We develop a general methodology to characterize equilibrium leverage dynamics in a tradeoff model when the firm can continuously adjust leverage and cannot commit to a policy ex ante. While the leverage ratchet effect leads shareholders to increase debt gradually over time, asset growth and debt maturity cause leverage to mean-revert slowly towards a target. Because investors anticipate future debt issuance, credit spreads are increased, fully offsetting the tax benefits from future debt issuance. Finally, although the target leverage and speed of adjustment depend critically on debt maturity, shareholders are indifferent toward the debt maturity structure.

Keywords: Capital Structure, Target Leverage, Tradeoff Theory, Credit Spreads, Debt Maturity, Debt Overhang, Dynamic Games, Coase Conjecture
1. **Introduction**

Understanding the determinants of a firm’s capital structure, and how its leverage is likely to evolve over time, is one of the central questions in corporate finance. Leverage and its expected dynamics are crucial to valuing the firm, assessing its credit risk, and pricing its financial claims. Forecasting the optimal response of leverage to shocks, such as the 2007-2008 financial crisis, is necessary to anticipate the likely consequences of a crisis and its aftermath, and to evaluate alternative policy responses.

Despite its importance, a fully satisfactory theory of leverage dynamics has yet to be found. Many models assume the absolute level of debt is fixed; for example, in the traditional framework of Merton (1974), as well as Leland (1994, 1998), the firm is committed not to change its outstanding debt before maturity, irrespective of the evolution of the firm’s fundamentals. As a result, the dynamics of firm leverage is driven solely by the stochastic growth in value of the firm’s assets-in-place. More recent work that allows the firm to restructure its debt over time typically assumes that all existing debt must be retired (at a cost) before any new debt can be put in place.¹

These assumptions are neither innocuous, as the constraints on leverage generally bind in the model, nor are they consistent with practice, where firms often borrow incrementally over time. See, for example, Figure 1, which shows how debt levels for American and United Airlines changed over time in response to fluctuations in their enterprise values (market value of equity plus book value of debt).

In this paper, we study a model in which equity holders lack the ability to commit to their future leverage choices and can issue or buyback debt at the current market price at any time. Aside from corporate taxes and bankruptcy costs, there are no other frictions or transactions costs in our model. In such a setting, when debt can be freely adjusted over time, it is feasible for the firm to avoid the standard leverage “tradeoff” by simply increasing debt to exploit tax shields when cash flows are high and reducing debt to avoid distress costs when cash flows fall.

But although such an ideal policy is feasible, absent commitment an important agency friction emerges with regard to the firm’s future leverage choices. As emphasized by Admati et al. (2018), equity holders will adjust leverage to maximize the current share price rather than total firm value. They demonstrate a “leverage ratchet effect,” in which equity holders are never willing to voluntarily reduce leverage, but always have an incentive to borrow more -- even if current leverage is excessive and even if new debt must be junior to existing claims. While the leverage ratchet effect is itself quite general, they numerically calculate a dynamic equilibrium only for a specialized model with perpetual debt and Poisson profitability shocks.

Solving the dynamic tradeoff model without commitment is challenging because of the dynamic interdependence of competitive debt prices today and the firm’s equilibrium leverage/default policies in the future. In this paper we develop a methodology to solve for such an equilibrium in a general setting that allows for finite maturity debt, asset growth, investment, and both Brownian and Poisson shocks. In this equilibrium, equity holders increase debt gradually over time, at a rate which increases with the current profitability of the firm. On the other hand, following negative shocks, equity holders never voluntarily reduce leverage, but do allow it to decline passively via debt maturity and asset growth.

Figure 1. Time-series of outstanding book debt and enterprise value for American and United Airlines, for fifteen years before their bankruptcies in 2011 and 2002, respectively. Book debt is calculated as the sum of “long-term debt” and “debt in current liabilities”, and market equity is calculated as “stock price” multiplying “common shares outstanding.” Data source: WRDS.
In our model, equity holders keep issuing debt to exploit tax benefits even after the firm’s leverage passes above the “optimal” level with commitment, leading to excessive inefficient default. This result holds even when there is no dilution motive to issue debt (either because there is zero recovery value in bankruptcy, or debt is prioritized so that newly issued debt must be junior to all existing debt). However, even without a direct dilution effect, there is an indirect “dilution” or devaluation effect associated with new debt issuance, as additional leverage raises the probability of default for all debt holders. Creditors anticipate the devaluation associated with future over-borrowing, and consequently lower the price they will pay for the debt. This price impact offsets the tax advantage of leverage so that, on the margin, equity holders are indifferent to leverage increases. We show that this effect is so strong that equity holders obtain the same value in equilibrium as if they committed not to issue any debt in the future. In other words, the extra tax shield benefits that tempt equity holders are exactly dissipated by the bankruptcy costs caused by excessive leverage.

The result that all gains from trade are dissipated in equilibrium is closely related to the Coase’s (1972) conjecture regarding durable goods monopoly. In this context, the firm is a monopoly supplier of its own debt, and creditors’ valuation of the debt depends on the total quantity supplied. In our model, as Coase conjectured, the debt price falls to the marginal (after-tax) cost to shareholders, and shareholders are thus unable to capture the benefits from trade.\(^2\)

We apply our methodology to the special case of geometric Brownian motion (as in Leland (1994)) and solve for the equilibrium debt price and issuance policy in closed form. Because equity holders refuse to buy back debt once it is issued, debt issuance becomes effectively irreversible, slowing its initial adoption. Debt accumulates over time at a rate that increases with profitability, while if profits decline sufficiently, new issuance drops below the rate of debt maturity and the debt level falls. Leverage is thus path dependent, and we show explicitly that the firm’s outstanding debt at any point of time can be expressed in terms of a weighted-average of the firm’s past earnings. The endogenous adjustment of leverage leads the firm’s interest coverage ratio to mean revert gradually in equilibrium, with the speed of adjustment decreasing with debt maturity and

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\(^2\) Though note that, unlike in Coase’s setting, marginal cost (which is the equilibrium debt price) is endogenous in our model, and investors share a common valuation for the asset. Similar results can be found in DeMarzo and Uroševic (2006) in the context of trading by a large shareholder, and Daley and Green (2018) in which a monopolistic buyer makes frequent offers to a privately informed seller.
asset volatility, providing a theoretical foundation for partial adjustment models (e.g., Jalilvand and Harris (1984); Leary and Roberts (2005); and more recently, Frank and Shen (2019)) that are widely used in the empirical capital structure literature. These dynamics differ from the abrupt adjustment to a “target” leverage level implied by models with an exogenous adjustment cost (for instance, Fischer, Heinkel, Zechner (1989); Goldstein, Ju, Leland (2001); and Streubalaev (2007); etc.).

We compare our model without commitment to the benchmark in which shareholders commit to never issue debt in the future. While equity prices coincide with models in which there is no new debt issuance, bond investors’ anticipation of future borrowing causes credit spreads to be bounded away from zero and much larger than in standard models. For a firm that is initially unlevered, we show that this high cost of debt might dissuade it from borrowing altogether, and thus help explain the zero-leverage puzzle (Strebulaev and Yang, 2013).

We also study the optimal debt maturity structure. Our model without commitment provides a fresh perspective on this question. We show that at every point in time, equity holders are indifferent to the maturity choice for future debt issuance. Short maturity debt leads to higher leverage on average, as equity holder issue debt more aggressively knowing leverage can be reversed more quickly when the short-term debt matures. Nevertheless, the gain from additional tax shields is offset by increased default costs. Thus, the agency costs associated with the leverage ratchet effect persist even for instantaneously maturing debt.

In our model firms with different debt maturity structures can have quite different leverage dynamics, yet firms are indifferent in the debt maturity choice. Thus, small perturbations or frictions that may lead firms to pick differing initial maturity structures can lead over time to dramatically different leverage outcomes. In addition, because leverage mean reverts slowly, even firms with the same debt maturity may have quite different leverage based on past shocks. These results provides a potential explanation for findings such as Lemmon, Roberts, and Zender.

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3 The choice of debt maturity structure does affect the value of equity if the firm is forced to borrow a fixed amount upfront. Indeed, this question has been studied in the Leland (1998) setting, and often long-term debt, which minimizes rollover risk, is preferred (He and Xiong, 2012; Diamond and He, 2014). In contrast, we show that without commitment firms prefer short-term debt for any positive targeted debt financing. Short-term debt has lower price impact because it allows the firm to quickly reduce leverage through maturity. (Another possible force favoring short-term debt is investors’ liquidity preference, which is modeled in He and Milbradt (2014).) On the other hand, if the firm is not forced to issue debt initially, we show that long-term debt is preferred from a social perspective because the firm will accumulate it more slowly, reducing the expected dead-weight losses from default.
(2008) that much of the cross-sectional variation in firms’ capital structure is persistent and largely unexplained by observable firm characteristics.

Finally, we consider the interaction of the firm’s leverage and investment policies. When the firm cannot commit to its investment policy, leverage distorts investment due to debt overhang. The firm tends to issue new debt more slowly, and targets a lower level of leverage in the presence of debt overhang. Near default, however, shareholders issue debt more aggressively when they have the option to cut investment. Ex ante, shareholders are not incrementally harmed by debt overhang, as underinvestment substitutes for default costs.

Our paper is most closely related to Admati et al. (2018). They demonstrate the leverage ratchet effect in the context of a one-time leverage adjustment, and then numerically evaluate a dynamic equilibrium in a stationary model with regime shocks and perpetual debt. Our paper studies leverage dynamics in a richer continuous-time framework that allows for both asset growth and debt maturity, as well as both Brownian and Poisson shocks. As one of the main contribution of the paper, we develop a general methodology to solve for an important class of equilibria. For the standard workhorse model of Leland (1994), we not only solve for the equilibrium in closed-form that allows for deeper analysis, but also establish the uniqueness of this equilibrium.

In Dangl and Zechner (2016), the firm can choose how much maturing debt to rollover, but covenants place a cap on the rate of new debt issuance that prevent it from increasing the face value of its debt outstanding without first repurchasing all existing debt (at par plus a call premium and a proportional transaction cost). Rolling over debt maintains the firm’s tax shields, as in in our model, and directly dilutes current creditors given their setting with a strictly positive recovery rate and pari-passu debt (which we analyze in Section 3.4). They show that when debt maturity is long, equity holders will rollover existing debt fully as it comes due, except for when leverage is so low that recapitalization to a higher face value of debt is imminent (in which case it is not worthwhile to issue debt that is likely to be replaced soon, at a cost). If debt maturity is sufficiently short, however, then when facing high leverage shareholders may rollover only a portion of the maturing debt so that the total face value of debt gradually declines. This behavior abruptly reverses when the firm approaches default as shareholders maximize dilution (and minimize equity
injections) by again rolling over debt fully.\footnote{In the extension of our model in which we allow for direct dilution, because there is no constraint on the rate of issuance, we show that the debt issuance rate increases only at the moment of default.} Importantly, they show that firm value is not monotonic in debt maturity; depending on parameters, an interior optimal maturity may exist that trades off the transactions costs of debt rollover (which favors long maturities) with the benefit from debt reductions given high leverage (which favors short maturities).\footnote{The same trade-off would apply in our model if we were to adopt the same assumption on transaction costs. A similar trade-off exists in Brunnermeier and Yogo (2009) who stress the advantage of short-term debt in providing the firm with flexibility to adjust debt quickly in the face of shocks to firm value, while long-term debt is more effective at reducing costs from rollover risk.} As in our model, the choice of debt maturity becomes an important commitment device that allows for future debt reductions in the face of negative shocks.

Benzoni, Garlappi, Goldstein, and Ying (2019) study the role of commitment when the firm faces a \textit{fixed} transaction cost when issuing new debt. They focus on the class of “s-$S$” capital restructuring policies as in Goldstein, Ju, and Leland (2001) and compare outcomes when the firm can commit to the restructuring boundary to the case when it cannot. Naturally, commitment leads to a strictly higher equity value. They also argue that when fixed costs are small, equity values without commitment are close to the outcome with commitment. This result is supported by using our equilibrium as a punishment outcome in a non-Markov equilibrium (as in the classic folk-theorem literature). In contrast, we focus on the class of Markov Perfect Equilibria, in which debt prices depend only on firm fundamentals, ruling out standard “grim trigger” strategies in Benzoni et al (2019).

In the literature of endogenous debt dynamics in the presence of real investment opportunities, Hennessy and Whited (2005) study one-period short-term debt and highlight the importance of a dynamic framework in testing trade-off theories empirically. Abel (2016) considers a dynamic model with investment in which firms adjust leverage by issuing debt with instantaneous maturity. Abel assumes i.i.d. regime shocks to profitability and shows that in response to a shock, (i) shareholders never reduce the amount of debt, and (ii) only firms that are borrowing constrained (i.e. have borrowed an amount equal to 100\% of firm value) choose to increase debt.

Finally, DeMarzo (2019) extends our model to incorporate collateral and related commitment mechanisms. Collateral is valuable because it resolves the non-exclusivity problem
underlying the leverage ratchet effect, making the value of secured debt insensitive to total leverage. Collateral lowers the cost of capital, and optimal leverage jumps discretely whenever new collateralizable assets are acquired. The model thus reconciles the empirical evidence regarding the persistence and slow adjustment of capital structure with the strong predictive power of collateral.

Our paper proceeds as follows. In Section 2 we introduce a general continuous-time model of the firm and develop our methodology for solving for an equilibrium in which shareholders adjust debt continuously. Section 3 applies our general results to the special case when cash flows are lognormal with possible jumps and derives a closed-form solution for security prices and debt issuance. We also consider the case of positive recovery and alternative default regimes with pari passu debt. Section 4 analyzes debt dynamics and shows that the firm gradually adjusts leverage towards a target level. We then evaluate the firm’s choice of debt maturity and on the share price and social welfare, and also relate our results to empirical puzzles regarding low leverage firms. Section 5 extends the model to include agency costs of investment, and Section 6 concludes.

2. A General Model

We consider a firm whose cash flows follow a general jump-diffusion process that encompasses typical settings used in the literature, and include both corporate taxes and bankruptcy costs as in a standard trade-off model. We depart from the existing literature by assuming that the firm cannot commit to its future capital structure choices, but instead is free to issue or repurchase debt at any time to maximize the current share price. We analyze the equilibrium no-commitment leverage policy in this setting.

In general, the optimal leverage policy depends on equilibrium debt prices, and debt prices depend on the firm’s anticipated future leverage choices, which determine the likelihood of default. Although this interdependence complicates the determination of an equilibrium, we show conditions for which we can construct and characterize the equilibrium leverage policy directly, and demonstrate that the rate of debt issuance is determined by the ratio of the tax benefits from new debt to its liquidity, or price sensitivity to new issues. Surprisingly, despite the issuance of new debt to exploit available tax shields, we show that shareholders do not benefit from this
activity -- the equilibrium share price is the same as if the firm committed not to issue any new debt in the future.

2.1. The Firm and Its Securities

All agents are risk neutral with an exogenous discount rate of \( r > 0 \).\(^6\) The firm’s assets-in-place generate operating cash flow (EBIT) at the rate of \( Y_t \), which evolves according to

\[
dY_t = \mu(Y_t)dt + \sigma(Y_t)dZ_t + \zeta(Y_t^-)dN_t, \tag{1}
\]

where the drift \( \mu(Y_t) \) and the volatility \( \sigma(Y_t) \) are general functions that satisfy standard regularity conditions; \( dZ_t \) is the increment of a standard Brownian motion; \( dN_t \) is an independent Poisson increment with intensity \( \lambda(Y_t) > 0 \); and \( \zeta(Y_t^-) \) is the jump size given the Poisson event.\(^7\)

Denote by \( F_t \) the aggregate face value of outstanding debt. The debt has a constant coupon rate of \( c > 0 \), so that over \([t, t + dt]\) debt holders receive coupon payments of \( cF_t dt \).\(^8\) The firm pays corporate taxes equal to \( \pi(Y_t - cF_t)dt \), where \( \pi(\cdot) \) is a strictly increasing\(^9\) function of the firm’s profit net of interest. When the marginal tax rate is positive (\( \pi' > 0 \), with “prime” indicating derivative), the net after-tax cost to the firm of the marginal coupon payment is \( 1 - \pi' \), reflecting the debt tax shield subsidy.

For simplicity, we assume that debt takes the form of exponentially maturing coupon bonds with a constant amortization rate \( \xi > 0 \). More specifically, each instant there are \( \xi F_t dt \) units of required principal repayments from maturing bonds, corresponding to an average bond maturity of \( 1/\xi \). Debt retirement in this fashion is similar to a sinking fund that continuously buys back debt

\(^6\) Alternatively, we can interpret the model as written under a fixed risk-neutral measure that is independent of the firm’s capital structure decision.
\(^7\) We have simplified notation by assuming the jump size \( \zeta(Y_t^-) \) conditional on cash flow \( Y_t \) is deterministic. We can easily generalize the model to allow a random jump size \( \tilde{\zeta}(Y_t^-) \), as long as the law of \( \tilde{\zeta}(Y_t^-) \) depends on \( Y_t \) only.
\(^8\) The coupon rate \( c \) is exogenously given in our model, so newly issued debt might not be issued at par. In practice, there may be limits/adjustments to the tax deductibility of the coupon if it is far from the par coupon rate. For simplicity, we ignore the tax consequences of non-par debt issuance for this paper.
\(^9\) Throughout, we use the terms increasing and decreasing in the weak sense, and add strictly as appropriate.
at par. Thus, combining interest and principal, equity holders are required to pay debt holders a total flow payment of \((c + \xi) F_t dt\) in order to avoid default.

In the main analysis, we assume investors recover zero value from the assets-in-place when equity holders default. The key implication of this assumption, which will simplify our analysis, is that debt seniority becomes irrelevant. Because there are no claims to divide in default, old debt holders are not directly diluted by new debt holders, where “direct dilution” refers to a decrease in the share of any bankruptcy proceeds going to prior creditors.

We make the zero recovery value assumption to emphasize that our results are not driven by the direct dilution that arises when issuing pari-passu debt (see, for example, Brunnermeier and Oehmke (2014) and Dangl and Zechner (2016)). Instead, there is an “indirect dilution” effect in our model because the value of existing debt is adversely affected by the increased likelihood of default once new debt is issued. This indirect dilution effect is a form of debt overhang: shareholders exercise their default option earlier if the firm is more indebted. It arises even if debt is fully prioritized, with new debt always strictly junior to existing debt.\(^\text{10}\) Nevertheless, in Section 3.4 we will consider the case with a positive recovery value in which the firm can issue new pari passu debt.

Equity holders control the outstanding debt \(F_t\) through an endogenous issuance/repurchase policy \(d\Gamma_t\), where \(\Gamma_t\) represents the cumulative debt issuance over time (which is a right-continuous-left-limit process and measurable with respect to the filtration generated by \(\{Y_s: 0 \leq s \leq t\}\)).\(^\text{11}\) Given our debt maturity assumption, the evolution of the outstanding face value of debt \(F_t\) is given by

\[
dF_t = d\Gamma_t - \xi F_t dt .
\]

\(^\text{10}\) Indeed, our qualitative results still hold with a positive recovery rate when new debt must be junior to existing claims. Extending the model in this way adds significant complexity, however, as debt securities issued at different times have distinct prices. In contrast, given zero recovery or pari passu debt, all debt is identical independent of the timing of issuance.

\(^\text{11}\) To rule out Ponzi schemes in which the firm avoids default by perpetually rolling over all debt, we must impose some upper bound \(F_t \leq \bar{F}(Y)\) on debt, where the bound \(\bar{F}(Y)\) exceeds the pre-tax unlevered value of the firm. This constraint will not bind in equilibrium and plays no role in the analysis.
Thus, the face value of debt will grow if the rate of issuance more than offsets the contractual retirement rate. To highlight the economic forces at play, and in contrast to much of the literature, we assume zero transaction costs in issuing or repurchasing debt.\textsuperscript{12}

Given the equity holders’ expected issuance/repurchase policy, debt holders price the newly issued or repurchased debt in a competitive market. Denote by $p_t$ the endogenous debt price per unit of promised face value. The debt price $p_t$ will reflect the information available up to date $t$, including the current debt issuance $d\Gamma_t$, and hence incorporates the price impact of new borrowing. Importantly, in equilibrium, $p_t$ also reflects creditors’ expectations regarding future leverage decisions. Then over $[t, t+dt]$ the net cash flows to equity holders are equal to

\[
\left( Y_t - \pi(Y_t - cF_t) - (c + \xi)F_t \right) dt + p_t d\Gamma_t. \tag{3}
\]

The firm continues to operate until the operating cash flow $Y_t$ drops low enough, relative to the outstanding debt level $F_t$, that equity holders find it optimal to default on their contractual obligation to debtholders. As in Leland (1994, 1998), shareholders cannot commit to a certain default policy, but instead default strategically. Again, for now we assume all investors receive zero cash flows post-default and consider alternative default payoffs in Section 3.4.

\section*{2.2. Smooth Equilibrium}

We focus on Markov-perfect equilibria (MPE) in which the two payoff-relevant state variables are the firm’s \textit{exogenous} operating cash flow $Y_t$, and the outstanding aggregate debt face value $F_t$, which is an \textit{endogenous} state variable. We will analyze value function $V(Y_t, F_t)$ for equity and the debt price $p(Y_t, F_t)$. Denote by $\tau_b$ the equilibrium default time; that is, the first time that the...
state \((Y,F_t)\) falls into the endogenous default region. We assume, for now, that all investors receive a payoff of zero if the firm defaults.

We begin by analyzing some general properties that must hold for any equilibrium. In particular, we show that shareholders never choose to repurchase debt, even if leverage is excessive from the perspective of total firm value. This result is a consequence of leverage ratchet effect (see Admati et al. (2018)). In addition, the debt price is decreasing in the amount of debt outstanding, so that debt issuance negatively impacts the debt price.

**Proposition 1 (Leverage Ratchet and Price Impact).** In any Markov-perfect equilibrium in \((Y,F)\), the firm never repurchases debt, and thus the issuance policy \(\Gamma_t\) is a monotonically increasing process. The equity value function \(V(Y,F)\) is convex and decreasing in \(F\), with the debt price as a subgradient:

\[
p(Y,F) \in -\partial_F V(Y,F).
\]

Hence, the debt price is decreasing in \(F\).

**Proof.** To see the convexity of \(V(Y,F)\) in \(F\), note that given any debt level \(F'\), equity holders have the option to adjust debt to \(F\) by issuing \(F - F'\) (buying back if this quantity is negative). Therefore, the value of the firm given \(F'\) must be at least as high as the value that equity would obtain by changing the debt level to \(F\):

\[
V(Y,F') \geq V(Y,F) + p(Y,F)(F - F') = V(Y,F) - p(Y,F)(F' - F),
\]

establishing that the (negative) debt price is a subgradient of \(V\). As a result, \(V\) is convex in \(F\) and debt price is weakly decreasing in \(F\). Because \(p \geq 0\), \(V\) is also decreasing in \(F\).

To see why shareholders would not benefit from a debt buyback, consider postponing a planned buyback by \(dt\). Shareholders would save cost of the buyback (the debt price) today, but then continue to pay the after-tax coupons and principal until they repurchase the debt at time \(t + dt\). Because the debt price today is equal to the present value of the pre-tax coupons, principal payments, and future debt price, shareholders profit from the delay by the amount of the tax shield. In addition, if they delay repurchasing the debt, they also maintain the option to default rather than repurchase. See the appendix for a formal proof.
Note that the negative impact of debt issuance on the debt price (i.e., the debt is traded at \( p(Y, F) \) with \( F \) being the post-trade debt obligation) will deter shareholders from issuing a large amount of debt at once. Indeed, if the value function is strictly convex (hence the debt price is strictly decreasing) in \( F \), it will be optimal for the firm to adjust debt in a continuous manner.

**Lemma (Continuous Adjustment).** If the equity value function \( V(Y, F) \) is strictly convex in \( F \), then the debt price is strictly decreasing in \( F \) and the optimal issuance policy \( \Gamma_t \) is continuous in \( t \).

**Proof.** With strict concavity, the inequality in (4) becomes strict. Hence, any discrete issuance with \( |F - F'| > 0 \) would be suboptimal for shareholders. ■

This result motivates us to consider a special class of equilibria in which equity holders find it optimal to adjust the firm’s outstanding debt smoothly, with \( d\Gamma_t = G_t dt \), where \( G_t \) specifies the rate of issuance at date \( t \). From now on, we call this equilibrium a “smooth” equilibrium, and call \( G_t \) the equity holders’ issuance policy (which must be nonnegative by Proposition 1).

**Definition (Smooth Equilibrium).** A smooth equilibrium is a Markov-perfect equilibrium in which the issuance policy is given by \( d\Gamma_t = G_t dt \) for some process \( G_t \).

In the remainder of this section, we develop a methodology to construct and characterize smooth equilibria. Later, in Section 3, we consider a specific setting and show conditions which rule out the existence of non-smooth equilibria.

### 2.3. Security Valuation

In a smooth equilibrium, given the debt price \( p(Y, F) \), the firm’s issuance policy \( G \) and default time \( \tau_b \) maximize the market value of equity:

\[
V(Y, F) = \max_{\tau_b, G} E_i \left[ \int_{\tau}^{\tau_b} e^{-r(s-t)} \left[ Y_s - \pi(Y_s - cF_s) - (c + \xi)F_s + G_s p_s \right] ds \right] |Y_i = Y, F_i = F].
\] (5)

---

13 Technically, there is also the possibility that the issuance policy might include a singular component, so that the sample path of \( \Gamma \) is continuous, but not absolutely continuous. We rule out such policies in our uniqueness proof in Section 3.
Because debtholders receive both coupon and principal payments until the firm defaults, and the firm recovery upon default is assumed to be zero, the equilibrium market price of debt must satisfy

\[
p(Y,F) = E[\int_t^T e^{-(r+\xi)(t-s)}(c+\xi)ds|Y_t=Y,F_t=F],
\]

where the expectation in (6) is under the evolution of \(F\) implied by \(G\).

**An Optimality Condition**

Recall that we are interested in an equilibrium when there is no commitment by equity holders to future leverage policies. Thus, at any point in time, the issuance policy \(G_s\) for \(s > t\) has to be optimal in solving the equity holders’ instantaneous maximization problem at time \(s\), given equity’s value function and equilibrium debt prices.

In this section we consider the necessary and sufficient conditions for the optimality of the debt issuance policy \(G_t\). The Hamilton-Jacobi-Bellman (HJB) equation for equity holders is

\[
rV(Y,F) = \max_G \left[ \frac{Y - \pi(Y-cF)}{\text{after-tax cash flow}} - \frac{(c+\xi)F}{\text{coupon & principal payment}} + \frac{Gp(Y,F)}{\text{new debt issuance}} + \frac{(G-\xi F)V_F(Y,F)}{\text{evolution of } dF} \right] + \frac{1}{2} \sigma(Y)^2 V_{YY}(Y,F) + \lambda(Y)\left[V(Y+\xi(Y))-V(Y)\right]
\]

In the first line, the objective is linear in \(G\) with a coefficient of \(p(Y,F) + V_F(Y,F)\), which represents the (endogenous) marginal benefit of the revenue from a debt sale net of the marginal cost of the future debt burden on shareholders. If equity holders find it optimal to adjust debt smoothly, then it must be that this coefficient equals zero, or equivalently,\(^{14}\)

\[
p(Y,F) = -V_F(Y,F).
\]

\(^{14}\) While (7) implies (8) in the non-default region, it is also true in default, as for defaulted firm the debt price \(p = 0\) and \(V(Y,F) = V_F(Y,F) = 0\). (We extend the model to the case with positive recovery in Section 3.4.) Also, the debt price for \(F = 0\) is relevant only if the firm were to buyback all of its debt (which is off-equilibrium according to \textbf{PROPOSITION 1}), and hence setting \(p(Y,0) \geq -V_F(Y,0)\) is sufficient for optimality.
No Trade Valuation

The first-order condition (8) is a necessary condition for a smooth equilibrium. While straightforward, it has deep implications for the equilibrium value of equity in any smooth equilibrium. Plugging condition (8) into the equity HJB equation (7), we have the following revised HJB equation for equity:

\[
 rV(Y,F) = Y - \pi(Y - cF) - (c + \xi)F + \mu(Y)V_Y(Y,F) - \xi F V_F(Y,F) \\
+ \frac{1}{2} \sigma(Y)^2 V_{YY}(Y,F) + \lambda(Y) \left[ V(Y + \xi(Y),F) - V(Y,F) \right].
\] (9)

This equation says that in the no-commitment equilibrium, the equity value can be solved as if there is no debt adjustment \((G \equiv 0)\), except for the natural retirement at rate \(\xi\).

Note that when \(G \equiv 0\), the value of equity is independent of the debt price, and thus is the solution to a standard optimal stopping problem in which \(Y_t\) follows (1) and \(dF_t = -\xi F_t dt\). These problems are well studied in the literature, as are conditions ensuring the uniqueness and smoothness of the solution.\(^{15}\) We denote the corresponding “no trade” equity value function by \(V^0(Y,F)\); here, “no trade” simply means equity holders do not participate in the debt market. In other words, \(V^0\) satisfies (9) and is the solution to (5) when \(G\) is constrained to be zero. The preceding argument then implies that the equity value must equal this no-trade value in any smooth equilibrium:

**Proposition 2 (No-Trade Equity Valuation).** In any smooth equilibrium, the value of equity is equal to the no-trade valuation: \(V(Y,F) = V^0(Y,F)\).

**Proof.** Because equity holders can always choose not to trade, \(V(Y,F) \geq V^0(Y,F)\) for any equilibrium. Next, given any smooth equilibrium with value function \(V\), (7) and (8) imply (9), so that setting the issuance policy to \(G = 0\) does not change the equity value \(V\). Hence, the value \(V\) under this equilibrium could be obtained with no trade. But because \(V^0\) is the optimal value with

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\(^{15}\) See, for example, Oksendal (2013) Chapter 10, which analyzes a case with a finite-dimensional diffusion. Recently, Ishikawa (2011) offers an analysis covering the case of jump-diffusion processes. We assume that the appropriate technical conditions hold throughout the paper.
no trade (with a potentially superior default policy), \( V(Y,F) \leq V^0(Y,F) \). Combining both we have
\[ V(Y,F) = V^0(Y,F) \]

Intuitively, because equity holders gain no marginal surplus from adjusting the debt level, their equilibrium payoff must be the same as if they never issue/repurchase any debt. This result, while perhaps striking at first, is analogous to the Coase (1972) conjecture for durable goods monopoly -- the firm is a monopolist issuer of its own debt. When a monopolist is unable to commit to restricting its future sales, it cannot resist the temptation to trade aggressively, so much so that price falls to marginal cost and any surplus from trading gets dissipated in equilibrium.\(^{16}\)

**Optimal Debt Issuance**

**PROPOSITION 2** implies that if a smooth equilibrium exists, we may compute the equity value function as if there is no further trade and the firm gradually retires its existing debt. Given the equity value \( V \), we can then invoke the first-order condition in (8) to obtain the equilibrium debt price \( p(Y,F) = -V(Y,F) \).

The question remains, however, whether this debt price is consistent with a smooth issuance policy \( G \) that is also optimal for shareholders. Note that \( G = 0 \) cannot be an equilibrium, as in that case the debt price would exceed its marginal cost to shareholders \( (p > -V) \) due to the interest tax shield, and so shareholders would find \( G > 0 \) optimal. But as the rate of issuance increases, the likelihood of default will increase and the price of debt will fall to the point that (8) holds.

To determine the equilibrium issuance policy, which we denote by \( G^* \), we see from (6) the debt price must satisfy the standard HJB equation,
\[
rp(Y,F) = \text{coupon} \left( 1 - p(Y,F) \right) + \text{debt retirement} \left( G^* - \xi F \right) p(Y,F) + \text{evolution of debt } dF \left[ \mu(Y)p_Y(Y,F) + \frac{1}{2} \sigma(Y)^2 p_{YY}(Y,F) + \lambda(Y) \left[ p(Y + \zeta(Y,F) - p(Y,F) \right) \right].
\]

\(^{16}\) A closely related result appears in DeMarzo and Urošević (2006) in a model of trade by a large shareholder trading off diversification benefits and price impact due to reduced incentives. In equilibrium, share prices are identical to those implied by a model with no trade. Similarly, the monopolist buyer in Daley and Green (2018) who cannot commit to his/her future strategy gains nothing from her ability to screen buyers over time.
Next, starting with the HJB equation (9) for \( V(Y,F) \), if we differentiate by \( F \) and use the optimality condition \( p = -V_F \), we obtain
\[
-rp(Y,F) = \pi'(Y-cF)c - (c + \xi) + \xi p(Y,F) + \xi Fp_F(Y,F) \\
-\mu(Y)p_Y(Y,F) - \frac{1}{2} \sigma^2(Y)p_{YY}(Y,F) + \lambda(Y) \left[ -p(Y + \zeta(Y),F) + p(Y,F) \right].
\] (11)

Although equation (11) is written in terms of the debt price \( p \), we emphasize that it follows mechanically from the equity valuation equation (9), together with the first-order condition (8) for the optimal issuance policy. Finally, adding (11) to (10), we obtain the following result on how to construct a smooth equilibrium and characterize the equilibrium debt issuance policy:

**Proposition 3 (Equilibrium Construction and Optimal Issuance).** Suppose the no-trade value \( V^0(Y,F) \) is continuously differentiable and strictly convex in \( F \) (outside the default region).\(^{17}\) Then \( V = V^0 \) is the unique smooth equilibrium value function, with debt issuance policy
\[
G^*(Y,F) = \frac{\pi'(Y-cF)c - p_F(Y,F)}{V_{FF}(Y,F)}.
\] (12)

Under this policy, the debt price given by (6) satisfies \( p = -V_F \).

**Proof.** Given a smooth policy, (7) and (8) imply (11), which combined with (12) implies that \(-V_F\) satisfies (10) and, because \( p = -V_F = 0 \) at the default boundary, \( p = -V_F \) satisfies (6). Finally, the global optimality of the issuance policy follows from \( p = -V_F \) and the strict concavity of \( V \).

While this result provides sufficient conditions for the existence of a smooth equilibrium and provides its characterization, it does not rule out non-smooth equilibrium. We give conditions for the smooth equilibrium to be the unique MPE in Section 3.

\(^{17}\) We note that a simple sufficient condition for convexity in \( F \) is a constant marginal tax rate (for a more general proof including investment see the proof of Proposition 11. Differentiability should follow given appropriate regularity conditions on the earnings process \( Y \) (and holds for standard models commonly used in finance).
We can interpret the policy (12) as follows. The rate of issuance of debt is such that the rate of devaluation of the debt induced by new issuances just offsets the marginal tax benefit associated with the coupon payments:

\[-G^*(Y, F) p_F(Y, F) = \pi'(Y - cF) c.\]  (13)

Note that the debt issuance is positive as long as the marginal tax rate is greater than zero. Again, this result is consistent with the leverage ratchet effect of Admati et al. (2018) – even if the firm’s current leverage is excessive, equity holders never actively reduce debt but always have an incentive to increase debt when it provides a marginal tax benefit.

**Discrete Optimization**

In equilibrium, because \( p = -V_F \), the value of equity is the same for any smooth issuance policy \( G \). The equilibrium policy \( G^* \) is then uniquely determined so that the resulting debt price makes shareholders indifferent. In a sense, our characterization of \( G^* \) is analogous to that of a mixed strategy equilibrium in which each player is indifferent to her choice of action, yet equilibrium strategies are uniquely determined to maintain that indifference.

Shareholder indifference regarding the issuance policy is, however, an artifact of the continuous-time limit. If we were to compute the equilibrium as the limit of a discrete-time model, the optimal policy \( G^* \) would arise as the result of a strict optimization by shareholders. To see this result heuristically, suppose that the firm issues debt \( \Delta \) which is fixed over the next \( dt \) instant, and let \( p \) and \( V \) be the end-of-period debt price and equity value functions, respectively. The firm would then pay additional interest of \( cd t \Delta \), and thus its earnings would decline by \((1 - \pi'(Y - cF))\Delta cd t\) on an after-tax basis.\(^\text{18}\) Because the bonds trade for a cum-coupon price of \( cd t + p(Y, F + \Delta) \), shareholders would choose \( \Delta \) to solve:\(^\text{19}\)

\[
\max_{\Delta} -\left(1 - \pi'(Y - cF)\right)\Delta cd t + \Delta \left[ cd t + p(Y, F + \Delta)\right] + V(Y, F + \Delta)
\]  (14)

\(^{18}\) Here we are ignoring terms of order \( dt^2 \) or higher which would arise if the marginal tax rate is not locally constant.

\(^{19}\) Recall that \( p \) is the end-of-period bond price. If sold earlier it will trade for a higher price that includes the initial coupons. Also, we assume the new debt issuance occurs after the current period’s default decision and principal repayments; changing the timing would introduce terms of order \( dt^2 \) without altering the conclusion.
Equation (14) has the first-order condition
\[
\Delta = \frac{\pi' (Y - c F) c dt + (p + V_F)}{-p_F} = \frac{\pi' (Y - c F) c}{-p_F} dt ,
\]
zero by equilibrium condition (8)
\[\text{(15)}\]
which exactly coincides with (12). Hence, we can interpret \( G^* \) as the strictly optimal issuance rate when the firm has “infinitesimal” commitment power over \([t,t + dt]\) in a discrete-time setting.\(^{20}\)

### Summary

In sum, for the general model in which equity holders are free to issue or repurchase any amount of debt at the prevailing market price, one can solve for the smooth no commitment equilibrium as follows:

(i) Use (9) to solve for the equity holder’s value function \( V(Y, F) = V^0(Y, F) \) by setting \( G = 0 \), i.e. as if equity holders commit to not issue any future debt.

(ii) Set the debt price \( p(Y, F) = -V_F(Y, F) \).

(iii) Check for global optimality by verifying that, up to the point of default, the debt price \( p(Y, F) \) is strictly decreasing in \( F \) (equivalently, \( V \) is strictly convex in \( F \)).

(iv) Given \( p(Y, F) \) solve for the equilibrium issuance policy \( G^*(Y, F) \) from (12).

In the remainder of the paper we will use this methodology to analyze several standard settings and consider the consequence for debt valuation and leverage dynamics. We will also show conditions ruling out any non-smooth equilibria.

### 3. An Explicit Solution

We now apply the general methodology developed in the previous section to the widely used framework of a lognormal cash flow process and derive an explicit solution.\(^{21}\) The results from

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\(^{20}\) See also DeMarzo (2019) for additional discussion of the convergence from discrete to continuous time, and why this convergence causes the gains from trade to vanish.

\(^{21}\) This setting is consistent with e.g. Merton (1974), Fischer et al. (1989), Leland (1994), Leland and Toft (1996), and follows the development of starting from cash flows rather than firm value as in Goldstein et al. (2001).
Section 2 allow us to fully characterize an equilibrium in closed form and evaluate the corresponding leverage dynamics. We further extend the model to allow for jumps to cash flows, and show that the solution is qualitatively unchanged. We also establish that only smooth equilibria exist in this setting, and thus our characterization is unique. Finally, Section 3.4 studies the case of a positive recovery value; equity holders’ ability to restructure the debt to their advantage using the threat of dilution makes the debt price more sensitive to new issuance, thereby reducing the equilibrium level of debt.

3.1. Lognormal Cash Flows

In the special case of lognormal operating cash flow, $Y_t$ follows a geometric Brownian motion:

$$\mu(Y_t) = \mu Y_t \text{ and } \sigma(Y_t) = \sigma Y_t, \text{ with } r > \mu. \quad (16)$$

To maintain homogeneity, we also assume a constant tax rate $\bar{\pi} > 0$ so that

$$\pi(Y_t - cF_t) = \bar{\pi} \cdot (Y_t - cF_t). \quad (17)$$

Given the scale invariance of the firm in this setting, the economically relevant measure of leverage is operating cash flow scaled by the outstanding face value of debt,

$$y_t \equiv Y_t / F_t, \quad (18)$$

which is proportional to the firm’s interest coverage ratio – that is, the ratio of operating income $Y_t$ to total interest expense $cF_t$ – a widely used measure of leverage and financial soundness. (An alternative, equivalent characterization is in terms of the debt-to-income ratio $f_t \equiv F_t / Y_t = 1 / y_t$.) Because all subgames with the same initial leverage $y_t$ are strategically equivalent, we will look for a MPE in this state variable, and show that it must be a smooth equilibrium.\(^\text{23}\) That is, with this restriction, the smooth equilibrium is the unique MPE.

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\(^{22}\) As with the existing literature, our model adopts an idealized version of the tax code. In practice, the debt tax shield is not strictly tied to the coupon rate, but includes an adjustment for any discount or premium at the time of issuance. In addition, tax shields may be deferred when earnings are negative.

\(^{23}\) Here we follow Maskin and Tirole (2001), who argue for defining MPE in terms of the coarsest partition such that equivalent subgames are “strategically equivalent.” A sufficient condition for strategic equivalence is that the payoffs are equivalent up to an affine transformation (as is the case here).
With this setting, the equity value function $V(Y,F)$ and debt price $p(Y,F)$ are homogeneous so that

$$V(Y,F) = V\left(\frac{Y}{F}, 1\right)F \equiv v(y)F \quad \text{and} \quad p(Y,F) = p\left(\frac{Y}{F}, 1\right) \equiv p(y). \quad (19)$$

We will solve for the (scaled) equity value function $v(y)$ and debt price $p(y)$ in closed form.

Given the evolution of our state variables $Y_t$ and $F_t$:

$$dY_t = \mu Y_t dt + \sigma Y_t dZ_t, \quad \text{and} \quad dF_t = (G_t - \xi F_t) dt, \quad (20)$$

the scaled cash-flows evolve as

$$\frac{dy_t}{y_t} = (\mu + \xi - g_t) dt + \sigma dZ_t, \quad \text{where} \quad g_t \equiv \frac{G_t}{F_t}. \quad (21)$$

As (21) shows, because the debt $F_t$ grows in a locally deterministic way, the scaled cash flow grows with the same volatility as total cash flow. The growth rate, however, is increased by the rate of debt amortization rate, $\xi$, net of the endogenous issuance rate, $g_t = G_t / F_t$. The higher the rate of debt issuance, the slower the growth rate of the scaled cash flow.

When the scaled cash flow $y_t$ falls below some endogenous default boundary $y^*_b$, equity holders are no longer willing to service the debt, and therefore choose to strategically default. In that event, we assume for now that both equity and debt holders receive zero liquidation value.

### 3.2. Model Solution

Recall from Section 2 that we can solve for the equilibrium equity value as if $g_t = 0$ and hence equity holders do not actively adjust the firm’s debt, even though they will do so in equilibrium. Using the fact that

$$V_y(Y,F) = v'(y), \quad V_{yy}(Y,F) = v(y) - y v'(y), \quad \text{and} \quad FV_{yy} = v''(y), \quad (22)$$

we can rewrite (9) with lognormal cash flows in terms of scaled cash-flow $y$ as follows:
\[(r + \xi)v(y) = (y - c - \xi) - \bar{\pi}(y - c) + (\mu + \xi) y v'(y) + \frac{1}{2} \sigma^2 y^2 v''(y). \tag{23}\]

Note that if the firm could not default, then the cash flows and debt payments could be evaluated as growing perpetuities. Thus, the “no default” value of equity would be

\[\bar{v}(y) = \frac{y(1 - \bar{\pi})}{r - \mu} + \frac{\pi c}{r + \xi} - \frac{c + \xi}{r + \xi} \equiv \phi y - \rho, \tag{24}\]

where \(\phi \equiv \frac{1 - \bar{\pi}}{r - \mu}\) is the unlevered valuation multiple for the firm, and \(\rho \equiv \frac{c(1 - \bar{\pi}) + \xi}{r + \xi}\) is the after-tax cost to the firm of a riskless bond.\(^{24}\) To compute the no trade equity value, we must add to \(\bar{v}\) the value of the default option. The next result characterizes the resulting value function, and establishes that this equilibrium is unique (smooth or otherwise).

**Proposition 4 (Equilibrium with Lognormal Cash Flows).** Given lognormal cash flows with constant tax rate \(\bar{\pi}\), let

\[\gamma \equiv \frac{(\mu + \xi - 0.5 \sigma^2) + \sqrt{(\mu + \xi - 0.5 \sigma^2)^2 + 2 \sigma^2 (r + \xi)}}{\sigma^2} > 0. \tag{25}\]

Then the unique Markov-perfect equilibrium in \(y\) is the smooth equilibrium, with the equity value function and optimal default boundary given by

\[v(y) = \phi y - \rho - \left(\frac{y}{y_b}\right)^{-\gamma} (\phi y_b - \rho) \text{ and } y_b = \frac{\gamma}{1 + \gamma} \phi. \tag{26}\]

**Proof.** Given shareholders’ option to default and receive zero, the no-trade value function equals

\[v(y) = \bar{v}(y) + E^0 \left[ e^{-(r + \xi) y} \right] (0 - \bar{v}(y_b)), \tag{27}\]

where \(E^0\) is the expectation given \(g \equiv 0\). The discount factor \(h(y) \equiv E^0 \left[ e^{-(r + \xi) y} \right]\) must solve the homogeneous version of (23),

\(^{24}\) The value \(\bar{v}\) is a particular solution to (23) that ignores the default boundary condition.
\[(r + \xi)h(y) = (\mu + \xi)y'h(y) + \frac{1}{2} \sigma^2 y^2 h''(y),\]

with boundary conditions \( h(y_b) = 1 \) and \( h(\infty) = 0 \), which has solution \( h(y) = \left(\frac{y}{y_b}\right)^{-\gamma} \). The optimal default boundary \( y_b \) maximizes the value of the default option and is determined by smooth-pasting, \( v'(y_b) = 0 \). Because \( v \) is smooth and strictly convex (prior to default), PROPOSITION 3 implies that \( v \) is the equity value function in the smooth equilibrium with trade. We prove in the appendix that there does not exist any non-smooth MPE in \( y \). □

PROPOSITION 4 establishes the unique equilibrium value function within the class of MPE given the firm’s interest coverage ratio (or debt-income ratio) measured by \( y \). While it is possible to construct other, non-smooth equilibria if we allow the firm and investors to condition on non-strategically relevant variables (such as the firm’s absolute size, or age, etc.), our equilibrium conforms with standard metrics used in practice to evaluate leverage and credit spreads.\(^{25}\)

Having solved for the value of equity, recall that we can determine the debt price from the first-order conditions (8). Using (22) and (26), we have

\[
p(y) = -V_r = yv'(y) - v(y) = \rho \left( 1 - \left(\frac{y}{y_b}\right)^{-\gamma} \right). \tag{28}
\]

Note that the debt price is strictly increasing in \( y \), and therefore strictly decreasing in \( F \), as required in PROPOSITION 3, which we can now apply to derive the equilibrium debt issuance policy. Note that the rate of debt issuance \( g^*(y) \) is strictly positive, and increasing in the scaled cash flow \( y \).\(^{26}\)

\(^{25}\) Including additional state variables allows investors to “punish” the firm discontinuously – via a discrete jump in credit spreads that depends on variables other than leverage – for even minor deviations from a proposed equilibrium path. Restricting debt prices to be continuous in firm leverage would be an alternative means to rule out such equilibria. See Maskin and Tirole (2001) for a formalization of the idea that MPE embody the principle that “minor causes should have minor effects.” (See also fn. 23 as well as our concluding comments.)

\(^{26}\) Note that \( g^* \) represents the issuance rate as a proportion of the current debt level \( F \); that is, total issuance is \( G^* = Fg^*(y) \). And although \( g^*(y) \) approaches infinity as \( F \rightarrow 0 \), in Section 4.1 we will derive the debt dynamics explicitly starting from \( F = 0 \) and show that under the optimal policy the firm’s outstanding debt follows a continuous sample path with no jumps.
**Proposition 5 (Equilibrium Debt Issuance).** Given lognormal cash flows with constant tax rate \( \pi \), the no commitment debt price is given by (28), and the equilibrium rate of debt issuance is

\[
g^*(y) = \frac{G^*}{F} = \frac{\pi c}{-Fp_F(Y, F)} = \frac{\pi c}{y p'(y)} = \frac{\pi c}{y^2 y'(y)} = \frac{\pi c}{\rho y'(y)}.
\]  

(29)

Thus, with lognormal cash flows, we can fully characterize equilibrium debt dynamics and security pricing in closed form. The equity value equals the value without future trade, implying that shareholders do not benefit from their ability to issue debt in the future. Without commitment, creditors anticipate future debt issuance at the rate given by (29), which depresses the current debt price. Indeed, as the following result demonstrates, the debt price falls by the value of the interest tax shield.

**Proposition 6 (Commitment vs. No Commitment Debt Price).** Let \( p^0(y) \) be the debt price if the firm committed not to issue future debt (\( g = 0 \)). Then

\[
p^0(y) = p(y) + \frac{\pi c}{r + \xi} \left[ 1 - \left( \frac{y}{y_h} \right)^{\gamma} \right].
\]

(30)

**Proof:** Because the equity value is unchanged, so is the default boundary \( y_h \). Thus,

\[
p^0(y) = \frac{c + \xi}{r + \xi} 
- E^0 \left[ e^{-(r + \xi) s} \right] \frac{c + \xi}{r + \xi} = \frac{c + \xi}{r + \xi} \left[ 1 - \left( \frac{y}{y_h} \right)^{-\gamma} \right]
\]

The result follows from comparison with (28).

Based on the equilibrium values for both equity and debt, total firm value (or total enterprise value, TEV) can be expressed as a multiple of the firm’s cash flow (i.e. TEV to EBIT) as

\[
\frac{V(Y, F) + p(Y, F)F}{Y} = \frac{v(y) + p(y)}{y} = v'(y) = \phi \left[ 1 - \left( \frac{y}{y_h} \right)^{-\gamma-1} \right],
\]

(31)
where the second equality follows from the equilibrium condition for the debt price, and the last equality uses the expression of $y^*_b$ in **Proposition 4**.

Note that the firm’s TEV multiple is strictly increasing with the scaled cash flow $y$, and thus, total firm value decreases with leverage. Although there are tax benefits associated with debt, the firm issues debt sufficiently aggressively that the cost of debt rises to offset the tax benefits. In equilibrium, the tax benefits of leverage are fully dissipated by the increase in expected default costs due to continued borrowing. This result is in stark contrast to standard trade-off theory models, such as Leland (1994), in which it is optimal for the firm to issue a large block of debt immediately. In these models, the firm is able to capture a tax benefit only because of its assumed commitment not to issue additional debt.

### 3.3. Cash Flow Jumps

In the no commitment equilibrium, the firm’s debt level evolves continuously according to (29). This smooth issuance policy might be thought to depend on continuity of cash flows and asset values in the diffusion setting. In this section we extend our model to allow the firm’s cash flows to jump discontinuously, for example in response to new product development, and show that our prior solution, in which shareholders issue debt smoothly, is essentially unchanged.

Consider a jump-diffusion model in which cash flows occasionally jump from $Y_t$ to $\theta Y_t$ for some constant $\theta > 1$. Specifically,

\[ dY_t = \mu Y_t dt + \sigma Y_t dZ_t + (\theta - 1) Y_t dN_t, \tag{32} \]

where $dN_t$ is a Poisson process with constant intensity $\lambda > 0$.\(^{27}\) In this extension, due to upward jumps, the effective expected asset growth rate becomes

\[ \mu \equiv \hat{\mu} + \lambda (\theta - 1), \tag{33} \]

and we continue to assume $\mu < r$ to ensure that the unlevered firm value is bounded.

---

\(^{27}\) While we focus on upward jumps with a fixed size, allowing the upward jump to be stochastic is straightforward. Downward jumps introduce an extra complication due to jump-triggered default, in addition to diffusion-triggered default. See footnote 29 for more details.
As before, we can solve for the equity value as if shareholders commit not to issue any new debt. Because (32) still maintains scale-invariance, 

\[ V(Y, F) = F \cdot v(y) \]

continues to hold, and the HJB equation for the equity value becomes

\[ (r + \xi) v(y) = (1 - \pi)(y - c) - \xi + (\hat{\mu} + \xi) v'(y) + \frac{1}{2} \sigma^2 y^2 v''(y) + \lambda (v(\theta y) - v(y)). \]  

(34)

The last term in equation (34) captures upward jumps. The usual boundary conditions apply: When \( y \to \infty \) so leverage is negligible, default risk disappears and \( v(y) \to \bar{v}(y) \); while at the point of default, we have value-matching \( v(y_d) = 0 \) and smooth-pasting \( v'(y_d) = 0 \).

Somewhat surprisingly, even with jumps, equilibrium security prices and debt dynamics have exactly the same form we derived in the diffusion-only case, and this smooth equilibrium is the unique MPE in state variable \( y \):

**Proposition 7 (Equilibrium with Cash Flow Jumps).** Suppose cash flows evolve as a log-normal diffusion with upward jumps as in (32). Then there exists a unique MPE in \( y \), in which the equity value, debt price, and issuance policy are given by (26), (28), and (29) respectively, with \( \gamma \) the unique positive root of

\[ W(\gamma) = \lambda \theta^{\gamma} + \frac{1}{2} \sigma^2 \gamma^2 - (\hat{\mu} + \xi - \frac{1}{2} \sigma^2) \gamma - (r + \xi + \lambda) = 0. \]  

(35)

Given parameters \( (\mu, \xi, r, \sigma, \lambda, \theta) \), with \( \hat{\mu} \) defined from (33), the solution \( \gamma \) is increasing in \( (\mu, \xi, r) \) and decreasing in \( (\sigma, \lambda, \theta) \).

**Proof.** See the Appendix.

Consequently, although the firm’s profitability (i.e., cash-flow \( Y_t \)) may jump up discretely, the equilibrium debt issuance policy continues to be smooth in the sense that it remains of order \( dt \). In response to positive jumps in the firm’s profitability, shareholders increase the speed of debt issuance, but do not issue a discrete amount of debt immediately. Consequently, leverage falls
discretely before gradually mean-reverting. This property holds even if $\sigma^2 \to 0$ so that the firm’s cash flows only grow with discrete jumps.\textsuperscript{28,29}

### 3.4. Positive Recovery with Pari Passu Debt

Thus far we have assumed that in the event of default the liquidation value of the firm is zero. Under this assumption, there is no difference between junior or senior debt, which rules out any direct dilution motive for issuing debt. Existing creditors are nonetheless harmed by the issuance of new debt due to its effect on the timing of default.

In this section we restrict attention to pari passu debt and consider the possibility that the firm may have a positive liquidation value in default.\textsuperscript{30} While the firm’s current creditors might hope to recover these liquidation proceeds, shareholders have an incentive to dilute their claim by issuing new debt prior to default. Indeed, given current debt $F$, shareholders could capture the fraction $\Delta / (F + \Delta)$ of any recovery value by issuing new debt $\Delta$ with equal priority just prior to default and using the proceeds to pay a dividend. Thus, by issuing an arbitrarily large block of debt just prior to default, shareholders could obtain the entire liquidation value, leaving existing creditors with zero recovery if they have no ability to restrict issuance.\textsuperscript{31}

In practice, creditors will naturally try to block such extreme dilution by attempting to seize assets, block shareholder payouts, and disrupt operations. Shareholders, in turn, may choose to liquidate assets to fund ongoing operations, or engage in asset substitution, in order to gamble for

\begin{itemize}
  \item When the diffusion term vanishes, we assume $\mu + \xi < 0$ so that cash flows decline faster than debt matures between jumps, allowing the equilibrium to be solved for via backward induction. Absent this assumption, the firm can sustain 100\% debt financing without risking default, and the first-best can be obtained. See DeMarzo (2019) for an analysis of the case with $\sigma^2 = 0$.
  \item If we allow for negative jumps, there is an additional complication that jumps may trigger default. Nonetheless, the analysis in Chen and Kou (2009), with certain special assumptions on jump distributions, suggests that one can still solve for the equity valuation in closed-form. As long as the equity value function remains convex, the key qualitative property of smooth debt issuance policy continues to hold in general jump-diffusion models.
  \item See DeMarzo (2019) for an analysis including senior secured debt. In that case, the firm can capture tax benefits associated with its collateralized debt.
  \item In other words, there would be a complete violation of absolute priority so that equity holders receive the entire recovery value of the firm (while debt holders recover nothing). Note that this dilution can be accomplished at the moment of default – there is no need for firm issue debt for the purpose of dilution beforehand. In contrast, Dangl and Zechner (2016) consider pari passu debt with positive recovery, but constrain the rate of debt issuance. Because of this constraint, shareholders issue debt at the maximum speed possible for some period prior to default. Note also that even if dividends are restricted, shareholders could use their ability to dilute to inefficiently continue the firm and gamble for resurrection, to similar effect.
\end{itemize}
resurrection. Shareholder-creditor conflicts throughout this process are likely to sacrifice efficiency and reduce the ultimate recovery value that can be achieved.\textsuperscript{32} Often, the resolution of default or distress is a restructuring in which both creditors and shareholders retain some value. Rather than model this complexity directly, we adopt a stylized reduced-form approach which can be calibrated to empirical data to allow for a tradeoff between creditor recovery and efficiency.

Suppose there are alternative bankruptcy or restructuring regimes shareholders may adopt, indexed by $j \in J$, which differ in terms of creditors’ expected recovery rate, $\beta_j \geq 0$, and expected efficiency, measured by the fraction of the firm’s unlevered value $\alpha_j \geq 0$ that is preserved. That is, in regime $j$, creditors receive a total expected payoff of $\beta_j \rho F_j$, while shareholders capture the expected residual value of the firm net of the creditors payoff, $\alpha_j \phi Y_j - \beta_j \rho F_j$.\textsuperscript{33} We assume that one possible outcome (regime 0) is that, as is Section 3.2, shareholders and creditors refuse to agree until all surplus is destroyed ($\alpha_0 = \beta_0 = 0$).\textsuperscript{34} Alternative regimes -- such as renegotiating or restructuring the debt prior to default, or Chapter 11 versus 7 bankruptcy proceedings -- may differ in terms of their efficiency and degree of creditor protections. More generally, the parameters $(\alpha, \beta)$ capture in reduced form the consequences of some “subgame” in which shareholders and creditors adapt their behavior in response to leverage (e.g., in Section 4 we will consider underinvestment due to debt overhang).

Given the available alternatives, shareholders choose the default or restructuring regime to maximize their payoff,

$$v^B(y) = \max_{j \in J} \alpha_j \phi y - \beta_j \rho.$$  \textsuperscript{(36)}

Compared to our initial setting, shareholders now choose both the timing and the mode of default. The existence of restructuring regimes which retain some firm value provides shareholders

\textsuperscript{32} In Section 5 we consider endogenous investment/disinvestment, which introduces an additional source of shareholder-creditor conflict.

\textsuperscript{33} We normalize the face value of debt by $\rho = \frac{c(1 - \pi) + \xi}{r + \xi}$ to account for the fact that the initial debt price (even with zero leverage) may differ from par. This normalization simplifies expressions but is not otherwise consequential.

\textsuperscript{34} We have already seen that by threatening to dilute existing creditors, shareholders can drive down their recovery rate ($\beta_0$) to zero. The assumption $\alpha_0 = 0$ presumes creditors can also block payouts to shareholders.
with an additional strategic option which can enhance shareholder value and change the cash flow threshold at which default will occur.

**PROPOSITION 8 (EQUILIBRIUM WITH RECOVERY).** In the unique MPE in \( y \), shareholders will choose the default regime \( j^* \) that solves

\[
\Gamma_j = \max_{j \in J} \frac{(1 - \beta_j)^{\gamma+1}}{(1 - \alpha_j)^{\gamma}} \geq 1.
\] (37)

The equity value function and optimal default boundary are

\[
v'(y) = \phi y - \rho - \Gamma_j \left( \frac{y}{y_b} \right)^{\gamma} (\phi y_b - \rho) \quad \text{with} \quad y'_b = \frac{1 - \beta_j}{1 - \alpha_j} y_b \geq y_b.
\] (38)

The equilibrium debt price and rate of debt issuance are given by

\[
p'(y) = \rho \left( 1 - \Gamma_j \left( \frac{y}{y_b} \right)^{\gamma} \right) \quad \text{and} \quad g'(y) = \frac{\pi c}{\Gamma_j \rho y} \left( \frac{y}{y_b} \right)^{\gamma}.
\] (39)

**PROOF.** See the Appendix.

**PROPOSITION 8** demonstrates the tradeoff between efficiency and debt recovery in restructuring as a function of the parameter \( \gamma \), which depends on the firm’s volatility, growth rate, and debt maturity according to (25) (or, in the case of jumps, (35)). The endogenous parameter \( \Gamma_j \geq 1 \), which takes the value of 1 for the baseline case \( \alpha_0 = \beta_0 = 0 \), then fully characterizes the impact of this decision on the equity and debt values. The existence of restructuring regimes with \( \Gamma_j > 1 \) increases efficiency and raises the value of equity. On the other hand, the threat of dilution makes the debt price (and total firm value) more sensitive to leverage, causing the equilibrium rate of debt issuance to decline proportionally with \( \Gamma_j \).
Note that the option to restructure will raise the default threshold to \( y_b' \geq y_b \). If \( y_b' \) is high enough, the firm may restructure even before cash flows become negative.\(^{35}\) In that case, in equilibrium the firm does not require future access to equity capital.

### 4. Debt Dynamics

Now that we have solved for the equilibrium debt issuance policy and security pricing, we can analyze the implications for observed debt dynamics. Although lack of commitment leads the firm to always have a positive rate of debt issuance, the countervailing effect of debt maturity and asset growth cause leverage to mean-revert gradually towards a target. We begin by characterizing this target as well as the speed of adjustment. We then consider the implications of alternative debt maturities. While the firm’s target leverage and rate of adjustment are greatly affected by maturity, we show that shareholders are indifferent to any maturity structure for future debt issuance. Thus, similar firms which are both maximizing shareholder value may nonetheless display very different debt dynamics. We conclude by considering welfare implications of debt maturity, as well as several low-leverage puzzles.

#### 4.1. Target Leverage and Adjustment Speed

As shown in Section 3, the equilibrium debt issuance rate \( g^*(y) \) is faster rate when cash flows are high, and slows as the firm approaches default. Because the mapping is monotonic, there is a unique level of leverage \( y^* \) such that the equilibrium issuance rate will equal the rate of debt maturity:

\[
g^*(y^*) = \xi.
\]  

\(^{35}\) The firm will have positive earnings at the time of restructuring if

\[
1 - \frac{\beta}{\gamma} \geq \left(1 + \frac{1}{\gamma} \right) \left( \frac{r + \xi}{r - \mu} \right).
\]

Even if this condition fails, it may have positive cash flows once the debt proceeds \( g^*(y^*) \beta \) are included.
We can interpret \( f_\xi^* \equiv 1/y_\xi^* \) as the firm’s leverage “target”, the ratio of debt to earnings at which new issuance exactly balances the retirement of existing debt, leaving the firm’s total indebtedness unchanged. Over time, leverage will mean revert towards this target level.\(^{36}\)

Figure 2 illustrates the net rate of debt issuance, given different debt maturities and asset volatilities, as a function of the firm’s debt-to-value ratio. Shorter debt maturity increases the speed of mean reversion, but has a non-monotonic impact on the target level of leverage. Lower volatility, on the other hand, raises both the target level of leverage and the speed of adjustment.

![Figure 2: Net Debt Issuance versus Firm Leverage for Different Maturities and Volatilities](image)

Baseline Parameters: \( \mu = 2\% , \sigma = 40\% , \overline{\pi} = 30\% , c(1 - \overline{\pi}) = r = 5\% , \xi = 20\% , \lambda = 0 , \Gamma_j = 1 \)

Without commitment, the firm’s debt is path dependent, with the current level of debt equal to the firm’s cumulative past issuance net of its debt retirement. Because the issuance rate varies with the level of cash flows, this path dependence can be quite complex. Somewhat surprisingly, we can derive the evolution of the firm’s debt explicitly as a function of the firm’s initial debt position and its earnings history, as shown next.

**Proposition 9 (Debt Evolution).** Let \( f_\xi^* \equiv 1/y_\xi^* \) be the firm’s target ratio of debt to earnings. Given the debt issuance policy \( g^* \) and initial debt face value \( F_0 > 0 \), the firm’s debt on date \( t \) given the cash-flow history \( \{Y_s : 0 < s < t\} \) is

\(^{36}\) Note that if the debt is perpetual or its maturity is very long, so that \( \xi \approx 0 \), then \( y_\xi^* < y_\xi \). In that case the firm’s total indebtedness \( F_t \) will strictly increase over time (prior to default).
\[ F_t = \left[ F_0^{-\gamma} e^{-\gamma \xi t} + \gamma \xi \int_0^t e^{\gamma \xi (s-t)} (f_x^* Y_s) d\xi \right]^{1/\gamma} . \] (41)

Equivalently, for \( dt \approx 0 \),
\[ F_{t+dt} = (f_x^* Y_t \xi dt) + F_t (1 - \gamma \xi dt) . \] (42)

**Proof.** Recall from (39) and (40) that
\[ g^*(y) = \frac{\pi c}{\Gamma_j \rho \gamma} \left( \frac{y}{y_b} \right)^{\gamma} \] and \( \xi = g^*(y^*_\xi) = \frac{\pi c}{\Gamma_j \rho \gamma} \left( \frac{y^*_\xi}{y_b} \right)^{\gamma} . \]

Then (41) and (42) are equivalent to
\[ \frac{\xi}{F} = 1 \frac{\xi}{\gamma} \frac{f_x^* Y}{F} = 1 \frac{\gamma \xi (f_x^* Y)^{\gamma} - \gamma \xi F^{\gamma}}{\gamma F} = \xi \left( \frac{y^*_\xi}{y_b} \right)^{\gamma} - \xi = g^*(y) - \xi . \]

This evolution of debt matches (20) with \( G = g^*(y)F \).

Equation (41) implies that the firm’s debt today is a type of “discounted moving average” of the firm’s initial debt and a target multiple \( f_x^* \) of the firm’s intervening cash flows. As (42) makes clear, the speed of adjustment toward the target level is determined by the product \( \gamma \xi \). Intuitively, shorter debt maturity (higher \( \xi \)) implies faster repayment of debt principal, allowing leverage to shrink more quickly in the face of declining cash flows. From (35), \( \gamma \) increases with shorter maturity, higher growth, or lower volatility, making the firm more aggressive in adding leverage in response to positive cash flows news. Finally, note from (39) that the only impact of the tax rate \( \pi \) or the default regime (via \( \Gamma_j \geq 1 \)) is on the target debt-to-income level \( f_x^* \).

**Proposition 9** demonstrates that once the firm is free to adjust leverage over time, equilibrium debt dynamics depart strongly from the predictions of the standard dynamic tradeoff theory literature. In particular, the result that debt slowly but continuously adjusts towards a target

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37 Indeed, the target debt level decreases with \( \Gamma_j \), as the increase in the value of the shareholders’ default option comes at creditors’ expense, lowering the debt price. The target debt ratio \( f_x^* \) increases with the tax rate if the debt maturity is long (\( \xi \approx 0 \)). But if the debt maturity is short and tax rates are high, a tax increase may raise the default boundary \( y_b \) sufficiently to cause the target debt-to-income ratio to fall.
leverage level differs from the models in which debt levels are fixed initially (say, Leland (1994, 1998), or jump periodically due to fixed adjustment costs (say, Goldstein, Ju, and Leland, 2001). Figure 3 simulates the evolution of debt of different maturities for an initially unlevered firm. In each case in this ten-year sample path, the shocks to earnings, and therefore the unlevered value of the firm, are the same.

The top panel of Figure 3 shows that the initial impact of these alternative debt maturities on total enterprise value is slight, with differences only emerging later when leverage becomes high. The slow adjustment of total debt in the middle panel highlights the long persistence and hysteresis in debt levels. This speed of adjustment declines with the debt maturity. Finally, the bottom panel makes clear that the primary driver of market leverage in the short term is due to fluctuation in the stock price.

These features resemble the evolution of debt most commonly observed in practice. Our model hence provides a theoretical foundation for partial adjustment models (Jalilvand and Harris, 1984; Leary and Roberts, 2005; etc) that are widely used in the empirical capital structure literature. Gradual “under-adjustment” also leads to the well-documented negative relation between leverage a profitability, even for frequent issuers of debt (see Frank and Goyal (2014) and Eckbo and Kisser (2018)).

Consider, for example the case of five-year debt. While the target level of market leverage is 42%, the firm does not issue this amount immediately. Debt increases quickly at first, but is soon outpaced by increases in firm value. When firm value declines after year four, leverage overshoots the target, as the firm can reduce leverage no faster than its debt matures. On the other hand, with two-year debt, the firm adjusts quickly to a target leverage ratio of 53%, and increase leverage significantly as firm value grows. But when firm value declines sharply in year five, the firm is unable to reduce leverage quickly enough and the firm defaults. Finally, with twenty-year debt, although the target leverage ratio is 60%, the speed of adjustment is very slow, and the firm gradually increases debt over the entire ten-year period.

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38 For example, Welch (2004) reports that market leverage changes primarily due to stock price fluctuations, and Baker and Wurgler (2002) document that the firm’s current leverage depends on its equity price over the past decade or more. Graham and Leary (2011) survey a number of studies which suggest an annual speed of adjustment towards a target leverage ratio of between 10% and 40%. Frank and Shen (2019) report that allowing for heterogeneity in the determination of leverage targets leads to much faster estimates of the mean-reverting adjustment speed.
Figure 3: Simulation of Debt Evolution
Evolution of firm value and debt level for different debt maturities. Top panel shows TEV, and middle panel shows debt level, versus unlevered value $V^U = \phi Y$. Bottom panel shows market leverage (with legend indicating target).
Note that 2-year debt defaults before year 5.
$(\mu = 2\%, \sigma = 40\%, \pi = 30\%, c(1 - \pi) = r = 5\%, \xi = 5\%, 10\%, 20\%, 50\%)$

### 4.2. Debt Maturity Indifference

Our model considers a constant maturity structure in which all debt has an expected maturity of $1/\xi$. This assumption is common in much of the dynamic capital structure literature which treats the debt maturity structure as a parameter. While it is beyond the scope of this paper to allow the firm full flexibility over maturity structures, we show that absent commitment, shareholders are indifferent to the maturity structure of the firm’s future debt issuance. While different maturity choices will lead to different future leverage levels, any increase in tax benefits is offset by an increase in default costs, and the firm’s current share price is unaffected.

Recall from **PROPOSITION 2** that we can compute the current value of equity as though the firm will not issue or repurchase debt in the future, and just repays its existing debt as it matures. This result immediately implies that for an initially unlevered firm ($F_0 = 0$), firm value does not depend on the choice of debt maturity structure $\xi$. This irrelevance result can be generalized further. Consider the following thought experiment, in which equity – facing the current cash flows and debt structure $(Y_t, F_t, \xi)$ – has a one-time opportunity to choose an alternative maturity $\xi'$ for the firm’s future debt. That is, the firm’s existing debts continue to retire at the old speed $\xi$, but the newly issued debts are with the new maturity and hence will retire at the new speed $\xi'$. We have the following proposition.

**PROPOSITION 10 (MATURITY INDIFFERENCE).** In a no-commitment equilibrium with smooth debt issuance, the firm’s current equity value is independent of the maturity $\xi'$ of new debt.

**PROOF:** See the [Appendix].
The intuition for the proof is straightforward. We can consider the implied future liabilities from the firm’s existing debt as a modification of the cash flow process for the firm and then apply our general methodology as in Section 2. For equilibria with smooth debt issuance polices, equity holders obtain zero profit by issuing future debt, and their value will be the same as if the firm does not issue any future debt. As a result, the current equity value only depends on the maturity structure $\xi$ of existing debt, but not on the maturity structure $\xi'$ of future debt. This logic and hence the indifference result can be further generalized to a setting in which the firm is free to choose any maturity structure for its newly issued debt any time. Again, the equity value will only depend on the maturity structure of the firm’s existing debt.

The indifference result can also be seen in Figure 3. Although the debt maturity choice leads to large differences in the evolution of debt and market leverage over time, the initial enterprise value of the firm is the same, and is equal to the unlevered firm value, in all four cases.

Figure 3 also provides a potential explanation for the finding in Lemmon, Roberts, and Zender (2008), that much of the cross-sectional variation in firms’ capital structure is persistent and largely unexplained by observable characteristics. From the perspective of our model, small perturbations or frictions that may lead firms to pick different initial maturity structures will lead over time to dramatically different leverage outcomes. See also DeMarzo (2019) for further discussion of this point together with the potential role of collateral.

### 4.3. Optimal Maturity: Price Impact and Welfare

In the previous section, we demonstrated that the firm’s shareholders are indifferent to the choice of debt maturity when debt issuance is unconstrained. This indifference result regarding debt maturity runs counter to the standard intuition that shareholder-creditor conflicts are ameliorated with short-term debt. While this intuition would hold if shareholders could commit ex-ante to maintain a given leverage policy, the analysis shows that without commitment this result is not correct: the use of short-term debt induces the firm to lever more aggressively, and the agency costs resulting from the leverage ratchet effect do not disappear. Indeed, as maturity shortens to zero ($\xi \to \infty$), the firm’s target leverage $y^*_\xi$ converges to the default boundary $y^*_b$, increasing the

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40 See Tserlukevich (2008) for further elaboration of this point with commitment. The flexibility offered by short-term debt is also studied in a recent paper by Geelen (2017).
firm’s tax shields while solvent but keeping the firm ever closer to default. Intuitively, as long as the opportunity to trade is sufficiently frequent relative to the maturity of the debt, leverage ratchet dynamics will emerge to reduce the gains from trade.\footnote{Given any fixed maturity $1/\xi > 0$ with continuous trading there is always the opportunity to issue new debt before the existing debt matures. The same issues arise even in a discrete time model, as long as new debt can be issued on the same date as the original debt; Bizer and DeMarzo (1992) demonstrate the agency cost associated with sequential rounds of simultaneous borrowing even in a one period model.}

In this section, we show that a benefit of short-term debt reemerges if the firm is constrained to raise a fixed amount of initial debt. But although short-term debt may be privately optimal, it may be inefficient from the perspective of social welfare.

**Constrained Borrowing**

We have assumed throughout that the firm has frictionless access to both debt and equity markets, and shown that debt issuance will occur gradually over time. Suppose instead the firm is forced to raise a fixed amount of capital using debt.\footnote{For example, suppose the firm requires capital to launch, and is restricted from using all equity financing due to governance concerns, illiquidity, or other temporary costs outside the model.} In that case, which choice of debt maturity would shareholders prefer?

As we have already shown, the convexity of the equity value function in $F$ implies that issuing a large block of debt is costly to shareholders. This cost arises because the firm faces a downward-sloping demand (i.e., $p_F < 0$) for its debt. The price sensitivity of the debt is, however, increasing with the debt’s maturity – because the initial debt issuance will not be unwound (due to the leverage ratchet effect), long maturity debt will be more subject to future dilution. We confirm this result in Figure 4, which shows the drop in total firm value (TEV) for a given initial amount borrowed as a fraction $d$ of the firm’s unlevered value:

$$d = \frac{p(Y_0 / F)F}{\phi Y_0} = \frac{p(y_0)}{\phi y_0}.$$  

Not only does short-term debt reduce the cost to firm value, it also increases the firm’s debt capacity – the maximum amount it is able to borrow – which is restricted because at a high level
of debt, the negative price impact from raising the face value more than offsets the increase in quantity.\footnote{See the NBER working paper version https://www.nber.org/papers/w22799 where we formally establish this result and show that the debt capacity approaches 100\% as $\xi \to \infty$. Of course, we have ignored other potential costs associated with short-term debt, such as rollover risk (see e.g. He and Xiong, 2012).}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure4.png}
\caption{Debt and Firm Value with Differing Maturities.}
If the firm must borrow a fixed amount, using short-term debt raises its debt capacity and reduces the cost to firm value. Parameters are $\mu = 2\%$, $\sigma = 40\%$, $\overline{\pi} = 30\%$, $c(1 - \overline{\pi}) = r = 5\%$.\end{figure}

\section*{Maturity and Welfare}

The result that shareholders are indifferent regarding debt maturity when issuance is smooth must imply that the expected tax benefits of new debt are exactly offset by the increase in default costs borne by shareholders. But from a social planner’s perspective, tax shields represent transfers, while bankruptcy costs reflect dead-weight losses which directly reduce welfare. Therefore, even if shareholders are indifferent to maturity, there will be a clear ranking in terms of expected welfare: choices that lead to higher expected tax benefits also imply higher expected default costs, and hence lower welfare.

We can use our model to calculate the expected future bankruptcy costs as a fraction of the current unlevered firm value, which we denote by $BC(y)$.

Starting with initial debt $F_0$, debt
evolves according to $dF_i = \left( g(y_i) - \xi \right) F_i dt$ and default occurs at $\tau_0$ with $Y_{\tau_0} = y_0 F_{\tau_0}$, with a loss proportional to the firm’s unlevered value at that time. Thus,

$$BC(y) = \frac{1}{\phi(y)} E \left[ e^{-\tau_0 Y_0} \left| Y_0 = y F_0, F_0 \right. \right] = \frac{y}{\phi(y)} E \left[ \exp \left( \int_0^{\tau_0} \left( g(y) - \xi - r \right) dt \right) \right] y_0 = y.$$ 

\[ \text{Figure 5. Expected bankruptcy costs } BC \text{ (as a fraction of unlevered firm value).} \]

For an initially unlevered firm ($d = 0$), expected bankruptcy costs decrease with both maturity and volatility, due to a slower issuance rate. But if the firm is constrained to borrow 40% of its unlevered value upfront ($p(y)/\phi(y) = d = 0.4$), then welfare is maximized for an intermediate debt maturity. ($r = c (1 - \pi) = 5\%, \mu = 2\%, \pi = 30\%$)

We compute $BC(y)$ for alternative debt maturities in Figure 5.\textsuperscript{44} As the figure illustrates, for an initially unlevered firm expected bankruptcy costs are higher if the debt maturity is shorter (i.e., both red curves decline with debt maturity). The intuition for this result is that, as we saw in Figure 3, with short-term debt the firm takes on debt more quickly and maintains higher leverage over time, taking greater advantage of interest tax shields but also leading to a higher risk of default. Interestingly, as the figure also shows, lower firm volatility also gives rise to greater expected bankruptcy costs (i.e., dashed lines sit below solid lines). The key to this counter-intuitive

\textsuperscript{44} We compute $BC$ numerically by defining $H(y) \equiv y \cdot BC(y)$, and noting that $H$ satisfied the ODE,

$$\left(r + \xi - g(y)\right) H(y) = \left( \mu + \xi - g(y) \right) y H'(y) + 0.5 \sigma y^2 H''(y),$$

with boundary conditions $BC(y_s) = 1$ and $BC(y) \to k$ when $y \to \infty$ for some constant $k \in (0,1)$. 

38
result is the endogenous debt issuance policy. Shareholders in a firm with lower volatility are more aggressive in levering up, so much so that we have greater expected bankruptcy costs.

The figure also reveals that if the firm is constrained to borrow an initial fraction \( d \in [0,1] \) of its unlevered value, then an intermediate debt-maturity will maximize welfare (i.e., green curves are minimized at an interior point). In this case, as we saw in Figure 4, given an initial borrowing requirement longer-term debt leads to a larger initial welfare loss even for shareholders (that is, expected bankruptcy costs exceed tax benefits). Intermediate maturity debt balances these effects by starting the firm further from default and reducing the aggressiveness with which it will issue debt in the future.

### 4.4. Low Leverage Puzzles

Two important empirical observations associated with leverage are the credit-spread puzzle and the zero-leverage puzzle. As we discuss below, the implications of our model for debt dynamics and pricing can help to resolve both of these apparent anomalies.

#### The Credit-Spread Puzzle

As Huang and Huang (2012) and others have observed, firms with low leverage often have much higher credit spreads than would be predicted by a standard structural models. In the context of our model, large credit spreads arise even for firms with low leverage because the debt is priced in anticipation of future debt issuance.

To see this effect, define the credit spread \( \delta(y) \) as the yield spread required to match the bond’s price absent default:

\[
\frac{c + \xi}{r + \delta(y) + \xi} \equiv p(y), \text{ or equivalently, } \delta(y) = \frac{c + \xi (1 - p(y))}{p(y)} - r. \tag{43}
\]

As a comparison, we can also define the credit spread \( \delta^0(y) \) that would apply in a model in which the firm commits not to issue more debt, by replacing \( p(y) \) with \( p^0(y) \) (from (30)) in the above definition.
It is easy to see that as leverage falls ($y$ increases), both credit spreads decline. But as we approach zero leverage, $y \to \infty$, the credit spread with commitment vanishes ($\delta^0(y) \to 0$), whereas in the no commitment case,

$$\lim_{y \to \infty} \delta(y) = \frac{\pi c}{\rho} > 0.$$  \hspace{1cm} (44)

In other words, even with very low leverage, we should expect significant credit spreads in our model. The reason, of course, is that even when the firm’s current debt level is very low, the future debt level is likely to be much higher given the mean-reverting leverage dynamics.

**The Zero-Leverage Puzzle**

A significant empirical puzzle for the standard tradeoff theory is that despite the tax benefits of debt, there exist a significant number of firms with zero leverage, as documented by Strebulaev and Yang (2013).

At first glance, this fact would also appear to contradict our model, in which firms issue debt repeatedly. However, our equilibrium dynamics, and the uniqueness result of Section 3, only apply in the range $y = Y / F < \infty$, and therefore $F > 0$. When $F = 0$, there are actually two possible Markov-perfect equilibrium outcomes: the firm can begin issuing debt, as in our model, or the firm can remain with zero-leverage. In either case, the payoff to shareholders is the same – they receive the unlevered value of the firm.

The zero-leverage equilibrium can exist, because without commitment the firm is unable to capture the tax shield benefits of debt. As Eq. (44) shows, the initial credit spread on the debt is sufficient to offset the tax benefits. But, this “no trade” equilibrium disappears once debt is in place. The reason is that, in order for the price of the firm’s outstanding debt to be consistent with equilibrium, the firm must be expected to continue to issue debt. This equilibrium constraint does not apply when $F = 0$, as there is no existing debt price.
5. Endogenous Investment and Debt Overhang

We now extend our model by adding an endogenous investment decision also under the control of shareholders. Including investment allows us to explore the interaction of shareholder-creditor conflicts over investment and leverage choices.

We adjust our model so that investment is efficient and leads to the same cash flow dynamics as before. But if shareholders have the option to cut investment, debt overhang will induce them to do so when leverage is high, as in Myers (1977). Anticipated agency costs lower the debt price and reduce the rate of debt issuance when the debt is low. But when debt is high, the option to cut investment delays default and ultimately raises the debt price and rate of issuance rate. In the end, the tax benefits of debt are dissipated by the combined costs of underinvestment and default, so that the share price is again equal to the no-trade value.

While these insights are general, for ease of illustration we consider a binary investment decision.\(^\text{45}\) When the firm invests optimally, taking all positive NPV projects, the cash flow process follows the same process as in the baseline model, i.e., \(dY_t = \mu Y_t dt + \sigma Y_t dZ_t\), where we now restrict attention to the case \(\mu > 0\). When shareholders decide not to invest, the drift of cash flow process drops to zero, so that \(dY_t = \sigma Y_t dZ_t\). The firm also saves the cost of investment (or earns proceeds from disinvesting) which generates additional after-tax cash flow equal to \(\kappa \mu Y_t dt\). In other words, the firm can invest \(\kappa > 0\) in capital today to permanently increase the firm’s expected profit rate by one dollar, with constant returns to scale up to a maximum growth rate of \(\mu\). We assume the investment cost \(\kappa < \phi = \frac{1-\pi}{r-\mu}\), so it is always positive NPV for an unlevered firm to invest. (The decision not to invest can also be interpreted as a decision to stop maintaining existing capital, saving on maintenance costs while sacrificing growth.)

Our model with investment thus matches the setting in Section 3 except for the fact that shareholders now have the option to cut investment. We consider an MPE with the same state variable \(y_t = Y_t / F_t\). As before, shareholders will default when leverage is too high and \(y\) drops to

\(^{45}\) See the NBER working paper version https://www.nber.org/papers/w22799 for a model with continuous investment and convex adjustment costs.
an endogenous default threshold \( y_{b}^{i} \). In addition, there will exist an endogenous investment threshold \( y_{i}^{i} \) such that if \( y \geq y_{i}^{i} \), leverage is sufficiently low so that shareholders will invest (and cash flows will evolve as in the baseline case). The new feature is that now, in the “distressed” region \( y \in (y_{b}^{i}, y_{i}^{i}) \), debt overhang is severe enough so that shareholders do not find it worthwhile to invest, but are not yet willing to default.

**Figure 6 Endogenous investment and debt issuance policies.**

We compare the extension with investment options and the baseline model by plotting equity value \( v \) (Panel A), debt price \( p \) (Panel B), enterprise value \( v + p \) (Panel C), and debt issuance policies (Panel D) for both cases. In the extension with investment options, the firm invests when \( y \geq y_{i}^{i} \), does not invest when \( y \in (y_{b}^{i}, y_{i}^{i}) \), and defaults when \( y \) hits \( y_{b}^{i} \). In the baseline model, the firm defaults earlier at \( y_{b} > y_{b}^{i} \). Parameters are

\[
 r = c(1 - \pi) = 5\%, \sigma = 10\%, \mu = 2\%, \xi = 0.01, \kappa = 22. 
\]

We solve this extension in closed-form in Appendix 7.2, where we show that the no-trade value remains strictly convex and thus can be used to characterize the smooth equilibrium. We demonstrate the interaction between investment and debt issuance in Figure 6 by comparing the
case with endogenous investment to the baseline setting in which investment is always first best. We plot the equity value $v^{inv}$, debt price $p^{inv}$, enterprise value $v^{inv} + p^{inv}$, and debt issuance rate in Panels A-D, respectively. The dashed vertical lines show the default boundaries for the two settings, whereas the solid vertical line shows the investment boundary when investment is endogenous.

Compared to the baseline case, the option to cut investment raises the equity value and lowers the default boundary ($v^{inv} > v$ and $y_b^{inv} < y_b$) as shown in Panel A. In contrast, the option to cut investment lowers the debt price and enterprise value when leverage is low, as creditors anticipate the cost of future underinvestment, but raises them when leverage is high, by delaying default, as shown in Panels B and C.

We plot debt issuance policies in both cases in Panel D. In the investment region $y \geq y_i^{inv}$, shareholders invest in both cases, but issue debt more slowly when there is the option to cut investment in the future. This slower debt issuance is a consequence of the greater sensitivity of the debt price to leverage increases when creditors anticipate that debt overhang will distort future investment. In the “distressed” region $y \in \left( y_b^{inv}, y_i^{inv} \right)$, because shareholders can cut investment to boost the firm’s current cash flow, the issuance rate declines more slowly as $y$ falls, and ultimately, close to default, the debt issuance speed without investment exceeds the baseline case. We summarize the key results below:

**Proposition 11 (Underinvestment).** The option to cut investment results in a higher equity price than in the case with fixed investment. Shareholders will cut investment when $y$ falls below $y_i^{inv} > y_b$, and default is delayed to $y_b^{inv} < y_b$. When $y \geq y_i^{inv}$, the firm invests optimally, and the value of equity satisfies

$$v'(y) = \phi y - \rho - \Gamma_y \left( \frac{y}{y_b} \right)^{-\gamma} (\phi y_b - \rho),$$

for some $\Gamma_y > 1$. In this region, the rate of debt issuance is proportionally lower, and the credit spread is proportionally higher, than in the fixed investment case.
**Proof:** See Appendix. The intuition is as follows. First, we establish that for an arbitrary investment technology, the no-trade value function remains convex, so that the characterization of the smooth equilibrium in Section 2 can be applied. For the binary investment case, we can solve in closed form for the value functions $(v^I, v^{NI})$ in the investment and no-investment regions, respectively, where the optimal boundary $y^{inv}_i$ is determined by the usual smooth pasting condition. The transition to no investment is equivalent to an alternative restructuring regime, with the parameters $(\alpha, \beta)$ determined endogenously from $v^{NI}$. We show in the appendix that $\Gamma_1 > 1$, where the inequality follows since $v^{NI}(y_b) > 0$. ■

6. Conclusions

When the firm cannot commit ex ante to future leverage choices, shareholders will adjust the level of debt to maximize the firm’s current share price. We develop a general methodology to solve for equilibrium debt dynamics in this setting, including endogenous investment. When earnings evolve as geometric Brownian motion (including possible upward jumps), we show the uniqueness of our Markov perfect equilibrium, and explicitly solve for the firm’s debt as a slowly adjusting weighted average of past earnings. The endogenous rate of debt issuance decreases with debt maturity and volatility, and decreases as the firm approaches default, so that the firm’s equilibrium leverage is ultimately mean-reverting.

Because creditors expect the firm to issue new debt in the future, credit spreads are wider in our model than in standard models with fixed debt, and remain wide even when firms are arbitrarily far from default. Lower debt prices dissipate the tax shield benefits of leverage, so that the equity value is identical to the case without no future debt issuance. This inability to capture tax benefits of leverage may provide a possible resolution for the zero-leverage puzzle (Strebulaev and Yang, 2013), as the potential tax benefit from leverage is offset by the high credit spread even for initial debt.

Finally, although shortening the maturity of debt raises the average level of leverage as well as its speed of adjustment, the increase in expected tax benefits is again offset by an increase in default costs so that there is no impact on the share price. As a result, even “instantaneous” debt does not resolve the agency problem, and equity holders have no incentive whatsoever to adjust
the firm’s debt maturity structure. Moreover, because debt adjusts gradually, similar firms may have very different leverage given their exposure to past shocks. These observations offers a potential explanation for findings such as Lemmon, Roberts, and Zender (2008) that much of the cross-sectional variation in firms’ capital structure is persistent and largely unexplained by firm characteristics.

There are many further extensions of our model worth considering. DeMarzo (2019) allows for a different risk-free rate across equity and debt markets, perhaps due to different investor level taxes, or a “moneyness” premium associated with debt. DeMarzo, He, and Tourre (2019) consider risk-averse creditors in the context of sovereign debt. As long as debt has a net funding advantage, all of the key results in our model continue to apply. The same holds true if we allow for proportional transactions costs associated with debt issuance (which can be interpreted as an additional wedge in the cost of capital across markets).

We have focused on Markov-perfect equilibrium in our analysis, in which debt pricing depends only on firm fundamentals. As in the folk-theorem literature, if we relax this constraint additional equilibria can be supported using “grim trigger” punishments in response to any deviation. Indeed, because our equilibrium produces the lowest possible equilibrium payoff for shareholders (Markov or not), all non-MPE equilibria can be supported by using our equilibrium off the equilibrium path. (See, for example, Benzoni et al. (2019) who use our results to support “commitment” to an “s-S” restructuring policy.)

Naturally, we expect that firms will try to reduce the agency costs resulting from the leverage ratchet effect and capture some of the funding advantages of debt by using alternative commitment mechanisms such as collateral or covenants that restrict future debt issuance. DeMarzo (2019) discusses these alternatives and demonstrates that collateral allows the firm to capture the funding advantages of debt because it can be exclusively promised to a single creditor. Other important commitment mechanisms in practice include regulations, restrictions on the tax-deductibility of leverage by corporations, and trading frictions. In addition, equity market imperfections may prompt the firm to actively manage its internal liquidity (cash) position (as in Hennessy and Whited, 2005; Bolton, Chen, and Wang, 2014). We leave for future work an exploration of the leverage dynamics that arise from the interaction of these additional forces with the leverage ratchet effects explored here.
7. Appendix

7.1. Remaining Proofs from the Main Text

**Proof of Proposition 7.** Note that the HJB equation (34) has the linear solution

\[ \bar{V}(y) = \phi y - \rho. \]  

(46)

The homogenous delayed differential equation

\[ (r + \xi + \lambda)f(y) = (\hat{\mu} + \xi)f'(y) + \frac{1}{2}\sigma^2 y^2 f''(y) + \lambda f(\Theta y) \]

has solutions of the form \( y^{-\hat{\gamma}} \) where \( \hat{\gamma} \) solves the characteristic equation (35). In (35), because \( W \) is convex, \( W(\infty) = W(-\infty) = \infty \), \( W(0) < -r - \xi < 0 \), and \( W(-1) = \hat{\mu} - r < 0 \), \( W \) has a unique positive real root (as well a unique negative real root \( \hat{\eta} < -1 \) that can be ruled out by the upper boundary condition). The remainder of the analysis follows exactly as in Section 3.2. (See also Section 7.1 for an alternative proof in terms of \( \frac{1}{f} F \)). For the comparative statics, note by convexity and the fact that \( \gamma \) is the largest root, \( W'(\gamma) > 0 \). Hence, for a given parameter \( x \), the sign of \( \frac{\partial}{\partial x} \gamma = -\frac{\partial}{\partial x} W(\gamma) \), where

\[ W(\gamma) = \lambda \theta^{-\gamma} + \frac{1}{2}\sigma^2 \gamma^2 - \left( \mu - \lambda (\theta - 1) + \xi - \frac{1}{2}\sigma^2 \right) \gamma - (r + \xi + \lambda) = 0. \]

So, for example, because \( \theta > 1 \) and \( \gamma > 0 \),

\[ \frac{\partial}{\partial \theta} W(\gamma) = -\gamma \lambda \theta^{-(1+\gamma)} + \lambda \gamma > 0, \]

we have that \( \gamma \) is strictly decreasing in \( \theta \).
PROOF OF PROPOSITION 8. As in the proof of PROPOSITION 4, the equity value function given no trade is given by

\[ v' (y) = \max_{\hat{y}_b} \phi y - \rho + \left( \frac{y}{\hat{y}_b} \right)^{\frac{\gamma}{\beta}} \left( v^b(\hat{y}_b) - (\phi \hat{y}_b - \rho) \right) \]

\[ = \max_{\hat{y}_b} \phi y - \rho + \left( \frac{y}{\hat{y}_b} \right)^{\frac{\gamma}{\beta}} \left( (\alpha_j \phi \hat{y}_b - \beta_j \rho) - (\phi \hat{y}_b - \rho) \right) \]

\[ = \phi y - \rho - y^{\gamma} \left[ \min_{\hat{y}_b} \hat{y}_b^{\gamma} \left( (1 - \alpha_j) \phi \hat{y}_b - (1 - \beta_j) \rho \right) \right]. \]

From the first-order condition for \( \hat{y}_b \),

\[ \hat{y}_b = \frac{(1 - \beta_j) \rho \gamma}{(1 - \alpha_j) \phi (1 + \gamma)} = \frac{1 - \beta_j}{1 - \alpha_j} y_b. \]

Hence,

\[ v' (y) = \phi y - \rho - y^{\gamma} \left[ \min_{\hat{y}_b} \hat{y}_b^{\gamma} (1 - \beta_j) (\phi y_b - \rho) \right] \]

\[ = \phi y - \rho - \left( \frac{y}{y_b} \right)^{\gamma} (\phi y_b - \rho) \left[ \max_{\hat{y}_b} \left( \frac{1 - \beta_j}{1 - \alpha_j} \right)^{\gamma} (1 - \beta_j) \right] \]

\[ = \phi y - \rho - \Gamma_j \left( \frac{y}{y_b} \right)^{\gamma} (\phi y_b - \rho). \]

The remaining results follow from PROPOSITION 3 as in Section 3.2.

PROOF OF PROPOSITION 10: The result follow from a straightforward extension of the methodology in Section 2. Let \( F = (F_o, F_n) \) be a vector showing the quantity of old and new debt, respectively. Let \( c = (c_o, c_n) \) and \( \xi = (\xi_o, \xi_n) \) be the corresponding coupon and amortization rates. Given \( (Y, F) \) the firm generates cash at rate \( u(Y, F) = Y - \pi (Y - c \cdot F) - (c + \xi) \cdot F \). Let \( p^n(Y, F) \) be the price of the new debt and \( V(Y, F) \) be the value of equity. Then the equity value function satisfies the HJB,
As before, if a smooth issuance policy is optimal it must be that \( p^n = V_{F_s} \). Thus the HJB becomes

\[
rV = u(Y,F) - \xi_n F_n V_{F_s} + \xi_o F_o V_{F_s} + \mu(Y)V_Y + \frac{1}{2} \sigma(Y)^2 V_{YY},
\]
equivalent to the case with no future trade. Note that before the firm issues new debt, \( F_n = 0 \), and hence the value of equity \( V(Y,(F_o,0)) = V(Y,F_o) \) is independent of coupon rate or maturity \((c_n, \xi_n)\) of the new debt. Finally, following the same approach as in Section 2, we can solve for the optimal \( G^* \) as before:

\[
G^* = \frac{\partial}{\partial F_n} u(Y,F) + c_n + \xi_n = \frac{\pi c_n}{-p^n}.
\]

Thus, the smooth equilibrium will exist under the same conditions as before (differentiability and strict convexity with respect to the debt face value).

### 7.2. Solving the Model in Terms of \( f \)

In some cases it is more convenient to solve the model in terms of the state variable \( f = F/Y = 1/y \), especially when considering an initially unlevered firm \((F = 0)\). Let \( \hat{v}(f) \) be the value function in terms of \( f \); that is \( \hat{v}(f) = V(1,f) = V(Y,F)/Y \). Note the given a jump, the value becomes \( V(0,Y,F)/Y = \theta \hat{v}(f/\theta) \). Hence, the HJB equation for \( \hat{v}(f) \) given no-trade is

\[
(r - \hat{\mu}) \hat{v}(f) = (1-cf) - \xi f - (\hat{\mu} + \xi) f \hat{v}(f) + \frac{1}{2} \sigma^2 f^2 \tilde{v}^*(f) + \lambda \left( \theta \hat{v} \left( \frac{f}{\theta} \right) - \hat{v}(f) \right). \tag{47}
\]

Recall we have defined \( \mu = \hat{\mu} + \lambda (\theta - 1) < r \). According to Liu (2018) Theorem 2.2, the general solution of \( \hat{v}(f) \) is the form of

\[
\hat{v}(f) = \phi f + A_1 f^\hat{\mu} + A_2 f^{-\hat{\mu}}, \tag{48}
\]
with $A_1$ and $A_2$ as coefficients to be determined. The two power functions of $f^\gamma$ and $f^{-\hat{\eta}}$ are derived as follows. The characteristic equation of the homogeneous part is

$$\hat{W}(x) = \lambda \theta^{1-x} + 0.5\sigma^2 x^2 - \left(\hat{\mu} + \xi + 0.5\sigma^2\right)x - (r - \hat{\mu} + \lambda) = 0.$$ 

Because $\hat{W}(x)$ is convex (note $\hat{W}''(x) = \lambda \theta^{1-x} (\ln \theta)^2 + \sigma^2 > 0$), $\hat{W}(-\infty) = \hat{W}(+\infty) = \infty$, and both $\hat{W}(0) = -(r - \hat{\mu}) < 0$ and $\hat{W}(1) = -\xi - r < 0$, we have two real roots $\hat{\gamma} > 1$ and $-\hat{\eta} < 0$ for the equation $\hat{W}(x) = 0$. It is easy to check that $\hat{\gamma} = 1 + \gamma$ from (35).

Because $\hat{\nu}(0)$ is bounded we must have $A_2 = 0$ and $\hat{\nu}(0) = \phi$. Then, $A_1$ is determined from the boundary condition $\hat{\nu}(f_b) = \hat{\nu}'(f_b) = 0$, which implies

$$\hat{\nu}(f) = \phi - pf - \left(\frac{f}{f_b}\right)^{\hat{\gamma}} (\phi - pf_b) \text{ with } f_b = \frac{\hat{\gamma} \phi}{\hat{\gamma} - 1}.$$

It is straightforward to check that $f_b = 1 / y_b$ and $\hat{\nu}(f) = \nu(y) / y$.

### 7.3. Equilibrium Uniqueness

We prove that the smooth equilibrium constructed in Section 3 is the unique equilibrium in the class of perfect Markov equilibria with the uni-dimensional state variable being $y = Y / F$. There are five steps in our proof strategy.

1) The equity value $\nu(y)$ is weakly convex, continuously differentiable $C^1$, and debt price function $p(y)$ is continuous and weakly increasing.

These regularity conditions satisfied by the equity value and debt price in any equilibrium; most of them are a straightforward implication of Proposition 1, but utilize the strength of having only a single state variable $y$ to rule out discontinuities in the debt price.

2) There will be no buybacks in any Markov equilibria with Markov state $(Y, F)$. 
We show that by deferring any planned buybacks, the firm will benefit from tax shields and its default option. This step is crucial, as it will allow us to focus on monotone issuance polices. It also does not depend on the assumption that the equilibrium is Markov in \( y \).

3) *The state variable* \( y_t \) *is a smooth process, plus some singular non-increasing process.*

Following on step 2, we use the Lebesgue decomposition which says a monotone process can be decomposed to an absolutely continuous part and a singular part.

4) *There are no gains from trade for an initially unlevered firm.*

Here we show that in any equilibrium, \( V(Y, 0) = \phi Y \), and thus there are no gains from trade for an initially unlevered firm.

5) *The equilibrium issuance policy is absolutely continuous.*

Having shown that the equilibrium payoff for an unlevered firm is unchanged, last step is to rule out the possibility of singular strategies in equilibrium.

Together, these steps verify that our constructed equilibrium – in which the debt issuance policy is absolute continuous in time (i.e., smooth) – is the unique MPE in \( y \).

**Step 1. Convexity and differentiability of the equity value**

We first prove some regularity properties for the scaled equity value function \( \nu(y) = \frac{V(Y, F)}{F} \). Denote \( y_b \) as the default policy in any equilibrium, which, with a slight abuse of notation, could differ from no-trade default boundary \( y_b^0 \) (in our constructed equilibrium, they coincide as in Eq. (26)).

**Lemma 1.** We have the following properties.

a) There exists a default boundary \( y_b \) so that shareholders continue whenever \( y > y_b \) with \( y_b < y_b^0 \).

b) \( \nu(y) \) is increasing, weakly convex, continuously differentiable \( C^1 \), and the debt price function \( p(y) = \nu y' - \nu(y) \) is continuous and weakly increasing over the region \( y \in [y_b, \infty) \).
c) When $v$ is linear in some interval, $p(y)$ is a constant and positive in that interval.

**Proof.** To start, because $v(y) \geq v^0(y) > 0$ for $y \geq y^0_b$, we know that it is optimal to continue for $y \geq y^0_b$, and implying that $y_b < y^0_b$. The optimality of the proposed threshold strategy follows from the monotonicity and convexity established in b).

For b), the monotonicity of $v(y)$ comes from $V(Y, F)$ being decreasing in $F$ shown in Proposition 1. And,

$$V(Y, F) \geq V(Y, \hat{F}) + (\hat{F} - F)p(Y, \hat{F}) \Rightarrow \frac{v(y)}{y} \geq \frac{v(\hat{F})}{\hat{y}} + \left(1 - \frac{1}{\hat{y}}\right)p(\hat{y})$$

Define $q(y) \equiv \frac{v(y) + p(y)}{y} > 0$; then the above inequality is equivalent to

$$v(y) \geq v(\hat{y}) + (y - \hat{y})q(\hat{y})$$

implying $v$ is weakly convex in $y$ and $q$ is a subgradient.

Next, the value function must be smooth (continuously differentiable) in $Y$; otherwise at a kink there would be an infinite expected rate of gain given the Brownian motion in $Y$ (note, shareholders have the option to issue no debt and hence reap this infinite gain, if exists). As a result, $v$ is $C^1$ in $y$, which implies that $v' = q$, and $p = yv' - v$ follows. Since Proposition 1 shows that $p(Y, F)$ is weakly decreasing in $F$, it follows that in our MPE $p(y)$ is weakly increasing in $y$. Finally, note that $p' = yv^*$, so $p$ is constant if $v$ is linear, and $p$ is positive as default must occur immediately if $p = 0$. □

**Step 2. No buybacks**

As mentioned in Proposition 1, the result of no debt buyback generally holds for any Markov equilibrium, with the Markov states being exogenous cash-flow $Y$ and endogenous debt face value $F$. Hence this constitutes the formal proof for the first result in Proposition 1.

The key idea behind the proof is as follows. Given a potential equilibrium debt issuance policy with buybacks, we show that equity holders can strictly improve their value by an alternative strategy without buybacks. For example, consider a strategy in which the firm buys back $1$ (face
value) of debt at date \( t \) and then issues $3 of debt at date \( t+1 \). Shareholders would strictly gain by postponing the buyback at \( t \) and issuing only $2 at \( t+1 \)-- the final debt level and debt price (by the Markov assumption) are the same, but the firm earns the incremental tax shield from $1 of debt between date \( t \) and \( t+1 \). In addition, shareholders retain the option to default on the additional $1 of debt, further enhancing their payoff.

Formally, suppose there exists an equilibrium starting from \( F_0 \geq 0 \) and with
\[
dF_t = -\xi F_t dt + d\Gamma_t, \quad \text{or equivalently,} \quad F_t = e^{-\xi t} F_0 + \int_0^t e^{\xi(s-t)} d\Gamma_s.
\]
The associated bond valuation equation implies that the bond price \( p_t = p(Y_t, F_t) \) satisfies
\[
(r + \xi) p_t dt = (c + \xi) dt + E_r[dp_t] \iff E_r[rp_t dt - dp_t] = cdt + \xi(1 - p_t) dt.
\]

Consider an alternative policy without buybacks defined by
\[
\hat{F}_t = \sup_{s \leq t} \left\{ e^{-\xi(t-s)} F_s \right\} \geq F_t;
\]

Because \( d\hat{\Gamma}_t = d\hat{F}_t + \xi \hat{F}_t dt \), one can show that
\[
d\hat{\Gamma}_t = \max(0, F_t - \hat{F}_t) = \max(0, d\Gamma_t + \int_0^t e^{\xi(s-t)} \left( d\Gamma_s - d\hat{F}_s \right)) \in [0, d\Gamma_t],
\]
which features no buybacks. Intuitively, this policy postpones any buybacks, issuing debt only as needed to prevent the new debt level \( \hat{F}_t \) from ever falling below the original debt level \( F_t \).

For ease of exposition let us define the following. Denote the new debt price by \( \hat{p}_t \equiv p(Y_t, \hat{F}_t) \) under the new debt policy. Let \( d\hat{\Gamma}_t \equiv d\hat{\Gamma}_t - d\Gamma_t \) be the deviation of the issuance policy, and \( F_t^\Delta \equiv \hat{F}_t - F_t \) be the deviation of the debt path which equals the cumulative deviation of debt issuances:
\[
F_t^\Delta = \int_0^t e^{\xi(s-t)} d\hat{\Gamma}_s - \int_0^t e^{\xi(s-t)} d\Gamma_s = \int_0^t e^{\xi(s-t)} d\Gamma_s^\Delta.
\]
As a result, we have
\[
dF_t^\Delta = d\Gamma_t^\Delta - \xi F_t^\Delta dt \iff d\Gamma_t^\Delta = dF_t^\Delta + \xi F_t^\Delta dt.
\]
Under the deviating issuance policy, shareholders may change their default policy accordingly. Our goal is to show that the non-negative issuance policy \( \hat{d} \Gamma \) dominates \( d \Gamma \), given the optimal default timing \( \tau_b \) under the original issuance policy. Note, this implies that \( \tau_b \), as a function of cash-flow history \( \{ Y_t \} \), is not affected by new debt policy \( \hat{F}_t \). Allowing shareholders to reoptimize the default policy can only further improve the value of the deviation. Hence it suffices to establish the deviation gain from \( d \Gamma^\Delta \) while fixing the original default timing \( \tau_b \).

**Lemma 2.** We have

\[
F_t^\Delta \geq 0 \quad \forall t \in [0, \tau_b], \quad \text{with} \quad F_0^\Delta = 0, \quad \text{and}
\]

\[
\hat{p}_t d\hat{\Gamma}_t - p_t d\Gamma_t = p_t (d\hat{\Gamma}_t - d\Gamma_t) = p_t d\Gamma_t^\Delta.
\]

**Proof.** The claim in (53) is obvious. Because issuance only occurs under the new policy in order to keep \( \hat{F}_t \) from falling below \( F_t \), we have \( d\hat{\Gamma}_t > 0 \Rightarrow \hat{F}_t = F_t \Rightarrow \hat{p}_t = p_t \), where the final implication follows from the Markov property that the debt price depends on \( Y \) and \( F \) only (and not their history). Although the debt price may differ when \( d\hat{\Gamma}_t = 0 \), the total issuance proceeds are not affected, and hence \( \hat{p}_t d\hat{\Gamma}_t = p_t d\hat{\Gamma}_t \), proving (54).

Now we are ready to show the gain from postponing buybacks (fixing the timing of default). The change in the value of equity can be expressed as

\[
V_0^\Delta = \hat{V}_0 - V_0
\]

\[
= E \left\{ \int_0^{\tau_b} e^{-\tau r} \left( (1 - \pi) c + \xi \right) F_t dt - e^{-\tau r} p_t d\Gamma_t - \int_0^{\tau_b} e^{-\tau r} \left( (1 - \pi) c + \xi \right) \hat{F}_t dt + e^{-\tau r} \hat{p}_t d\hat{\Gamma}_t \right\}
\]

Eq (1.5)
\[
= E \int_0^{\tau_b} \left[ e^{-\tau r} p_t d\Gamma_t^\Delta - e^{-\tau r} \left( (1 - \pi) c + \xi \right) F_t^\Delta dt \right]
\]

Eq (1.4)
\[
= E \int_0^{\tau_b} \left[ e^{-\tau r} \left[ dF_t^\Delta + \xi F_t^\Delta dt \right] - e^{-\tau r} \left( (1 - \pi) c + \xi \right) F_t^\Delta dt \right]
\]

\[
= E \int_0^{\tau_b} e^{-\tau r} p_t dF_t^\Delta - E \int_0^{\tau_b} e^{-\tau r} \left( (1 - \pi) c + \xi (1 - p_t) \right) F_t^\Delta dt
\]

Integration by parts for the first term \( E \int_0^{\tau_b} e^{-\tau r} p_t dF_t^\Delta \) gives
\[
E\left\{\int_{0}^{\tau} e^{-rt} p_dF_{\Gamma}^d\right\} = E\left\{e^{-r\tau} p_{\tau_0} F_{\tau_0}^\Delta - p_0 F_{0}^\Delta - \int_{0}^{\tau} F_{\tau}^\Delta \cdot e^{-rt} \left[ E_t\left(dp_t\right) - rp_t dt \right] \right\} \\
= E\left\{e^{-r\tau} p_{\tau_0} F_{\tau_0}^\Delta - p_0 F_{0}^\Delta + \int_{0}^{\tau} F_{\tau}^\Delta \cdot e^{-rt} \left(c + \xi (1 - p_t)\right) dt \right\}
\]

Plugging in this result back into our calculation of \( V_0^\Delta \), and using (53) together with the fact that \( p_{\tau_0} = 0 \) under the original policy, we have

\[
V_0^\Delta = E\left\{e^{-r\tau} p_{\tau_0} F_{\tau_0}^\Delta - p_0 F_{0}^\Delta + \int_{0}^{\tau} e^{-rt} \pi c F_{\tau}^\Delta dt \right\} \\
= \pi c E\int_{0}^{\tau} e^{-rt} F_{\tau}^\Delta dt \geq 0,
\]

where the inequality is strict unless \( \hat{F}_t = F_t \) and the original policy had no buybacks. Otherwise, shareholders strictly gain by capturing the debt tax shield on any deferred buybacks. This completes the proof that there are no buybacks in equilibrium.

**Step 3. Smooth versus singular issuance**

The important no-buyback result implies that the equilibrium issuance policy \( \Gamma_t \) must be nondecreasing over time. By Lesbesgue's decomposition for monotonic functions (cf. Proposition 5.4.5, Bogachev, 2013), we can decompose \( \Gamma_t \) into a “smooth” and “singular” process

\[
\Gamma_t = \Gamma_{t,a.c.} + \Gamma_{t,c.s.} + \Gamma_{t,jump} \\
\]

Here, \( \Gamma_{t,a.c.} \) is an absolutely continuous process, so that there exists some positive process \( G_t \) so that \( d\Gamma_{t,a.c.} = G_t dt \). The component \( \Gamma_{t,c.s.} \) captures continuous but singular increases which occur at isolated points on the state space (e.g. a “devil’s staircase”); and \( d\Gamma_{t,jump} \) captures discrete jumps. We aim to rule out both \( \Gamma_{t,c.s.} \) and \( \Gamma_{t,jump} \), which we call \( \Gamma_{t,singular} \), in any equilibrium, and thereby establish the desired result.

Under our assumption that \( Y_t \) evolves according to a geometric Brownian motion and that the equilibrium is Markov perfect in \( y_t \), the decomposition in (55) implies that we can write
\[ dy_t = d \left( \frac{Y_t}{F_t} \right) = y_t \left( \tilde{\mu} - \xi + g(y_t) dt \right) + y_t \sigma dZ_t + (\theta - 1) y_t dN_t - dL_{\text{singular}} \]

for some singular increasing process \( L_{t}^{\text{singular}} = L_{t}^{c.s.} + L_{t}^{\text{jump}} > 0 \). Here \( L_{t}^{c.s.} ( L_{t}^{\text{jump}} ) \) occurs if and only if \( \Gamma_{t}^{c.s.} ( \Gamma_{t}^{\text{jump}} ) \) occurs.

We now show that the existence of a non-zero singular component to the optimal strategy implies the existence of \( \hat{y} > y_b \) such that \( \nu'(\hat{y}) = 0 \). There are two cases to consider.

First, consider \( L_{t}^{\text{jump}} \). By Lemma “Continuous Adjustment” in Section 2.2, the optimal issuance strategy can jump only in an interval for which \( V(Y, F) \) is linear in \( F \) and therefore, by PROPOSITION 1, the debt price is constant. In our homogenous setting \( y^2 \nu''(y) = FV_{FF} \), and so a jump can only occur in an interval for which \( \nu(y) \) is linear. Letting \( \hat{y} \) be the lower bound of this linear segment. Because it is not optimal to jump to default, we can assume \( \hat{y} > y_b \). Note that \( p'_{t}(\hat{y}) = 0 \), and therefore \( p'_{t}(\hat{y}) = 0 \) as otherwise the debt price would have an infinite expected loss rate at \( \hat{y} \), violating no arbitrage. Thus, \( p'_{t}(\hat{y}) = \nu''(\hat{y}) = 0 \).

Second, consider an isolated singular point \( \hat{y} \) for which \( L_{t}^{c.s.} \geq 0 \) (note \( L_{t}^{c.s.} > 0 \iff \Gamma_{t}^{c.s.} > 0 \) and \( L_{t}^{c.s.} = 0 \iff \Gamma_{t}^{c.s.} = 0 \); see the discussion at the end of Step 3). Based on the study of “skew Brownian motion” in Harrison and Shepp (1981), if \( l_{\hat{y}} (t) \) is the local time that the process \( \{y_t\} \) spends at \( y = \hat{y} \), then we have for some positive constant \( \chi \in (0,1] \),

\[ L_{t}^{c.s.} = \chi \cdot l_{\hat{y}} (t). \]

The speed of change is proportional to local time (continuous in time but faster than \( dt \), as it is not absolutely continuous in time). As a result, the excursions of the state variable are asymmetric, with probability \( \frac{1}{2}(1+\chi) \) to the left and with probability \( \frac{1}{2}(1-\chi) \) to the right. The no arbitrage condition for bond investors requires that the expected local gain at \( \hat{y} \) is zero, therefore

\[ \frac{1}{2}(1-\chi) p'_{t}(\hat{y}) = \frac{1}{2}(1+\chi) p'_{t}(\hat{y}). \]
Because the singular point \( \hat{y} \) must be isolated, there is an open ball \( B = (\hat{y} - \varepsilon, \hat{y} + \varepsilon) \) so that 
\[ dL_{\text{singular}} = 0 \] on \( B \setminus \hat{y} \). Because \( v \) and \( v' \) are continuous, and since the HJB equation for \( v \) holds and is identical on the left hand and right hand sides of \( \hat{y} \), we must have \( v'' \) continuous at \( \hat{y} \), and therefore \( p'_v(\hat{y}) = p'_w(\hat{y}) = v' v''(\hat{y}) = 0 \).

**Step 4. No gains from trade**

Next we prove that in any equilibrium an initially unlevered firm achieves no gains from trade and thus has value \( \phi Y \). Specifically,

\[
\frac{V(Y,0)}{Y} = \lim_{f \to 0} \frac{V(Y,F)}{Y} = \lim_{y \to \infty} \frac{v(y)}{y} = \lim_{y \to \infty} v'(y) = \phi. \tag{58}
\]

We must consider three cases regarding the trading strategy for the unlevered firm: (i) trading is smooth in a neighborhood of \( F = 0 \), (ii) the firm immediately issues debt \( F > 0 \), or (iii) trading is singular in any neighborhood of \( F = 0 \).

Case (i): Trading is smooth in a neighborhood of \( F = 0 \).

In this case, we can use the fact that \( V(Y,0) = \hat{v}(0)Y \) and apply the results of Section 7.1. Because trading is smooth, the HJB equation (47) applies in a neighborhood of \( f = 0 \) and has solution of the form (48). Because \( V(Y,0) \) is bounded, \( A_2 = 0 \), and therefore \( \hat{v}(0) = \phi \).

Case (ii): The firm immediately issues debt \( F > 0 \).

For a jump to be optimal, there exists some largest \( \hat{y} \) such that \( v(y) \) is linear for \( y \leq \hat{y} \). Differentiating the HJB equation (34) and using linearity plus the fact that \( v''(\hat{y}) = 0 \) and therefore \( v''(\hat{y}) \leq 0 \) from Step 3,

\[
(r + \xi)v'(\hat{y}) = 1 - \pi + \left( \mu + \xi \right) \left( \hat{y} v''(\hat{y}) + v'(\hat{y}) \right) + \frac{1}{2} \sigma^2 \left[ \hat{y}^2 v''(\hat{y}) + 2 \hat{y} v'(\hat{y}) \right] + \lambda \left( \frac{\theta v'(\hat{y}) - v'(\hat{y})}{\hat{y}} \right) \\
\leq 1 - \pi + \left( \mu + \xi \right) v'(\hat{y}) + \lambda (\theta - 1) v'(\hat{y}) \\
= 1 - \pi + \left( \mu + \xi \right) v'(\hat{y})
\]

Therefore,
\[ v'(\hat{y}) \leq \frac{1 - \pi}{r - \mu} = \phi. \quad (59) \]

But because \( V(Y, 0) \geq \phi Y \) since the firm always has the option not to trade, we must have \( v'(y) = \phi \) for \( y \geq \hat{y} \).

Case (iii): Trading is singular in any neighborhood of \( F = 0 \).

In this case, using the results of Step 3, there exists an increasing sequence of points \( \{\hat{y}_n \to \infty\} \) such that \( v''(\hat{y}_n) = 0 \) and \( v'''(\hat{y}) \leq 0 \). Differentiating the HJB equation (34) as in Case (ii), but without assuming linearity in the jump region,

\[
\begin{align*}
(r + \xi)v'(\hat{y}_i) &\leq 1 - \pi + (\hat{\mu} + \xi)v'(\hat{y}_i) + \lambda(\theta - 1)v'(\hat{y}_i) + \lambda \theta(v'(\hat{y}_i) - v'(\hat{y}_i)) \\
&= 1 - \pi + (\mu + \xi)v'(\hat{y}_i) + \lambda \theta(v'(\hat{y}_i) - v'(\hat{y}_i))
\end{align*}
\]

Taking the limit as \( \hat{y}_i \to \infty \), and because \( \lim \limits_{y \to \infty} v'(y) \) converges monotonically (from convexity and the fact that \( V(Y, 0) \) is bounded in (58)), we get

\[
\lim_{y \to \infty} v'(y) \leq \frac{1 - \pi}{r - \mu} = \phi.
\]

Again, because the firm has the option not to trade, we must have \( \lim_{y \to \infty} v'(y) = \phi \).

**Step 5. Smooth issuance**

Now that we have established there are no gains from trade, we complete the proof that the smooth equilibrium is the unique MPE by showing that the equilibrium trading strategy does not involve singularities.

We begin with a result that bounds the potential gain from a cash flow jump:

**Lemma (Gains from Cash Flow Jumps).** For \( \theta \geq 1 \) and any two points \( y \) and \( y_i \) so that \( y_i \geq y \geq y_h \), we have

\[
v(y) \geq v'(\theta y) - v'(\theta y)(\theta - 1) y \geq v(\theta y) - \phi(\theta - 1) y \quad (60)
\]

\[
v(\theta y_i) \geq v(\theta y) + v'(\theta y)(y_i - y) \geq v(\theta y) + \theta v'(y)(y_i - y) \quad (61)
\]
Proof. In both cases the first inequality follows from convexity, and the second from the fact that \( v' \) is monotonically increasing and, from Step 4, bounded by \( \phi \).

Next, suppose there exist a singular component to the optimal trading strategy. From Step 3, there exists \( \hat{y} \) with \( v''(\hat{y}) = 0 \), and for which trading is smooth (and the HJB holds) in the neighborhood just below \( \hat{y} \). Evaluating the HJB of \( v(\cdot) \) at \( \hat{y} \),

\[
(r + \xi) v(\hat{y}) = (1 - \pi)(\hat{y} - c) - \xi + (\hat{\mu} + \xi) \hat{v}'(\hat{y}) + \frac{1}{2} \sigma^2 \hat{y}^2 \hat{v}''(\hat{y}) + \lambda (v(\theta \hat{y}) - v(\hat{y}))
\]

\[
= (1 - \pi)(\hat{y} - c) - \xi + (\hat{\mu} + \xi) \hat{v}'(\hat{y}) + \lambda v(\theta \hat{y}) - v(\hat{y}))
\]

\[
\leq (1 - \pi)(\hat{y} - c) - \xi + (\hat{\mu} + \xi) \hat{v}'(\hat{y}) + \lambda (\theta - 1) \phi \hat{y}
\]

\[
= (1 - \pi) \hat{y} + (\hat{\mu} + \lambda (\theta - 1) + \xi) \hat{v}'(\hat{y}) - (c(1 - \pi) + \xi) + (\hat{\mu} + \xi) \hat{v}'(\hat{y}) - \phi
\]

\[
= (r + \xi)(\phi \hat{y} - \rho) + (\hat{\mu} + \xi) \hat{v}'(\hat{y}) - \phi
\]

where the inequality follows from (60). Now because any equilibrium must be at least as good as no trade, \( v(y) \geq v^0(y) > \phi y - \rho \), and so the above implies

\[
(\hat{\mu} + \xi)(v'(\hat{y}) - \phi) > 0.
\]

(62)

Note from Step 4, \( v'(y) \leq \phi \), and therefore (62) implies an immediate contradiction if \( \hat{\mu} + \xi > 0 \). We have not restricted the sign of \( \hat{\mu} + \xi \), however, and so we argue next that at a singular point \( v'(\hat{y}) \geq \phi \). Then, combined with Step 4, \( v'(\hat{y}) = \phi \), and thus (62) cannot be satisfied.

There are three cases to consider.

Case (i): The firm immediately issues debt \( F > 0 \).

In this case, \( v(y) = v(\hat{y}) + v'(\hat{y})(y - \hat{y}) \) for \( y > \hat{y} \). Then,

\[
0 \leq v(y) - v^0(y) < v(y) - (\phi y - \rho) = (v'(\hat{y}) - \phi) y + (v(\hat{y}) - v'(\hat{y}) \hat{y} - \rho).
\]

Letting \( y \to \infty \), the above implies \( v'(\hat{y}) \geq \phi \).

Case (ii): Interior jump.
Suppose there is an interior interval \([\hat{y}, y_1]\) on which \(y\) is linear (and strictly convex at each end). Taking the difference between the HJB equations at \(\hat{y}\) and \(y_1\),

\[
(r + \xi)v'(y_1) = (1 - \pi)(y_1 - c) - \xi + (\hat{\mu} + \xi)y_1 v'(y_1) + \frac{1}{2}\sigma^2 y_1^2 v''(y_1) + \lambda(v(\theta y_1) - v(y_1))
\]

\[
(r + \xi)v'(\hat{y}) = (1 - \pi)(\hat{y} - c) - \xi + (\hat{\mu} + \xi)\hat{y} v'(\hat{y}) + \frac{1}{2}\sigma^2 \hat{y}^2 v''(\hat{y}) + \lambda(v(\theta \hat{y}) - v(\hat{y}))
\]

and using the fact that \(v(y_1) = v(\hat{y}) + v'(\hat{y})(y_1 - \hat{y})\) and \(v''(y_1) \geq 0 = v''(\hat{y})\),

\[
\left(r - \hat{\mu} + \lambda(\theta - 1)\right)v'(y_1 - \hat{y}) \geq (1 - \pi)(y_1 - \hat{y}) + \lambda(v(\theta y_1) - v(\theta \hat{y}) - \theta v'(\hat{y})(y_1 - \hat{y}))
\]

\[
\geq (1 - \pi)(y_1 - \hat{y})
\]

where the second inequality follows from (61). Dividing both sides by \((r - \mu)(y_1 - \hat{y})\) we have \(v'(\hat{y}) \geq \phi\).

Case (iii): Isolated singular point.

From Step 3, we know that \(v''(\hat{y}) = 0\). Since it is isolated, the HJB holds on both sides of \(\hat{y}\). By taking the derivative of the HJB and looking at the right hand side of \(\hat{y}\), since \(v''(y) \geq 0\) at \(\hat{y}+\), we have \(v''(\hat{y}) \geq 0\) and hence

\[
(r + \xi)v'(\hat{y}) = 1 - \pi + (\hat{\mu} + \xi)\left[\hat{y}v''(\hat{y}) + v'('\hat{y})\right] + \frac{1}{2}\sigma^2 \left[\hat{y}^2 v'''(\hat{y}) + 2\hat{y}v''(\hat{y})\right] + \lambda\left(\theta v'(\theta \hat{y}) - v'(\hat{y})\right)
\]

\[
\geq 1 - \pi + \left(\hat{\mu} + \lambda(\theta - 1) + \xi\right)v'(\hat{y})
\]

which implies that \(v'(\hat{y}) \geq \phi\). This completes the proof.

### 7.4. Endogenous Investment

We begin with a general proof that with a constant marginal tax rate, the no-trade value function is convex in \(F\).
**Lemma (Convexity of No-Trade Value Function with Investment).** Suppose the tax rate is constant and let \( \hat{Y}_t^i \) be any after-tax cash flow process given some investment policy \( i_t = i(Y,F) \). Then the no-trade value function is convex in \( F \).

Proof: Given initial debt levels \( (F_0,F_1) \) with maturity \( \xi \), define

\[
F_s^a = \alpha F_s^1 + (1 - \alpha) F_s^0 = \alpha F_s^1 e^{-\xi_s} + (1 - \alpha) F_s^0 e^{-\xi_s}.
\]

Let \( \hat{Y}_t^i \) be the after-tax cash flow process given investment policy \( i_t = i(Y,F) \), and let \( \hat{c} = c(1 - \pi) + \xi \) be the total after-tax debt burden. Then the no-trade value function, given the optimal investment \( i(Y,F) \) and default \( \tau(Y,F) \), policies satisfies

\[
V(Y,\alpha F_1 + (1 - \alpha) F_0) = \max_{i,\tau} E \left[ \int_0^\tau e^{-r(s-t)} \left( \hat{Y}^i_s - \hat{c} F_s^a \right) ds \right]
\]

\[
= \max_{i,\tau} \alpha E \left[ \int_0^\tau e^{-r(s-t)} \left( \hat{Y}^i_s - \hat{c} F_s^1 \right) ds \right] + (1 - \alpha) E \left[ \int_0^\tau e^{-r(s-t)} \left( \hat{Y}^i_s - \hat{c} F_s^0 \right) ds \right]
\]

\[
\leq \max_{i,\tau} \alpha E \left[ \int_0^\tau e^{-r(s-t)} \left( \hat{Y}^i_s - \hat{c} F_s^1 \right) ds \right] + (1 - \alpha) \max_{i,\tau} E \left[ \int_0^\tau e^{-r(s-t)} \left( \hat{Y}^i_s - \hat{c} F_s^0 \right) ds \right]
\]

\[
= \alpha V(Y,F_1) + (1 - \alpha) V(Y,F_0),
\]

establishing the convexity of the no-trade value function with investment.

With the binary investment choice \( i_t \in \{0,1\} \), we have

\[
dY_t = i_t \mu Y_t dt + \sigma Y_t dZ_t \quad \text{and} \quad \hat{Y}_t^i = (1 - \pi) Y_t + (1 - i_t) \kappa \mu Y_t.
\]

The no-trade value function satisfies the HJB,

\[
(r + \xi) \hat{v}(y) = \max_{i \in \{0,1\}} \left( 1 - \pi \right) \left( y - c \right) - \xi + (1 - i) \kappa \mu y + (i \mu + \xi) \hat{v}'(y) + \frac{\sigma^2 y^2}{2} \hat{v}''(y).
\]

Investment is therefore optimal if \( \hat{v}'(y) \geq \kappa \). Convexity of the value function together with \( \kappa < \phi \) therefore implies it is optimal to invest whenever \( y \) exceeds some threshold \( y_{inv}^{i_t} \). Because at default \( \hat{v}'(y_{b}^{inv}) = 0 < \kappa \), the option to stop investing is valuable and hence \( y_{b}^{inv} < y_b \).
To characterize \( \hat{v} \), it is easiest to consider separately the value functions \( (v', v_{NI}) \) in the investment and no-investment regions, respectively. The value function \( v' \) can be solved for using the methods of Section 3, where now shareholders have the option to stop investing rather than to default. Because \( v_{NI}(y_{inv}^i) > 0 \),

\[
v'(y) = \max_{\hat{y}_i} \phi y - \rho +\left(\frac{y}{\hat{y}_i}\right)^{-\gamma} \left(v_{NI}(\hat{y}_i) - (\phi \hat{y}_i - \rho)\right)
= \phi y - \rho + y^{-\gamma} \left[\max_{\hat{y}_i} \hat{y}_i' \left(v_{NI}(\hat{y}_i) - \phi \hat{y}_i + \rho\right)\right] > \phi y - \rho + y^{-\gamma} \left[\max_{\hat{y}_i} \hat{y}_i' \left(0 - \phi \hat{y}_i + \rho\right)\right]
= \phi y - \rho + y^{-\gamma} \left[y_{b}^{i'} (\rho - \phi y_{b})\right]
= v(y),
\]

where \( v(y) \) is the value function from Section 3 without the investment option. Therefore,

\[
v'(y) = \phi y - \rho - \Gamma_i \left(\frac{y}{y_{b}}\right)^{-\gamma} \left(\phi y_{b} - \rho\right) \quad \text{with} \quad \Gamma_i = \frac{\max_{\hat{y}_i} \hat{y}_i' \left(v_{NI}(\hat{y}_i) - \phi \hat{y}_i + \rho\right)}{y_{b}^{i'} (\rho - \phi y_{b})} > 1.
\]

Note that \( \Gamma_i > 1 \) implies \( v'(y) > v'(y) \geq \kappa > 0 \) for \( y \geq y_{inv}^i \). Because \( v(y_{b}) = 0 \), we therefore must have \( y_{inv}^i > y_{b} \).

We can solve for \( v_{NI} \) explicitly as follows. From the HJB in the no investment region, it is straightforward to show that

\[
v_{NI}(y) = \phi_d y - \rho + A_d y^{-\gamma_d} + B_d y^{\eta_d}
\]

where \( \phi_d = \frac{1 - \pi + \mu \kappa}{r} \), and \( \eta_d = \gamma_d + 1 - 2\xi \sigma^{-2} > 1 \) with

\[
\gamma_d = \frac{(\xi - 0.5 \sigma^2) + \sqrt{(\xi - 0.5 \sigma^2)^2 + 2 \sigma^2 (r + \xi)}}{\sigma^2} > 0.
\]

To solve the model, note that smooth pasting and the optimality condition for investment imply

\[
v'(y_{inv}^i) = v_{NI}(y_{inv}^i) \quad \text{and} \quad v'(y_{inv}^i) = v_{NI}'(y_{inv}^i) = \kappa.
\]
Combined with the usual default boundary conditions,

\[ v^{NI}(y_{b}^{inv}) = v^{NI}'(y_{b}^{inv}) = 0 , \]

we have five equations to solve for five unknowns \( \{ \Gamma_i, A_d, B_d, y_i^{inv}, y_{b}^{inv} \} \). The solution to this system can be characterized as follows. The default boundary \( y_{b}^{inv} \) can be solved from the following uni-dimensional nonlinear equation:

\[
\left[ \frac{(\rho \eta_d - \phi_d y_{b}^{inv}(1+\gamma_d))\left(y_{b}^{inv}\right)^{-\eta_d}}{(1+\gamma_d)(\phi - \phi_d) + (\gamma_d / \gamma - 1)(\phi - \kappa)} \right]^{1-\gamma_d} = \left[ \frac{(\rho \eta_d - \phi_d y_{b}^{inv}(\eta_d - 1))\left(y_{b}^{inv}\right)^{\gamma_d}}{(\eta_d - 1)(\phi - \phi_d) + (\eta_d / \gamma + 1)(\phi - \kappa)} \right]^{1+\gamma_d}
\]

Once we obtain \( y_{b}^{inv} \), the remaining unknowns are

\[
y_{i}^{inv} = \left( \frac{(\rho \eta_d - \phi_d y_{b}^{inv}(\eta_d - 1))\left(y_{b}^{inv}\right)^{\eta_d}}{(\eta_d - 1)(\phi - \phi_d) + (\eta_d / \gamma + 1)(\phi - \kappa)} \right)^{1+\gamma_d}, \Gamma_i = \left( 1 - \frac{\kappa}{\phi} \right) \left( \frac{y_{i}^{inv}}{y_{b}^{inv}} \right)^{1+\gamma_d}
\]

\[
A_d = -\frac{\phi_d y_{b}^{inv}(\eta_d - 1) - \rho \eta_d\left(y_{b}^{inv}\right)^{\gamma_d}}{y_{d} + \eta_d}, B_d = -\frac{\phi_d y_{b}^{inv}(1+\gamma_d) - \rho \gamma_d\left(y_{b}^{inv}\right)^{-\eta_d}}{y_{d} + \eta_d}
\]

Given the solution for \( (v', v^{NI}) \), we can then calculate the equilibrium debt price \( p = yv' - v \) and issuance rate \( g^* = \frac{\pi c}{y^2v'} \) as in the baseline model.

8. References


Geelen, Thomas, 2017, Debt Maturity and Lumpy Debt, working paper, Swiss Finance Institute and EPFL.


