Krull Dimension of Malcev-Neumann Rings

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Abstract

It is known that if $R$ is a right noetherian ring, then the corresponding Malcev-Neumann ring $R * ((G))$ is right noetherian and $r.K.dim(R) = r.K.dim(R * ((G)))$. We prove that the right noetherianness is a necessary condition for a Malcev-Neumann ring to have right Krull dimension. Also, we consider uniform dimension of Malcev-Neumann rings. The results obtained are applied to some other ring constructions.

1 Introduction

The Malcev-Neumann construction appeared for the first time in the latter part of the 1940's (the Laurent series ring, a particular case of Malcev-Neumann rings, was used before by Hilbert). Using them, Malcev and B. Neumann independently showed (in 1948 and in 1949 resp.) that the group ring of an ordered
A Malcev-Neumann group over a division ring can be embedded in a division ring. Since then, the construction has appeared in many papers, mainly in the study of various properties of division rings and related topics. For instance, Makar-Limanov in [11] used a particular skew-Laurent series division ring to prove that the skew field of fractions of the first Weyl algebra contains a free noncommutative subalgebra. The study of Malcev-Neumann group rings over arbitrary rings was initiated in [10] by Lorenz while investigating properties of group algebras of nilpotent groups.

In [3] Goodearl and Small used ordinary Laurent series rings to prove that the Krull dimension of any noetherian P.I. ring is bounded above by its global dimension (when finite). In particular, they showed that if $R$ is a right noetherian ring, then $r.Kdim(R((θ))) = r.Kdim(R)$. In [13] this theorem was extended to arbitrary Malcev-Neumann rings over right noetherian rings.

In this paper, we prove that if the Malcev-Neumann ring over a ring $R$ has Krull dimension then $R$ is right noetherian. Also, we study the Krull dimension of Malcev-Neumann rings as modules over generalized power series rings, particular unitary subrings defined by the positive cone of an ordered group. It seems reasonable to generalize slightly the construction to obtain Malcev-Neumann modules over Malcev-Neumann rings.

In section two of the paper we briefly develop the definition of Malcev-Neumann modules and introduce our notation. Also, we prove some facts concerning Krull and uniform dimensions.

In the third section we show that Malcev-Neumann modules with Krull dimension are noetherian. This result is well known for polynomial and power series modules. The main difference is that the Malcev-Neumann ring over a ring need not contain a non-invertible non-zero-divisor.

The fourth section is purely technical. Properties of an ordered group with deviation are investigated.

In the fifth section we consider Malcev-Neumann modules as modules over generalized power series rings.

In the final section we show that there exists a uniserial commutative ring $R$ such that the Laurent series ring $R((θ))$ has infinite uniform dimension.
2 Definitions and notation

Throughout the paper all rings are associative with unit and all modules are right unitary.

We construct the Malcev-Neumann (group) ring in the following way. Let $R$ be a ring, let $G$ be an ordered group, and suppose that $\sigma$ is a map from $G$ into the group of automorphisms of $R$, $x \mapsto \sigma_x$. Suppose also that we are given a map $t$ from $G \times G$ to $U(R)$, the group of invertible elements of $R$. Now $R((G, \sigma, t))$ is the set of all formal sums $r = \sum_{x \in G} r_x \bar{x}$ with $r_x \in R$ such that $\text{Supp}(r) = \{ x \in G | r_x \neq 0 \}$ is well ordered. Addition is defined as usual, that is $\sum_x a_x \bar{x} + \sum_y b_y \bar{y} = \sum(a_x + b_x)\bar{x}$, and multiplication is defined by

$$(\sum a_x \bar{x})(\sum b_y \bar{y}) = \sum c_x \bar{x} \text{ where } c_x = \sum_{\{ x, y | xy = x \}} a_x \sigma_x(b_y)t(x, y).$$

It is necessary to impose two additional conditions on $\sigma$ and $t$ to insure associativity, namely that for all $x, y, z \in G$

(i) $t(xy, z)\sigma_z(t(x, y)) = t(x, yz)t(yz)$, \hspace{1cm} (ii) $\sigma_y \sigma_z = \sigma_{yz}\delta(y, z)$

where $\delta(y, z)$ denotes the automorphism of $R$ induced by the unit $t(y, z)$ (see [15, Lm. 1.1]). It is now routine to check that $R((G, \sigma, t))$ is a ring which we call the Malcev-Neumann (group) ring. We make no explicit use of conditions (i) and (ii), so we will denote the construction simply by $R*((G))$. Basic properties of it (without twisting $t$), and the original Malcev-Neumann theorem can be found in [15] or [8].

Dealing with the Krull dimension, it seems more reasonable to use modules instead of one-sided ideals, so we develop the notion of the Malcev-Neumann (group) module similarly. If $A$ is a module over $R$, then the Malcev-Neumann module $A*((G))$ is the set of all formal sums $\sum_{x \in G} a_x \bar{x}$ with coefficients in $A$ and well-ordered supports. With operations defined as above, one can easily check that (i) and (ii) insure that $A*((G))$ is a right unitary module over $R*((G))$.

We will keep the notations $B = A*((G))$ and $T = R*((G))$ throughout the paper. Also, we will use $R((\theta))$ to denote the Laurent series ring over
a ring $R$ (i.e. the Malcev-Neumann ring over $R$ with $G = \mathbb{Z}, \sigma_z = id$ \forall x \in G, t(x, y) = 1$ \forall x, y \in G). If $f \in B$, then we denote by $m(f)$ the minimal element in $\text{Supp}(f)$, and write $\lambda(f)$ for the coefficient of $m(f)$ in $f$. If $C$ is a subset of $B$, then $\lambda(C)$ is the set $\{\lambda(f) \mid f \in C\}$. If $G$ is an ordered group, then the positive cone $P$ of $G$ is the set $\{x \in G \mid x > 1\}$.

We will always assume that $G$ is non-trivial (i.e. $G \neq \{1\}$).

The construction of ordinary Laurent series rings is also a particular case (when $\delta = 0$) of another ring-theoretical construction, we refer to as the ring of formal linear pseudo-differential operators. Given a ring $R$ with a derivation $\delta$, we denote by $R((\theta^{-1}, \delta))$ the set of all formal expressions $a = \sum_{i=0}^{\infty} a_i \theta^{m_i}$ where $n \in \mathbb{Z}$ and $a_i \in R$. The addition is defined as usual, and given any $a = \sum_{i=0}^{\infty} a_i \theta^{m_i}, b = \sum_{j=0}^{\infty} b_j \theta^{m_j} \in R((\theta^{-1}, \delta))$, the product $ab$ is defined as $ab = \sum_{k=0}^{\infty} c_k \theta^{n+m-k}$ where each

$$c_k = \sum_{i=0}^{k} \sum_{j=0}^{k-i} \left( \binom{n-i}{k-i-j} a_i \delta^{k-i-j}(b_j) \right).$$

We refer to [1] for an extended discussion of the construction (in particular, for properties of generalized binomial coefficients used in the preceding definition).

We only note that if $R$ is a right noetherian ring, then $R((\theta^{-1}, \delta))$ is also right noetherian (e.g. [4, p.19]).

We use $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{D}$ to denote the set of positive integers, the additive group of integers, the additive group of rationals, and the set $\{m2^{-n} \mid 0 \leq m2^{-n} \leq 1, m, n \in \mathbb{N}\}$ respectively.

The Krull (resp., uniform (Goldie)) dimension of a module $A$ over a ring $R$ is denoted by $\text{K.dim}(A_R)$ (resp., $\text{u.dim}(A_R)$). We refer to [12, Ch. 2, 6] for the definitions and basic properties of these two dimensions and of the deviation of a poset. We only note that if $\text{Lat}(A_R)$ denotes the lattice of submodules of a module $A$, partially ordered by inclusion, then $\text{K.dim}(A_R)$ is equal (by definition) to the deviation of $\text{Lat}(A_R)$. If $x, y$ are elements of a poset $X$ such that $x \geq y$, then $z/y$ denotes the set $\{z \in X \mid x \geq z \geq y\}$.

**Definition.** A subset $\{a_\gamma\}_{\gamma \in \Gamma}$ of a module $A$ over a ring $R$ is called irredundant provided that for any $\mu \in \Gamma, a_\mu \notin \sum_{\gamma \in \Gamma \setminus \mu} a_\gamma R$.

Further, we will need the following easy lemma:
Lemma 1. (1) For a module $A$ over a ring $R$, the following conditions are equivalent: (i) Every homomorphic image of $A$ has finite uniform dimension.

(ii) $A$ contains no infinite irredundant set.

(2) Any module with Krull dimension contains no infinite irredundant set.

Proof. (1), (i) $\Rightarrow$ (ii). Suppose that $\{a_{\gamma}\}_{\gamma \in \Gamma}$ is an infinite irredundant set in $A$. For each $\mu \in \Gamma$, let $A_{\mu} = \sum_{\gamma \neq \mu} a_{\gamma} R$. Set $C = \sum_{\gamma \in \Gamma} a_{\gamma} R$, $D = \cap_{\gamma \in \Gamma} A_{\gamma}$. Let $\varphi : A/D \rightarrow \prod_{\gamma \in \Gamma} A/A_{\gamma}$ be the homomorphism such that $(\varphi(a + D))_{\gamma} = a + A_{\gamma}$. It is straightforward to verify that the restriction of $\varphi$ to $C + D/D$ is an isomorphism and $\varphi(C + D/D) = \oplus_{\gamma \in \Gamma} A/A_{\gamma}$.

(ii) $\Rightarrow$ (i) is straightforward.

(2). Observe that every homomorphic image of a module with Krull dimension has finite uniform dimension. \qed

3 Malcev-Neumann modules with Krull dimension are noetherian

For the convenience of the reader, we state the main result of [13] in terms of Malcev-Neumann modules.

Proposition 2. Let $A$ be a module over a ring $R$, let $B$ (resp. $T$) be the corresponding Malcev-Neumann module (resp. ring). Then $A_R$ is right noetherian if and only if $B_T$ is right noetherian. Moreover, in this case $K.dim(A_R) = K.dim(B_T)$. \qed

The proof is the same as the proof of Theorem 2.1 in [13] (where $A = R, B = T$), which in turn extends the proof of Proposition 2 in [3].

Now we shall need the following Lemma 3, which is the crucial step in the proof of our main Theorem 4.

Lemma 3. If a module $A_R$ is not noetherian, then the corresponding Malcev-Neumann module $B_T$ contains an infinite irredundant set. Moreover, this set can be chosen so that any element $f$ in it has $m(f) \geq 1$.\qed
Proof. Let \( x \) be any element of \( G \) such that \( 1 < x \) (recall that we assume \( G \) to be non-trivial.) Then \( 1 < x < x^2 < \ldots \), and so the set \( \{x^i \mid i = 0,1,2,\ldots \} \) is well-ordered. Since \( A \) is not noetherian, there exists a sequence

\[
a_1, a_2, \ldots \quad \text{with } a_n \in A \quad \text{such that } \quad a_{n+1} \not\in \sum_{k=1}^{n} a_k R \quad \forall n \in \mathbb{N}.
\]

We will use it to obtain a family \( \{f^{(i)} = \sum_{k=0}^{\infty} f^{(i)}(x^k) \}_{i \in \mathbb{N}} \) of elements in \( B \) such that

\[
f^{(j)} \not\in \sum_{i \in \mathbb{N}\setminus \{j\}} f^{(i)}T \quad \forall j \in \mathbb{N}.
\]

Let \( \tau : \mathbb{N} \to \mathbb{N} \times \mathbb{N}, \quad n \mapsto \tau(n) = (\tau_1(n), \tau_2(n)) \) be any bijective map, and let a map \( \alpha : \mathbb{N} \cup \{0\} \to \mathbb{N} \) be defined recursively by \( \alpha(0) = 1 \) and \( \alpha(n) = \tau_2(n) + \alpha(n-1) + 1 \) for all \( n > 0 \). Note that \( \alpha \) is an increasing function. We construct \( f^{(i)}, \ i \in \mathbb{N} \) simultaneously by induction, in each step \( n \) putting \( f^{(\tau_1(n))} = a_n \). All the other coefficients of \( f^{(i)}, i \in \mathbb{N} \) are assumed to be zero.

Now it remains to show that the set \( \{f^{(i)}\}_{i \in \mathbb{N}} \) is irredundant. Assume not. Then there exists some \( j \in \mathbb{N} \) such that \( f^{(j)} \in \sum_{i \in \mathbb{N}\setminus \{j\}} f^{(i)}T \), that is, there exist some series \( f^{(1)}, \ldots, f^{(k)} \) (we may change numbers so that \( j \notin \{1,\ldots,k\} \)) and \( g^{(1)}, \ldots, g^{(k)} \in T \) satisfying

\[
f^{(j)} = f^{(1)}g^{(1)} + \ldots + f^{(k)}g^{(k)}.
\]

By [15, Lm. 13.2.9], the set \( X = \bigcup_{i=1}^{k} \text{Supp}(g^{(i)}) \) is well-ordered. Since \( f^{(j)} \) has support in \( \langle x \rangle \), we have \( X \cap \langle x \rangle \neq \emptyset \), and so we can put \( v = \min \{ s \mid x^s \in X \} \). Because \( \tau \) is a bijection, there exists \( n = \tau^{-1}(j, \mid v \mid) \) where \( \mid v \mid \) stands for the absolute value of \( v \). Now consider the coefficient

\[
f^{(j)}_{\alpha(n)} = a_n = \sum_{i=1}^{k} (f^{(i)}_{\alpha(n)-1}g^{(i)}_v + \ldots + f^{(i)}_{\alpha(n)}g^{(i)}_v),
\]

where \( g^{(i)}_v \) are some coefficients of the series \( g^{(1)}, \ldots, g^{(k)} \). We have

\[
\alpha(n) - v \leq \alpha(n) + \mid v \mid < \alpha(n + 1),
\]
and thus \( a_n \) is a linear combination of coefficients \( f_m^{(i)} \) with \( i \neq j \) and \( m < \alpha(n + 1) \), that is, of \( a_1, a_2, \ldots, a_{n-1} \). This contradicts the choice of \( a_n \). Thus \( f^{(j)} \notin \sum_{i \in \mathbb{N}(j)} f^{(i)}T \), contradicting the assumption. Therefore, \( B \) contains an infinite irredundant set. It remains to note that \( m(f^{(i)}) \geq 1 \) for all \( i \).

**Theorem 4.** Let \( G \) be a non-trivial ordered group, let \( A \) be a module over a ring \( R \), and let \( B = A \ast ((G)) \) be the Malcev-Neumann module over \( T = R \ast ((G)) \). If the module \( B_T \) has Krull dimension, then \( B_T \) (and hence \( A_R \)) is noetherian.

**Proof.** Assume \( A_R \) is not a noetherian module. By Lemma 3, the Malcev-Neumann module \( B_T \) contains an infinite irredundant set. By Lemma 1, this contradicts the fact that \( B_T \) has Krull dimension. \( \square \)

The same argument gives us the similar assertion for rings of pseudo-differential operators. Indeed, the multiplication rule above allows us to use the construction of Lemma 3 in this case as well (we use \( \theta^{-1} \) instead of \( x \)). Then we apply Lemma 1 again to get the following

**Proposition 5.** Let \( R \) be a ring with a derivation \( \delta \). If the ring of formal linear pseudo-differential operators \( R((\theta^{-1}, \delta)) \) has right Krull dimension, then \( R \) is right noetherian. \( \square \)

### 4 Deviation of ordered groups

The main aim of this section is purely technical. For the purposes of the following section, we need to establish some properties of deviation of ordered groups. For abelian groups the assertions 6-8 can be found in [9]. Throughout the section \( G \) (resp. \( P, Z(G) \)) denotes an ordered multiplicative group (resp. the positive cone of \( G \), the center of \( G \)). We will write \( \text{dev}(G, <) \) when it is necessary to distinguish different orders on \( G \). We say that a group \( G \) has deviation if \( G \) admits some full order and has a deviation with respect to this order.
Lemma 6. (1) Given any \( x, y, z \in G \) such that \( y \geq z \),
\[ \text{dev}(xy/xz) = \text{dev}(yx/zx) = \text{dev}(y/z). \]
(2) Given any \( x, y \in P \), \( \text{dev}(x/1) > \text{dev}(y/1) \) if and only if \( x > y^n \) for all \( n \in \mathbb{N} \).
(3) For any \( x \in P \) such that \( \text{dev}(x/1) = 0 \), \( x \in Z(G) \).

Proof. Straightforward. \( \square \)

We note that (2) in Lemma 6 shows that for any \( x \in P \), \( \text{dev}(x/1) \) is in fact the archimedian class of \( x \) (see [8, p.245]).

Proposition 7. (1) (a) If \( \alpha \) is a non-limit ordinal, then \( \text{dev}(G) = \alpha + 1 \) if and only if \( \text{dev}(P) = \alpha \).

(b) If \( \alpha \) is a limit ordinal, then \( \text{dev}(G) = \alpha + 1 \) if and only if \( \text{dev}(P) = \alpha \) and there exists \( x \in P \) such that \( \text{dev}(x/1) = \alpha \). Otherwise, \( \text{dev}(G) = \alpha \).

(2) If \( \text{dev}(G) = 1 \), then \( G \) is isomorphic to \( \mathbb{Z} \) both as a poset and as a group.

(3) If \( G \) is an additive subgroup of reals and \( G \) has deviation, then \( G \cong \mathbb{Z} \).

Proof. (1): First, we observe that for any ordinal \( \alpha \), \( \text{dev}(P) = \alpha \) implies that \( \alpha \leq \text{dev}(G) \leq \alpha + 1 \). This follows from [12, 6.1.3(i)].

(a): Now let \( \alpha \) be a non-limit ordinal, and let \( \beta \) denote the predecessor of \( \alpha \). Suppose that \( \text{dev}(P) = \alpha \). We must show that \( \text{dev}(G) > \alpha \). Assume on the contrary that \( \text{dev}(G) = \alpha \). There exists a chain \( x_1 > x_2 > \ldots > x_i > \ldots > 1 \) such that each factor of this chain has deviation \( \beta \). Since an ordered group has no maximal element, we can take by induction \( y_{i+1} \) such that \( y_{i+1} > x_{i+1}y_i \) \( (y_i \in P \) is chosen arbitrarily). Then for all \( i, n \in \mathbb{N} \) we get \( y_i^{-1}x_{i+n} > y_i^{-1} = y_i^{-1}x_{i+1} > y_i^{-1}x_{i+1} > \ldots > y_i^{-1}x_1 \). We note that \( y_i^{-1} > y_1^{-1} > \ldots \) implies that \( y_i^{-1}x_1 > y_i^{-1}x_2 > \ldots > y_i^{-1} > \ldots \). Since \( \text{dev}(G) = \alpha \), \( \text{dev}(y_i^{-1}x_i/y_i^{-1}x_{i+1}) \leq \beta \) for all but finitely many indices \( i \). For any \( i \in \mathbb{N} \), consider the chain \( y_i^{-1}x_i > y_i^{-1}x_{i+1} > \ldots > y_i^{-1}x_{i+n} > \ldots > y_i^{-1}x_{i+1} \). For all \( n \), \( \text{dev}(y_i^{-1}x_{i+n}/y_i^{-1}x_{i+n+1}) = \text{dev}(x_{i+n}/x_{i+n+1}) = \beta \). Thus for all \( i \), \( y_i^{-1}x_i/y_i^{-1}x_{i+1} \) contains an infinite descending chain with factors of deviation.
Lemma 8. Let $\text{dev}(G) = \alpha$ where $\alpha$ is some ordinal. For any ordinal $\gamma \leq \alpha$, let $H_\gamma$ be a subgroup of $G$ generated by all $x \in P$ such that $\text{dev}(x/1) < \gamma$. Then:

(i) $H_0 = \{1\}$, $H_\alpha = G$.
(ii) For any ordinal $\gamma < \alpha$, $H_{\gamma+1}/H_\gamma \cong \mathbb{Z}$.

Proof. Let $\gamma$ be any ordinal, $\gamma < \alpha$. Clearly, $H_\gamma$ is convex (as an ordered set). By Lemma 6(1), $H_\gamma$ is normal in $G$. Now assume that $H_\gamma$ is not a maximal convex subgroup of $H_{\gamma+1}$. Then there are some element $x \in H_{\gamma+1} \setminus H_\gamma$, $x > 1$ and a convex subgroup $K$ of $H_{\gamma+1}$ such that $\langle x, H_\gamma \rangle \subseteq K$. Observe that $\text{dev}(x/1) > \gamma$, and thus $\text{dev}(K) = \gamma + 1$. Since $K$ is convex, $K = H_{\gamma+1}$, contradicting the assumption, and so $H_\gamma$ is a maximal convex subgroup of $H_{\gamma+1}$. Then the factor-group $H_{\gamma+1}/H_\gamma$ has no non-trivial convex subgroups. By Holder's theorem [7, Ch. 2, Th. 2.1], $H_{\gamma+1}/H_\gamma$ is isomorphic (as an ordered group) to a subgroup of the naturally ordered group of real numbers. It is straightforward to check that $H_{\gamma+1}/H_\gamma$ has deviation. Then Proposition 7(3) implies that $H_{\gamma+1}/H_\gamma \cong \mathbb{Z}$, as claimed.

Corollary 9. If $(G, <)$ has finite deviation $n$, then

(1) $G$ is $n$-generated.
(2) $G$ is nilpotent of a class at most $n$.
(3) $n = h(G)$, the Hirsch number of $G$.
(4) For any other (full) order $<'$, if $\text{dev}(G, <')$ exists, then it is equal to $n$.

Proof. Straightforward. □

We provide the reader with some examples with obvious proofs. In particular, we will show that the cases (a) and (b) in Proposition 7(1) are indeed
different. Examples in 10(1), (2) are abelian groups, so the group operation there is written as addition.

Examples 10. (1) By Proposition 7(3), the set of rationals \( \mathbb{Q} \) ordered as usual has no deviation. The subgroup of reals generated by \( \{1, \sqrt{2}\} \) with the natural ordering has no deviation as well.

(2) ([9]) Let \( \alpha \) be any ordinal, and let \( \Delta_\alpha \) denote the set of all ordinals \( \gamma \) such that \( 1 \leq \gamma \leq \alpha \). Then let \( G_\alpha \) be a free \( \mathbb{Z} \)-module with basis \( \Delta_\alpha \), and set

\[
P_\alpha = \{ m_1 \delta_1 + \ldots + m_n \delta_n \mid m_1 > 0, \ m_i \geq 0, \ \delta_1 > \delta_2 > \ldots > \delta_n \} \text{ where } i \leq n, n \in \mathbb{N}.
\]

It is proved in [9, Prop. 23], that \( \text{dev}(G_\alpha) = \alpha \). Moreover, any ordered abelian group of deviation \( \alpha \) is isomorphic to \( G_\alpha \) ([9, Th. 25]). However, in [5, Ex. 17.2.9] there are presented two non-isomorphic finitely generated torsion-free nilpotent groups each of which is isomorphic to a subgroup of the other (and hence they have the same finite deviation, namely 3).

(3) Let \( G = \text{UT}_m(\mathbb{Z}) \), the multiplicative group of unitriangular \( m \times m \) matrices with integer entries. We claim that \( \text{dev}(G) = m(m - 1)/2 \) (with respect to any full order, for which the deviation exists). By Corollary 9(3), it suffices to exhibit some particular full order \( < \) on \( G \) such that \( \text{dev}(G, <) = m(m - 1)/2 \). We order "matrix units" \( E_{ij}, \ i < j \) in the following fashion:

\[
E_{m-1,m} > E_{m-2,m-1} > \ldots > E_{12} > E_{m-2,m} > \ldots > E_{13} > \ldots > E_{1m}.
\]

Given any \( x \in G \), we have the formal expression \( x = \sum_{i<j} x_{ij} E_{ij} \) with \( x_{ij} \in \mathbb{Z} \). Now for \( x, y \in G \), we define \( x < y \) if and only if \( x_{km} < y_{km} \) for some \( E_{km} \) and \( x_{ij} = y_{ij} \) for all \( E_{ij} > E_{km} \). It is straightforward to verify that \( G \) with \( < \) becomes an ordered group and is isomorphic as a poset to \( G_{m(m-1)/2} \) (cf. Example 10(2) above). Then \( \text{dev}(G, <) = m(m - 1)/2 \), as claimed.

(4) Let \( G = \prod_{m=1}^{\infty} \text{UT}_m(\mathbb{Z}) \) where each factor ordered as in (3), and then the product is ordered reverse lexicographically. One can check that \( G \) has deviation with respect to this order. However, \( G \) is neither nilpotent, nor even solvable. \( \square \)
Proposition 11. If $G$ is a finitely generated torsion-free nilpotent group, then $G$ has finite deviation with respect to some full order. Moreover, for any two orders $<, <'$, if a deviation with respect to $<'$ exists, $\text{dev}(G, <) = \text{dev}(G, <')$.

Proof. By Malcev's theorem [7, App., Th. 4.1], any finitely generated torsion-free nilpotent group admits a representation by unitriangular matrices with integer entries. That is, $G$ is isomorphic to a subgroup of $\text{UT}_n(Z)$ for some $n \in \mathbb{N}$. By Example 10(3) above, $\text{UT}_n(Z)$ has finite deviation, whence $G$ has finite deviation with respect to the induced order. Now the assertion follows from Corollary 9.

Remark. By Lemma 8, every group with deviation is hypercyclic, and hence is locally nilpotent (see [18, p. 351]). In fact, the class of groups with deviation coincides with the class of hypercyclic groups with infinite cyclic factors of a generalized central upper series.

5 Malcev-Neumann modules over power series rings

In this section we consider one possible generalization of the construction of power series rings. Set $S = R \ast [[P]] = \{ f \in T \mid \text{Supp}(f) \subseteq P \cup \{1\} \}$. Then $S$ is a unitary subring of $T$, and hence $B$ can be considered as a (right) $S$-module. Another possible generalization of the ordinary power series ring construction (commutative case) can be found in [17].

It was shown in [13] (and in the case of Laurent series in [3]) that if $C \subseteq D$ are right ideals of $T$, and the right ideal $\lambda(C) = \lambda(D)$ is finitely generated, then $C = D$. One may note that in fact a slightly more general result is obtained in [13, Lm. 2.3]. The proof is exactly the same, so we will omit it.

Lemma 12. Let $C_S \subseteq D_S$ be submodules of a Malcev-Neumann module $B$ over $S = R \ast [[P]]$, suppose that

$$\lambda(D) = \lambda(C) = \sum_{\{f(i) \in C \mid m(f(i)) = x, i = 1, \ldots, m\}} \lambda(f(i))R$$
for some \( x \in G \), and suppose that \( m(g) \geq x \) for all \( g \in D \). Then \( C = D \). \( \square \)

**Proposition 13.** \( B_S \) contains no infinite irredundant set if and only if \( A_R \) is a module of finite length.

**Proof.** \( \Rightarrow \): If \( A_R \) is not noetherian, then \( B_T \) contains an infinite irredundant set by Lemma 3. Hence \( B_S \) contains an infinite irredundant set. Now assume that \( A_R \) is not artinian. Then there exists an infinite sequence \( a_1, a_2, \ldots \in A \) such that \( a_j \notin \sum_{i=j+1}^\infty a_iR \). Take any \( x \in P \). Clearly, \( \{a_i x^{-i}\}_{i \in \mathbb{N}} \) is an infinite irredundant set, a contradiction.

\( \Leftarrow \): Assume that \( B \) contains some infinite irredundant set \( V = \{f(i)\}_{i \in \mathbb{N}} \).

Since \( A_R \) is noetherian, there exists \( n_1 \) such that \( \lambda(\sum_{i=1}^\infty f(i)S) = A_1 = \lambda(\sum_{i=1}^{n_1} f(i)S) \). Set \( x_1 = \min_{i=1, \ldots, n_1} \{m(f(i))\} \). Since \( V \) is irredundant, none of the \( f(i) \) lies in \( \sum_{i=1}^{n_1} f(i)S \) when \( i > n_1 \). If the set \( \{f \in V \mid m(f) < x_1\} \) is empty, then there exists an \( x \in \{m(f(1)), \ldots, m(f(n_1))\} \) such that the set \( W = \{f \in V \mid m(f) = x\} \) is infinite. Then one can choose a finite subset \( g^{(1)}, \ldots, g^{(k)} \in W \) such that

\[
\lambda\left( \sum_{f \in V \mid m(f) = x} fS \right) = \lambda\left( \sum_{j=1}^k g^{(j)}S \right)
\]

and apply Lemma 12 to obtain a contradiction. Therefore \( \{f \in V \mid m(f) < x_1\} \) is a non-empty set. There exists \( n_2 \) such that \( \lambda(\sum_{f \in V \mid m(f) < n_2} f(i)S) = A_2 = \lambda(\sum_{i=n_1}^{n_2} f(i)S) \). If \( A_1 = A_2 \), then

\[
\sum_{i=n_1}^{n_2} f(i)S = \sum_{i=n_1}^{n_2} f(i)S + \sum_{i=1}^{n_1} f(i)(m(f(i)))^{-1}x_1S
\]

by Lemma 12, a contradiction. Thus \( A_1 \supset A_2 \) is a strict inclusion. Continuing in this way, we obtain a strictly descending infinite chain \( A_1 \supset A_2 \supset \ldots \), which is impossible because \( A_R \) is an artinian module. \( \Box \)

**Theorem 14.** The following conditions are equivalent:

1. \( B_S \) has Krull dimension.
2. \( A_R \) is a module of finite length, and \( G \) has deviation as an ordered set.
When this conditions hold, $\text{K.dim}(B_S) = \text{dev}(G)$. The Krull dimension of $B_S$ is finite if and only if $G$ is a finitely generated torsion-free nilpotent group.

**Proof.** (1) $\Rightarrow$ (2): Since $B_S$ has Krull dimension, Lemma 1 implies that $B_S$ contains no infinite irredundant set. By Proposition 13, $A_R$ has finite length. Now assume that $G$ has no deviation. Then $G$ contains some subset $X = \{x\}$ isomorphic (as a poset) to $D$ ( [12, 6.1.13] ). Consider the set $\{xS\} \subseteq \text{Lat}(B_S)$, and note that the poset $\{xS\}_{x \in X}$ is also isomorphic to $D$. Again by [12, 6.1.13], $B_S$ has no Krull dimension, contradicting the assumption.

(2) $\Rightarrow$ (1): We proceed by induction on $n$, the length of $A$. Suppose that $n = 1$, so $A_R$ is a simple module. We claim that in this case for any $f \in B$, $fS = \text{am}(f)S$ for all $a \in A$. In particular, $B_S$ is uniserial (that is, $\text{Lat}(B_S)$ is linearly ordered by inclusion). Moreover, we claim that $\text{Lat}(B_S)$ is isomorphic to $G$ as a poset.

To prove the first claim, set $C = \text{am}(f)S$, $D = C + fS$ and apply Lemma 12. Indeed, if $y, h \in B$ and $m(g) < m(h)$, then there exists $x \in P$ such that $m(g)x = m(h)$, and so $gS = \lambda(g)S$, $hS = \lambda(h)S$ implies that $hS \subseteq gS$. Therefore $B_S$ is uniserial.

Let $\varphi$ be the map $G \rightarrow \text{Lat}(B_S)$ that takes each $x \in G$ to the submodule $AxS$. If $y, z \in P$ and $y < z$, then $xy < xz$, whence $AyzS \supseteq AxzS$ and the above argument implies that this inclusion is strict. Obviously, $\varphi$ is surjective. Thus $\varphi$ is an isomorphism of posets. Therefore $K\text{.dim}(B_S) = \text{dev}(G)$. This completes the first step of induction.

Now assume that the assertion is proved for $n - 1$, and let $A_R$ be a module of finite length $n$. Then $A$ contains some submodule $A'$ such that $A/A'$ is a simple $R$-module, and $A'$ has length $n - 1$. Set $B' = A' \ast ((G))$. By hypothesis, $K\text{.dim}(B'_S) = \text{dev}(G)$. Note that $B/B'_S \cong (A/A') \ast ((G))_S$ and that $A/A_R$ is simple, whence $B/B'_S$ also has Krull dimension equal to $\text{dev}(G)$. By [12, 6.2.4],

$$K\text{.dim}(B_S) = \sup\{K\text{.dim}(B'), K\text{.dim}(B/B')\} = \text{dev}(G),$$
and the induction step is completed.

The last assertion follows from Corollary 9. □

6 Uniform dimension

The main purpose of this section is to apply the construction of an irredundant set given in Lemma 3 to uniform dimension of Laurent series rings. A well-known theorem due to Shock asserts that if $I$ is a right uniform ideal of a ring $R$, then $I[\theta]$ is a right uniform ideal in $R[\theta]$ (we refer to the statements and proofs in Goodearl's book [2, 3.21-23]). This is the crucial step in the proof of the following theorem ([2, 3.23]): Let $\Theta$ be any collection of indeterminates. If $R_R$ has finite uniform dimension, then so does $R[\Theta]|_R$, and $\dim(R_R) = \dim(R[\Theta]|_R)$. The proof (in [2]) can be easily extended to the case of a right uniform module $A$ over a ring $R$. Moreover, [4, Th. 9.17] asserts that the cited theorem remains true for modules of Laurent polynomials. Here we show that it fails for Laurent series modules and rings (cf. Remark below).

Example 15. Let $p$ be any prime number, let $A = \mathbb{Z}_{p\infty}$ be the Prufer abelian group (represented as the $p-$ component of the additive group $\mathbb{Q}/\mathbb{Z}$ ), and let $R = \mathbb{Z}$. Set $B = A(\theta), T = R(\theta))$. Note that $A_R$ is uniserial, whence $\dim(A_R) = 1$. We claim that $B_T$ contains an infinite direct sum of submodules, and so $\dim(B_T) = \infty$.

Constructing an infinite direct sum of submodules of $A$, we follow the way of Lemma 3. Set $a_i = p^{-(1+2+\ldots+i)}$ for all $i \in \mathbb{N}$. Clearly, this sequence satisfies the required property. Then we construct $\{f(i)\}_{i \in \mathbb{N}}$ as in Lemma 3. Now we must show that the sum $\sum f(i)_T$ is direct. Assume not. Then there exist some series $f(1), \ldots, f(k)$ and some nonzero series $g(1), \ldots, g(k) \in T$ such that $f(1)g(1) + \ldots + f(k)g(k) = 0$. Set $m_j = m(g(j))$ for $j = 1, \ldots, k$. Then we have $\sum_{j=1}^k (f(i)g_{m_j} + \ldots + f(0)g_i) = 0$ for all $i \geq 0$. One may assume without loss of generality that $m_1 = \min\{ m_j \}$. Let $m \in \mathbb{N}\{0\}$ be the maximal number such that $p^m$ divides $g^{(1)}_{m_1}$. By construction, there exists some $n \in \mathbb{N}$ such that $n > m$, $\tau_1(n) = 1$, and $\tau_2(n) = \max_{j=1,\ldots,k}\{m_j\}$. Take $i = a(n) + m_1$ and consider the equality

$$f_{a(n)}^{(1)}g_{m_1} = -f_{a(n)-1}^{(1)}g_{m_1+1} - \ldots - \sum_{j=2}^k (f_{i-m_j}^{(j)}g_{m_j} + \ldots + f_0^{(j)}g_i^{(j)}).$$
Reasoning as in Lemma 3, we obtain that in the right side of the equation all the coefficients of series \( f^{(j)} \) are either zero or contained in the set \( a_1, \ldots, a_{n-1} \).

So the absolute value of the exponent of the denominator in the right side of the equality does not exceed \( 1 + \ldots + (n-1) \). On the other hand, the absolute value of the exponent in the left side is equal to \( 1 + \ldots + (n-1) + n - m > 1 + \ldots + (n-1) \) (recall that \( n > m \)), a contradiction. Thus \( B_T \) has infinite uniform dimension, as claimed. \( \square \)

One may prefer to have an example of a uniserial ring \( R \) (instead of a uniform module) such that the Laurent series ring over \( R \) has infinite right uniform dimension. The preceding example can be rearranged to achieve this goal as follows.

**Example 16.** Let \( \mathbb{Z}(p) \) be the ring of \( p \)-adic integers for some prime integer \( p \). Then let \( R \) be a ring isomorphic to \( \mathbb{Z}(p) \oplus \mathbb{Z}_p^\infty \) as an abelian group with multiplication defined by \( (r, a) \ast (s, b) = (rs, rb + sa) \). The commutative ring \( R \) was used by Osofsky in [14] as an example of an injective cogenerator without chain conditions. Also, it was proved in [14] that the only ideals of \( R \) are additive subgroups of \( \mathbb{Z}_p^\infty \) and ideals generated by \( ((p^i, 0)) \) for all \( i \in \mathbb{N} \). Thus \( \text{u.dim}(R_R) = 1 \). Moreover, \( R \) is uniserial (as a module over itself). Now observe that for any \( r \in R \) and \( a \in \mathbb{Z}_p^\infty \), the denominator of \( ra \) does not exceed the denominator of \( a \). Thus the argument of the previous example is still valid. Therefore \( R((\theta)) \) (and hence also \( R[[\theta]] \)) has infinite uniform dimension. \( \square \)

**Remark.** In [6] was given an example of a right Ore domain \( R \) such that the power series ring \( R[[\theta]] \) (and hence the Laurent series ring \( R((\theta)) \)) is not right Ore. Now [4, Th. 5.15] asserts that the domains \( R[[\theta]] \) and \( R((\theta)) \) have infinite right uniform dimension. Note that the ring \( R \) in this example is not right uniserial.

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