L$, as desired.

It remains to prove that $\tilde{\pi}(F)(f) = 0$. As with the proof of 4.2, it's enough to consider sections of $\pi$ and show all corresponding maps are zero. If $s$ is some section, we have $(\tilde{\pi}(F))(f) \circ s = F(f \circ s)$. In fact, by the construction any section will be of the form $s = i_{f_k}$ for some $k$. We have

$$f(i_{f_k}(x)) = f((f_i)_x) = f_i(p_{\mathcal{F}}((f_i)_x)) = f_i(x)$$

where $(f_i)_x$ denotes the corresponding element of the stalk $\mathcal{F}_x$. Since $F(f_i) = 0$, this shows that $F(f) = 0$. Thus, the proof is complete. □

It is possible to prove that the functor $F$ is fully faithful, so $F$ in fact induces an equivalence of categories:

**Theorem 4.7** ([8] Theorem 3.3.7)**.** *The functor $F$ defined above gives a contravariant equivalence of categories between compact connected Riemann surfaces as branched covers of $X$ and finite field extensions of $\mathcal{M}(X)$. In particular, finite Galois branched covers correspond to finite Galois extensions of the same degree.*

In particular, we have the following:

**Corollary 4.8.** *Let $Y$ be a Riemann surface which is a Galois branched cover of $X$. The group of automorphisms of $Y$ over $X$ is canonically isomorphic to the Galois group of $\mathcal{M}(Y)$ over $\mathcal{M}(X)$.*

*Proof.* The functor $F$ is an equivalence of categories. In particular, for every $X$, it induces isomorphisms between the automorphism group of $X$ and that of $F(X)$. □

With a little additional effort, Theorem 4.7 effectively solves the inverse Galois problem over $\mathbb{C}(t)$, showing that every finite group occurs as a Galois group for some $L/\mathbb{C}(t)$. The reader is encouraged to consult section 3.4 of [8] for details.

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