FLows and Performance with Optimal Money Management contracts

Job market paper

For the latest version please visit
https://voices.uchicago.edu/pegoraro/research/

Stefano Pegoraro*

February 9, 2020

Abstract

Previous literature documents that mutual funds’ flows increase more than linearly with realized performance. I show this convex flow-performance relationship is consistent with a dynamic contracting model in which investors learn about the fund manager’s skill. My model predicts that flows become more sensitive to current performance after a history of good past performance. It also predicts that the effect of past performance on the current flow-performance relationship is weaker for managers with longer tenure. I consider an optimal incentive contract for money managers, and I provide an explanation for common compensation practices in the industry, such as convex pay-for-performance schemes and deferred compensation. In the optimal contract, flows become more sensitive to performance when the manager faces stronger incentives from the compensation contract. With learning, the manager’s incentives become stronger after good performance, so that a manager exerts more effort when his assessed skill is higher. However, the relation between past performance and incentives becomes weaker over the manager’s tenure. Using mutual fund data, I test the predictions of the model on the dynamic behavior of the flow-performance relationship, and I find empirical support for the theory.

*Kenneth C. Griffin Department of Economics and Booth School of Business, The University of Chicago, Chicago IL – pegoraro@uchicago.edu. I am immensely grateful to Lars Peter Hansen, Zhiguo He, Douglas Diamond, and Pietro Veronesi. I would also like to thank Benjamin Brooks, Nicola Limodio, Lubos Pastor, Panagiotis Souganidis, Ehsan Azarmia, Jian Li, Luis Simon, Yuyao Wang, and participants at SITE (Banks and Financial Frictions), Chicago Booth Asset Pricing Conference, IEA meeting, and several working groups at the University of Chicago.
1 Introduction

Mutual funds face a double challenge. On the one hand, they cope with volatile flows of money as investors react to funds’ performance: Well-performing funds experience money inflows, whereas poorly-performing funds experience money outflows. On the other hand, funds may struggle to generate good performance if their portfolio managers face inadequate incentive schemes. These two challenges are tightly connected. Funds collect fees on their assets under management. They therefore aim to maximize performance in order to increase money inflows and fee revenues. However, funds need to delegate investment decisions to managers whose objective is not to maximize performance but to pursue their own interests. In this paper, I study the connection between these two challenges and show how funds can overcome them using optimal money management contracts.

I develop a dynamic contracting model that explains common patterns in the money management industry, including a convex relation between fund flows and performance, as well as convex pay-for-performance schemes for portfolio managers. Existing literature studies these patterns in isolation, often viewing them as puzzling, and sometimes viewing them as problematic. Brown et al. (1996) and Chevalier and Ellison (1997) interpret the convex flow-performance relationship as an implicit incentive scheme for mutual funds, and investigate its implications for funds’ risk-taking behavior. Basak and Pavlova (2013), Cuoco and Kaniel (2011), and Panageas and Westerfield (2009) study how common compensation schemes distort managers’ portfolio choices. Using a mechanism-design approach, I show that a convex flow-performance relationship and a convex compensation scheme are jointly consistent with an optimal dynamic contract for money managers. To provide empirical support for my theory, I derive novel statistical predictions on the dynamic behavior of the flow-performance relationship and test them in mutual fund data.

In the model, I consider the two layers of incentives that characterize the mutual fund industry. In the first layer, competitive investors supply capital and pay proportional fees to a fund advisor who represents the fund family or the fund management company. In the second layer, the advisor hires a portfolio manager and sets the terms of the manager’s compensation contract. Because the advisor captures the value added of the fund through fee revenues, she faces implicit incentives to maximize performance and assets under management. To align the manager’s incentives to her own, she offers an explicit performance-based compensation contract to the manager.

To generate an increasing and convex relation between fund flows and performance,
the model relies on two assumptions. First, the manager possesses an unknown skill to generate excess returns. The manager, the advisor, and the investors learn about the manager’s skill by observing realized returns. According to this assumption, investors expect better future performance from a manager who performed better in the past. Second, the manager is subject to moral hazard in his private choice of costly effort: He may exert low effort and reduce returns for investors. The advisor designs an optimal incentive contract to prevent this possibility. In the optimal contract, the advisor specifies the incentives of the manager as an increasing function of the manager’s expected performance, so that a manager exerts more effort when his assessed skill is higher. Moreover, flows into and out of the fund become more sensitive to performance when the manager faces stronger incentives. Fund flows thus become more sensitive to current performance when future expected performance increases. Therefore, as good returns accumulate and investors expect increasingly better future performance, fund flows become increasingly more sensitive to current performance. As a result, over any period of time, cumulative fund flows respond in a positive and convex way to cumulative performance. The empirical literature has repeatedly documented a positive and convex response of fund flows to performance (Chevalier and Ellison, 1997; Del Guercio and Tkac, 2002; Sirri and Tufano, 1998).

The model produces a managerial compensation scheme that reflects three common practices in the money management industry. First, managers are compensated for their performance. Second, similar to capital flows, the manager’s compensation becomes more sensitive to performance after a history of good returns, thus resulting in a convex compensation scheme on an annual basis. These first two results are consistent with the widespread use of convex pay-for-performance contracts in the industry (BIS, 2003; Ma et al., 2019). According to the model, advisors opt for these contracts so that managers with higher assessed skill face stronger incentives to exert effort. Third, the optimal contract includes a deferred compensation feature. After good performance, the advisor partially postpones the delivery of the promised compensation in order to provide stronger incentives to the manager. In the money management industry, several practices effectively postpone the payout of performance-based compensation to future periods.\(^1\)

I provide empirical support for my theory by testing a novel prediction of the model regarding the dynamic behavior of the flow-performance relationship: Fund flows react more strongly to current performance after a history of good performance. I measure

---

\(^1\)Ma et al. (2019) document that 30% of the mutual fund managers in their sample are subject to explicit deferred compensation contracts. Moreover, they find that managers’ bonuses depend on their average performance over multiple years in the past. This last practice effectively implements a deferred compensation scheme.
a mutual fund’s performance by comparing its return with the average return of funds with the same investment objective. For every month, I compute the average past performance over the previous months. At a monthly frequency, past performance has a positive and statistically significant effect on the extent to which fund flows respond to current performance.

I test an additional novel prediction of the model, which establishes a connection between the dynamic behavior of the flow-performance relationship and the tenure of the manager. Under the assumption that the advisor and the manager can fully commit to the terms of the optimal long-term contract, the slope of the flow-performance relationship depends more weakly on past performance if the manager has a longer tenure. I verify this prediction holds in mutual fund data. According to the model, the ex ante optimal contract imposes constraints on the incentives and on the skill of the manager as the manager’s tenure becomes longer. If the manager’s skill plays a minor role in generating returns, past performance provides less information about future performance, thus weakening the link between past performance and the slope of the flow-performance relationship.\(^2\)

I derive my results in a continuous-time contracting problem with learning about the manager’s skill. This problem poses some challenges, which I overcome by using duality methods. Given an optimal incentive contract with learning, the manager could acquire an ex post information rent after shirking. Intuitively, a manager who shirks and appears unskilled has better career prospects than a manager who is actually unskilled. If a contract induces larger information rents, the manager has weaker incentives to maximize returns and the advisor obtains lower revenues. Therefore, one could formulate an optimization problem in which the advisor controls the manager’s information rent as an independent state variable. Unfortunately, this problem cannot be feasibly solved. To overcome this challenge, I use duality methods and offer a tractable and intuitive formulation of the contract-design problem. In the optimal contract, the advisor commits to ex post inefficient incentives in order to reduce the ex ante information rent of the manager. In the dual formulation, this commitment is captured by a multiplier that, over time, distorts the terms of the contract towards a lower risk exposure for the manager. By considering the dynamics of the multiplier and of beliefs, I provide intuitive interpretations for my results on the flow-performance relationship and on the manager’s compensation scheme.

\(^2\)Almazan et al. (2004) provide evidence that more experienced managers are subject to more investment constraints. My model suggests this evidence could simply represent the outcome of an optimal contract that restricts the use of the manager’s skill over time.
The rest of the paper is organized as follows. In section 2, I review the related literature. In section 3, I present the setup of the model. In section 4, I characterize the optimal contract. In section 5, I show the implications of the optimal contract for flows, performance, and compensation and provide the economic intuition behind the results. In section 6, I empirically test model’s predictions. Section 7 concludes.

In Appendix A, I show the model’s key results hold also when I relax the assumption that the manager and the advisor fully commit to the terms of the optimal contract. Appendix B contains a two-period model that illustrates the trade-off between ex ante and ex post efficiency. All proofs are in Appendix C, Appendix D, and Appendix E. Appendix F contains robustness checks for the empirical analysis.

2 Related Literature

My paper extends the current asset management literature by studying the connection between agency frictions, managerial compensation, and the flow-performance relationship. Existing models of asset management (Basak and Pavlova, 2013; Cuoco and Kaniel, 2011; Kaniel and Kondor, 2013; Vayanos and Woolley, 2013) focus on the implications of common fee schedules on portfolio choices and asset prices. Compared to this literature, my model builds on the recent empirical evidence in Ibert et al. (2018) and Ma et al. (2019), who show that portfolio managers face compensation contracts that differ substantially from the fee revenues that fund management companies receive. Although my paper does not study the asset-pricing implications of managerial incentive schemes, it provides a theoretical foundation for common compensation practices in the industry. My paper is therefore related to the literature that adopts a mechanism-design approach to study investment delegation (Bhattacharya and Pfleiderer, 1985; Cadenillas et al., 2007; Dybvig et al., 2010; He and Xiong, 2013; Heinkel and Stoughton, 1994; Ou-Yang, 2003; Palomino and Prat, 2003). I further develop this literature by studying the connection between managerial incentives and the flow-performance relationship.

Compared to previous theoretical studies of the convex flow-performance relationship, my research highlights the dynamic nature of such a relationship. In particular, the model produces a relation between flows and performance that depends on the history of past performance. This dynamic aspect of the flow-performance relationship is novel to the literature. My model therefore complements Berk and Green (2004), in which the flow-performance relationship changes as a function of the fund’s age, but not of past performance. To derive my results, I rely on learning about the manager’s skill and
on the agency frictions inside the fund. I abstract from other determinants of the flow-performance relationship that previous studies have considered, for example, decreasing returns to scale (Berk and Green, 2004), changes in investment strategies (Lynch and Musto, 2003), participation costs (Huang et al., 2007), or bias in the social transmission of information (Han et al., 2018).

In my empirical analysis, I stress the dependence of the flow-performance relationship on the history of the fund’s past performance and on the manager’s incentives. My analysis therefore differs from Franzoni and Schmalz (2017), who study how the flow-performance relationship changes with aggregate risk factors, rather than with the fund’s idiosyncratic performance. My research complements also the analysis in Chevalier and Ellison (1997). Whereas they study how the shape of the flow-performance relationship differs across young and old funds, I study how the history dependence of the flow-performance relationship varies with the tenure of the manager. Although my empirical analysis is primarily related to the extensive literature on the flow-performance relationship (Chevalier and Ellison, 1997; Del Guercio and Tkac, 2002; Ippolito, 1992; Sirri and Tufano, 1998; Zheng, 1999), I design my empirical tests on the basis of a model that connects fund managers’ incentives to the flow-performance relationship. In this sense, my paper adds a new perspective to the literature that explores the connection between managers’ incentives and fund performance (Agarwal et al., 2009; Almazan et al., 2004; Chen et al., 2008; Chevalier and Ellison, 1999; Khorana et al., 2007).

From a modeling perspective, my paper builds on the literature about dynamic contracting with learning (Bergemann and Hege, 2005; DeMarzo and Sannikov, 2017; Halac et al., 2016; He et al., 2017; Hörner and Samuelson, 2013; Prat and Jovanovic, 2014). I contribute to this literature along two dimensions. First, I allow for capital flows and for smooth changes in the rate of learning. By doing so, I obtain testable predictions on the relation between capital flows, performance, and managers’ tenure. Second, I show how to use duality methods to overcome technical challenges posed by this class of contracting models. In particular, I use duality methods to solve a dynamic programming problem with an endogenous bound on the state space, which would be otherwise unfeasible. Sannikov (2014) and Miao and Zhang (2015) also use duality methods to solve contracting problems, although with the purpose of obtaining linear partial differential equations.
3 Model Setup

I consider a setting that involves a portfolio manager, a fund advisor, and a population of investors as players. No player knows the true alpha-generating skill of the manager, and a standard agency friction exists within the fund, because the manager could shirk and gain a private benefit at the expense of investors. Time is continuous and starts at 0.

3.1 Players

I consider the following players: a population of investors, one fund advisor (the principal/she), and a portfolio manager (the agent/he). I use the term “fund” to refer to the organization formed by the portfolio manager and the fund advisor. The advisor collects capital from investors on the spot market and hires the manager to actively manage this capital.

Investors are risk neutral, competitive, and cannot commit to long-term contracts. They interact with the advisor through a series of spot contracts that, at every time \( t \), specify the assets investors supply to the fund, \( K_t \), and the proportional fee the advisor receives, \( f_t \). Investors then collect the returns that the fund produces, \( R_t \). Through their interaction with the fund, investors obtain a utility

\[
E \left[ \int_0^\infty e^{-rt} \left( K_t dR_t - (f_t + r)K_t dt \right) \right| \mathcal{F}_0 ],
\]

where \( r > 0 \) is the market risk-free rate and represents both the discount rate of investors, as well as their opportunity cost of capital.\(^3\) Investors are therefore willing to provide any amount of capital as long as they expect the net-of-fee return to weakly exceed the risk-free rate. \( \mathcal{F}_0 \) captures the initial information, which is common across all players.

Through most of the paper, I assume that the fee is variable over time and that \( K_t \) represents the amount of assets that the fund actively manages. Alternatively, one could assume, as in Berk and Green (2004), that the fee is fixed at some sufficiently low value, \( \bar{f} \), and that investors supply additional capital, \( \bar{K}_t \), which the fund invests in a passive benchmark. Investments in the passive benchmark can be easily monitored, so they are not subject to the moral-hazard friction that I describe shortly. These two assumptions yield equivalent outcomes, as long as the passively and actively managed parts of the portfolio are contractible and the implied transfers coincide, that is, \( f_t K_t = \bar{f}(K_t + \bar{K}_t) \).

\(^3\)These preferences reflect the assumption that investors are active in a complete asset market. The fund offers an idiosyncratic return that they are able to fully diversify. The fund’s returns are thus uncorrelated with their aggregate consumption and carry no risk premium for investors.
The advisor collects the fees paid by investors, \( f_t K_t \), and offers a compensation \( \tilde{C}_t \) to the manager. The advisor is risk neutral and her objective is to minimize the cost of running the fund,

\[
E \left[ \int_0^\infty e^{-rt}(\tilde{C}_t - f_t K_t) \, dt \right]_{\mathcal{F}_0}.
\]

Unlike the investors, the advisor has the power to commit to a long-term contract with the manager. Investors observe this long-term contract and understand the consequences of the contract on the manager’s incentives. In particular, investors adjust their supply of capital and their willingness to pay fees in response to the terms of the contract. Therefore, in designing the contract, the advisor accounts not only for the incentives of the manager, but also for the investors’ responses.

The manager controls the fund’s active portfolio. However, he cannot be directly monitored by the advisor. In particular, the advisor cannot verify whether the manager exerted full effort in making his investment and trading choices. Because of the imperfect monitoring, a moral-hazard friction emerges. More formally, I assume the agent may shirk at rate \( m_t \) and reduce the fund’s cash flow rate by \( m_t K_t \). The manager obtains a private consumption value of \( \lambda m_t K_t \) from shirking, where \( 1 - \lambda \in (0, 1) \) represents the inefficiency of shirking. Full effort coincides with \( m_t = 0 \), and the consumption value of shirking can be equivalently interpreted as the cost of effort. If a fund manager consumes \((C_t)_{t \geq 0}\) and shirks at a rate \((m)_{t \geq 0}\), his expected utility is given by

\[
V_0 = E \left[ \int_0^\infty e^{-\delta t} u(C_t + m_t \lambda K_t) \, dt \right]_{\mathcal{F}_0},
\]

(1)

where \((K_t)_{t \geq 0}\) are the fund’s assets under management. I assume \( \delta \geq r \) and \( u(x) = \frac{x^{1-\rho}}{1-\rho} \) with \( \rho \in (0, 1/2) \).

The manager possesses a specific skill, which is unobservable to the advisor and the investors, as well as to the manager himself. However, he can generate informative signals about his skill through costly experimentation. For example, he can search for profitable investment opportunities or implement new trading strategies. If the manager is skilled, he will produce additional excess returns. If he is not, his experimentation efforts will

---

4 I relax the full-commitment assumption in Appendix A.

5 The assumption that \( \rho < 1/2 \) is needed to obtain a finite solution to the model. If \( \rho > 1/2 \), the manager’s marginal utility of consumption would decline quickly enough that the advisor would find it profitable to give infinite capital and infinite consumption to the manager and overcome the incentive problem. This assumption can be relaxed if I extend the model to allow the manager to privately save. However, this extension would substantially complicate the model along several dimensions. Moreover, it would not add any additional insight about the economic mechanism that determines the flow-performance relationship.
be worthless. By observing realized returns, all players will be able to learn over time whether the manager possesses a superior investment skill. I assume experimentation can be represented by a variable, $\eta_t$, that takes values in a bounded set, $\eta_t \in [0, \bar{\eta}]$. Experimentation is fully observable and contractible. To introduce costs for experimentation, I assume experimentation reduces the consumption value of the manager’s compensation. If the manager receives compensation $\tilde{C}_t$ from the advisor, but he is required to undertake experimentation $\eta_t$, his final consumption is given by

$$C_t = q(\eta_t) \tilde{C}_t,$$

(2)

for a function $q(\cdot)$ such that $q(\cdot) \in (0, 1)$, $q'(\cdot) < 0$ and $q(\bar{\eta}) \geq \lambda$. We can interpret the quantity $(1 - q(\eta_t))\tilde{C}_t$ as the cost of searching for new investment ideas. If the manager searches more assiduously for new investment ideas, then his experimentation rate $\eta$ increases, and the consumption value of his compensation decreases.

### 3.2 Returns

The advisor hires the manager to actively manage the fund’s portfolio and generate excess returns. However, the manager’s skill is unknown to all players, including the manager. I assume the manager could be either skilled or unskilled, as indexed by his hidden type $h \in \{0, 1\}$. If the manager experiments with investment ideas and trading strategies, his skill will be reflected in the fund’s returns. Players then learn over time about the manager’s skill by observing the fund’s performance.

I assume returns follow a stochastic process,

$$dR = (r + \mu(\eta_t, h, m_t))dt + \sigma dW_t$$

$$\mu(\eta_t, h, m_t) = \alpha + \sigma \eta_t h - m_t,$$

(3)

where $r$ is the risk-free rate, and $\alpha \geq 0$ and $\sigma \geq 0$ are known parameters.

Returns depend on the uncertain skill of the manager, $h$, and on the manager’s hidden action, $m_t \geq 0$, which is positive if the agent shirks and does not exert full effort in managing the assets. If a manager is skilled ($h = 1$), he obtains superior returns by experimenting, that is, by setting $\eta_t > 0$. If he is unskilled ($h = 0$), his experimentation efforts will not be reflected in returns. By shirking at rate $m_t$, the manager reduces cash flow for investors by $m_tK_t$. However, his private benefit of shirking is only $\lambda m_t K_t$ for $\lambda < 1$. Shirking is therefore inefficient because the manager destroys more value that what he obtains. The advisor, in order to maximize her revenues, designs a contract that enforces
full effort. When investors observe this contract, they understand that the manager has incentives to exert full effort and they account for these incentives when deciding how much capital to provide to the advisor for a given level of fees.

Besides affecting the distribution of returns, experimentation determines the information content of returns as a signal for the manager’s skill. Experimentation $\eta_t$ coincides with the signal-to-noise ratio of returns at $t$, which is defined as

$$\frac{\mu(\eta_t, 1, m_t) - \mu(\eta_t, 0, m_t)}{\sigma}.$$ 

This quantity measures how informative returns are regarding the manager’s skill. Suppose a skilled manager produces, on average, much larger returns than an unskilled manager, that is $\mu(\eta_t, 1, m_t) \gg \mu(\eta_t, 0, m_t)$. In this case, a good (bad) return realization will be a strong signal that the manager possesses high (low) skill. However, if the volatility of returns, $\sigma$, is very large, a skilled manager could generate a very poor return due to bad luck, whereas an unskilled manager could deliver a superior return due to good luck. Therefore, volatility reduces the information content of return signals. By increasing experimentation, $\eta_t$, the fund increases the signal-to-noise ratio of returns and hence generates more information about the manager’s skill. Although experimentation is beneficial in the short run, I show that in the long run future experimentation worsens the moral-hazard problem. In the optimal contract, the advisor will therefore trade off the benefits of current experimentation with the costs of any future experimentation that she promises.

### 3.3 Contracting Environment and Learning

The two main elements of the model are the incentive contract between the advisor and the manager, and the learning process. Returns constitute the only source of information for the players. First, returns allow players to draw inferences about the manager’s skill. Second, because return realizations change depending on the hidden action of the manager, they can be used as the basis of an incentive contract. Therefore, the terms of the contract between the manager and the advisor, as well as the spot contracts between the advisor and the investors, can solely depend on the history of returns. In this framework, players’ information is generated by the history of returns and I denote with $(\mathcal{F}_t)_{t \geq 0}$ the filtration generated by the history of returns. Let $(R_s)_{0 \leq s \leq t}$ denote the history of returns up to time $t$, then $\mathcal{F}_t = \{(R_s)_{0 \leq s \leq t}\}$ is the smallest $\sigma$-algebra for which $(R_s)_{0 \leq s \leq t}$ is measurable.
A contract between the advisor and the manager specifies the manager’s consumption, the size of his actively managed portfolio, and the experimentation that he undertakes. Moreover, for completeness, I assume the contract also specifies the effort that the advisor expects. Although the agent’s effort cannot be verified, the advisor will form a conjecture about the manager’s hidden action at any point in time. I let the contract specify this conjecture. To keep the notation parsimonious, I omit compensation from the definition of contract. Given consumption $C_t$ and experimentation $\eta_t$, the compensation $\tilde{C}_t$ of the manager is determined by equation (2).

**Definition 1 (Contract).** A contract $\mathcal{C}$ is a set of $\mathcal{F}_t$-adapted processes $((C_t)_{t \geq 0}, (K_t)_{t \geq 0}, (\eta_t)_{t \geq 0}, (m_t)_{t \geq 0})$.

Although the advisor cannot directly control the manager’s hidden action, she understands the implications of a contract on the manager’s incentives to exert effort. If her conjecture about the manager’s hidden action coincides with the action that the manager has incentives to take, the contract is called incentive compatible.

**Definition 2 (Incentive Compatible Contract).** A contract $\mathcal{C} = ((C_t)_{t \geq 0}, (K_t)_{t \geq 0}, (\eta_t)_{t \geq 0}, (m_t)_{t \geq 0})$ is incentive compatible if

$$(m_t)_{t \geq 0} \in \arg\max_{(\hat{m}_t)_{t \geq 0}} E^{\mathcal{C},(\hat{m}_t)_{t \geq 0}} \left[ \int_0^\infty e^{-\delta t} u(C_t + \hat{m}_t \lambda K_t) \, dt \bigg| \mathcal{F}_0 \right].$$

The notation $E^{\mathcal{C},(\hat{m}_t)_{t \geq 0}}[\cdot|\mathcal{F}_0]$ explicitly expresses the fact that the distribution of returns depends on the contract $\mathcal{C}$ and the hidden action strategy $(\hat{m}_t)_{t \geq 0}$. Without loss of generality, we can consider only contracts that are incentive compatible. Because equilibrium strategies are common knowledge, the advisor can always change any given contract to another one in which her conjecture about the manager’s hidden action is consistent with the manager’s best response to the contract.

Given an incentive-compatible contract, players will symmetrically learn about the skill of the manager by observing the fund’s returns. Suppose all players possess a common prior, $E[h|\mathcal{F}_0] = p \in [0, 1]$. Because the advisor and the investors correctly anticipate the hidden action of the manager, at any time $t$, all players have common beliefs about the manager’s skill given by

$$\phi_t = E[h|\mathcal{F}_t].$$

Beliefs are an important state variable of the model. Beliefs determine investors’ expectations about the fund’s returns,

$$E \left[ \mu(\eta_t, h, m_t) | \mathcal{F}_t \right] = \mu(\eta_t, \phi_t, m_t).$$
and, through expected returns, beliefs determine the investor’s willingness to supply capital.

**Proposition 1.** Given an incentive-compatible $C = ((C_t)_{t \geq 0}, (K_t)_{t \geq 0}, (\eta_t)_{t \geq 0}, (m_t)_{t \geq 0})$, beliefs $\phi_t = E[h|\mathcal{F}_t]$ evolve as

$$d\phi_t = \eta_t \phi_t (1 - \phi_t) dW_t^e,$$

(4)

where

$$dW_t^e = \frac{1}{\sigma} [dR_t - (r + \mu(\eta_t, \phi_t, m_t)) dt]$$

(5)

is an increment to a standard Brownian motion under the measure of returns induced by $C$.

Equation (4) shows the role of experimentation in the production of information. If the advisor requires higher experimentation $\eta_t$, the signal-to-noise ratio increases, together with the response of beliefs to any given return shock $dW_t^e$.

Given their beliefs $\phi_t$ and the contract $C$, competitive and risk-neutral investors provide capital to the fund through a series of spot-market contracts. These contracts specify the proportional fees that investors pay to the advisor. Because investors are competitive, they take the fund’s net expected returns as given and they compare them to the interest rate $r$, which represents their outside option. Given their risk neutrality, they are not concerned about the idiosyncratic risk of the fund.

**Definition 3 (Spot-Market Contract).** Given a contract $C$ and beliefs $\phi_t$, a spot-market contract, $C^S$, is a pair of $\mathcal{F}_t$-measurable variables, $(f_t, K_t)$ such that for all times $t$ investors are willing to supply capital $K_t$ and pay fees $f_t$

$$K_t \in \arg\max_K (\mu(\eta_t, \phi_t, m_t) - f_t) K,$$

and such that the advisor obtains revenues $f_t K_t$.

From this definition, we can see that competitive investors offer a perfectly elastic supply of capital at rate $r$. The advisor can increase fees up to the point at which they coincide with expected excess returns. Given these fees, individual investors are willing to supply any amount of capital, whereas the advisor maximizes the revenues she collects per unit of assets under management. I call a spot market contract optimal if it maximizes the spot revenues of the advisor.

**Lemma 1.** In any optimal spot-market contract, $f_t = \mu(\eta_t, \phi_t, m_t)$ for all $t \geq 0$.

Since the proof is standard, I omit it. The lemma is based on the intuition that, if investors are willing to provide at least $K_t$ units of capital and pay a fee $f'_t < \mu(\eta_t, \phi_t, m_t)$,
they are also willing to provide at least $K_t$ units of capital for any fee $f_t \in [f'_t, \mu(\eta_t, \phi_t, m_t)]$. A revenue-maximizing advisor therefore chooses the maximum fee investors that are willing to pay. This fee coincides with the expected return $\mu(\eta_t, \phi_t, m_t)$.

Whereas investors’ preferences pin down fees $f_t$, the size of the fund is pinned down by the advisor’s demand. By setting fees equal to expected excess returns, the advisor is capturing the value added of the fund, $K_t \mu(\eta_t, \phi_t, m_t)$, through her revenues. She therefore has incentives to design a compensation contract that maximizes the value of the fund.

The compensation contract of the manager will therefore be optimal for the advisor if it minimizes the costs of running the fund, after taking into account the incentives that the manager receives from the contract. Formally, an optimal contract can be defined as follows:

**Definition 4 (Optimal Contract).** A contract $\mathcal{C} = (C_t)_{t \geq 0}, (K_t)_{t \geq 0}, (\eta_t)_{t \geq 0}, (m_t)_{t \geq 0}$ is optimal for initial beliefs $\phi_0$ and for initial promised value $V_0$ if

$$\mathcal{C} \in \arg \inf_{\hat{\mathcal{C}}} \mathbb{E}^{\hat{\mathcal{C}}, (\tilde{m}_t)_{t \geq 0}} \left[ \int_0^\infty e^{-rt} \left( \frac{\hat{C}_t}{q(\hat{\eta}_t)} dt - \hat{K}_t \mu(\hat{\eta}_t, h, \tilde{m}_t) \right) dt \bigg| \mathcal{F}_0 \right]$$

$$\text{s.t. } (\tilde{m}_t)_{t \geq 0} \in \arg \max_{(m'_t)_{t \geq 0}} \mathbb{E}^{\hat{\mathcal{C}}, (m'_t)_{t \geq 0}} \left[ \int_0^\infty e^{-\delta t} u(\hat{\mathcal{C}}_t + m'_t \lambda \hat{K}_t) dt \bigg| \mathcal{F}_0 \right]$$

$$\mathbb{E}^{\hat{\mathcal{C}}, (\tilde{m}_t)_{t \geq 0}} \left[ \int_0^\infty e^{-\delta t} u(\hat{\mathcal{C}}_t + \tilde{m}_t \lambda \hat{K}_t) dt \bigg| \mathcal{F}_0 \right] \geq V_0$$

$$\mathbb{E}[h|\mathcal{F}_0] = \phi_0.$$

In the optimal contract, the advisor explicitly takes into account that the manager might face incentives to shirk and gain private benefits. By shirking, the manager affects expected returns $\mu(\hat{\eta}_t, h, \tilde{m}_t)$, as well as the distribution of returns. Because a contract’s terms are written as functions of the history of returns, the expectations in the definition involves the probability measure of returns induced by the contract $\mathcal{C}$ and by the agent’s shirking process $(\tilde{m}_t)_{t \geq 0}$. Finally, the optimality of a contract is always defined with respect to a promised value for the manager $V_0$, which can be seen as an initial outside option for the manager.

In this paper, I am interested in optimal contracts. Although verifying optimality in

---

6The logic would be slightly different in the alternative scenario in which fees are fixed. In this case, the advisor would determine the amount of actively managed assets. Investors would then supply additional capital to be invested in the passive, monitored portfolio that does not involve agency frictions. In this framework, investors determine the total size of the fund because they keep supplying capital until the fund’s total net return coincides with the interest rate. As already mentioned, these two scenarios imply identical outcomes for the size of the active portfolio and for the value of the advisor’s revenues.
the sense of Definition 4 appears intractable, the following proposition shows that it suffices to search over a restricted class of contracts.

**Proposition 2.** The optimal contract is incentive compatible with full effort, that is, \( m_t = 0 \) for all \( t \).

The intuition for this proposition is the following. First, as already discussed, we lose no generality if we restrict attention to incentive-compatible contracts. Second, optimal contracts must align the manager’s incentives to the advisor’s objectives. Because shirking is inefficient and the advisor obtains revenues from the fund’s value added, it is intuitive that the optimal contract will be designed to implement full effort.

From an operational point of view, Proposition 2 asserts that, in searching for an optimal contract, we can simply derive the conditions under which the manager has no incentive to shirk and, within the class of contracts that satisfy this condition, we can select the optimal one. In the remainder of this section, I derive the conditions that an optimal contract must satisfy. I use these conditions in section 4 to derive the optimal contract.

### 3.4 Incentive Compatibility and Information Rent

In the previous subsection, I argued that we can restrict our attention to contracts that are incentive compatible and that induce the manager to exert full effort. In this subsection, I provide conditions for a contract to achieve this target. As in static principal-agent problems, these conditions require the manager to be exposed to some fund-level risk. Whereas in static models the principal exposes the agent to risk by giving him a performance-contingent pay, in a dynamic model the principal exposes the agent to risk by adjusting his future continuation value.

To design an optimal contract, the advisor needs to account for the incentives of the manager. At any time \( t \), the manager has incentives to shirk because he obtains consumption value from shirking. However, he could be deterred from shirking if shirking causes a loss in future utility. The manager’s future utility coincides with his continuation value \( V_t \), which is a function of the continuation contract\(^7\) and belies, and which can be expressed as

\[
V_t = V(\xi_t, \phi_t) = E \left[ \int_t^\infty e^{-\delta(s-t)} u(C_s) \, ds \bigg| \mathcal{F}_t \right], 
\]

\(^7\)A continuation contract a time \( t \), \( \xi_t \), is a set of \( \mathcal{F}_s \)-adapted processes \((C_s)_{s \geq t}, (K_s)_{s \geq t}, (\eta_s)_{s \geq t}, (m_s)_{s \geq t}\), where \( \mathcal{F}^t_s = \{(R_u)_{t \leq u \leq s}\} \).
for an incentive-compatible contract enforcing full effort. The continuation value \( V_t \) represents the present value of the future utility that the manager expects to receive from the contract, given his current beliefs \( \phi_t \).

Using the martingale representation approach developed in previous literature (Sannikov, 2008; Williams, 2009), I obtain the law of motion for the manager’s continuation value \( V_t \).

**Lemma 2.** The manager’s continuation value evolves as

\[
dV_t = (\delta V_t - u(C_t))dt + \beta_t dW_t^C
\]

for some \( \mathcal{F}_t \)-adapted process \((\beta_t)_{t \geq 0}\).

The proof of Lemma 7 is standard and it is therefore omitted.

If \( \beta_t \) is different from zero, the manager is facing a risky consumption path, which is inefficient. In a fully efficient allocation, the risk-neutral investors and advisor should fully insure the risk-averse manager. However, because of moral hazard, the advisor designs a performance-based contract \( C \) that exposes the manager to some risk in order to provide incentives. If \( \beta_t \) is positive, the utility of the manager increases if he delivers a return that exceeds expectations. If the manager shirks, expected returns decline and the manager suffers a loss of future utility.

As a benchmark, assume for the moment that the skill of the manager, \( h \), is known. In this case, the advisor can prevent shirking by offering a contract in which the manager’s exposure to returns, \( \beta_t \), offsets the marginal consumption value of shirking.

**Lemma 3.** If \( h \) is common knowledge, a necessary and sufficient condition for incentive compatibility is

\[
u'(C_t)\lambda \sigma K_t \leq \beta_t.
\]

A proof for Lemma 3 can be found in Di Tella (2017) and Di Tella and Sannikov (2018).

The condition in Lemma 3 is intuitive. If the manager shirks, he reduces returns by \( m_t \), and hence \( dW_t^C \) acquires a negative drift \(-\frac{m_t}{\sigma}\). The manager therefore suffers a loss of continuation value equal to \( \beta_t \frac{m_t}{\sigma} \). However, his current utility increases from \( u(C_t) \) to

---

\(^8\)Given a continuation contract \( C_t \), beliefs are a sufficient statistic for the probability measure of future returns. To see why, write

\[
V_t = (1 - \phi_t)E \left[ \int_t^\infty e^{-\delta(s-t)}u(C_s) \, ds \bigg| h = 0 \right] + \phi_t E \left[ \int_t^\infty e^{-\delta(s-t)}u(C_s) \, ds \bigg| h = 1 \right].
\]

The conditional expectations on the right-hand side of this equation are functions of the continuation contract \( C_t \) only.
The condition in Lemma 3 ensures that $m_t = 0$ is the best response of the manager to the contract, that is,

$$0 = \arg \max_{m \geq 0} \left( u(C_t + m \lambda K_t) - \beta_t \frac{m}{\sigma} \right).$$

If the skill of the manager is uncertain, the advisor faces some additional challenges in designing an incentive-compatible contract. For example, suppose the manager deviates to $m_t = m > 0$ for a small amount of time between $s$ and $s + \Delta s$. With learning, the manager not only gains consumption value from the deviation, but he also earns an informational advantage over the advisor and the investors. Unaware of the manager’s deviation, the advisor and the investors update beliefs according to equations (4) and (5). Because the true drift of returns is now $\mu(\eta_t, h, m)$, their beliefs, $\phi_t$, will acquire a negative drift relative to the manager’s beliefs, $\bar{\phi}_t$. Immediately after the deviation, the difference between the manager’s and the other players’ beliefs, $\bar{\phi}_t - \phi_t$, will be given by

$$\bar{\phi}_{s+\Delta s} - \phi_{s+\Delta s} \approx \eta_s \phi_s (1 - \phi_s) \frac{m}{\sigma} \Delta s.$$

This difference in beliefs is persistent and causes persistent distortions in the provision of incentives. Following a deviation, the manager will always be more optimistic than the other players, that is, $\bar{\phi}_t > \phi_t$ for all $t > s + \Delta s$. The manager is not only more optimistic, but also aware of possessing correct beliefs. By having more accurate and optimistic beliefs, the manager earns an information rent over the other players. Intuitively, a skilled manager who is believed to be unskilled is better off than a manager who is actually unskilled. The skilled managers can expect to surprise the market in the future thanks to his superior skill. The truly unskilled manager cannot expect to surprise anyone. To see how a manager earns an information rent, consider equation (7). If $\bar{\phi}_t > \phi_t$, shocks $\frac{1}{\sigma} \left[ dR_t - (r + \mu(\eta_t, \phi_t, 0))dt \right]$ have a positive drift given by

$$\eta_t (\bar{\phi}_t - \phi_t) dt > 0.$$

Therefore, the manager’s continuation value acquires a positive drift

$$\beta_t \eta_t (\bar{\phi}_t - \phi_t) dt.$$

The drift $\beta_t \eta_t (\bar{\phi}_t - \phi_t)$ captures the surprise that the manager expects other players to receive. Suppose that, after a (hidden) deviation, the advisor promises a continuation
value $V_{t+\Delta s}$ to the manager. However, the true continuation value of the manager is larger than what the advisor explicitly promises, and it includes the additional surplus that accrues to the manager through the future drift $\beta_t \eta_t (\phi_t - \phi_t)$. The present value of this additional surplus constitutes the information rent that the manager has over the advisor and investors.

This reasoning suggests that when players are learning, an incentive-compatible contract should provide incentives $\beta_t$ to offset the manager’s marginal utility of shirking as well as the information rent that he could earn. I formalize this argument using the stochastic maximum principle introduced in the contract-design literature by Williams (2011).

**Proposition 3.** In any optimal contract,

$$u'(C_t)K_t \lambda \sigma \leq \beta_t - \eta_t \xi_t,$$  \hspace{1cm} (8)

where $\xi_t$ follows

$$d\xi_t = (\delta \xi_t - \eta_t \beta_t \phi_t (1 - \phi_t)) dt + \omega_t dW_t^c$$  \hspace{1cm} (9)

for some $\mathcal{F}_t$-adapted process $(\omega_t)_{t \geq 0}$.

The term $\eta_t \xi_t$ in the incentive-compatibility condition (8) accounts for the information rent that accrues to the manager after a deviation. To interpret the variable $\xi_t$ more clearly, consider expression (6). In particular, consider the fact that at any point in time, the manager’s continuation value is a function of the continuation contract $C_t$ and his beliefs $\phi_t$, that is $V_t = V(C_t, \phi_t)$. For a given contract, $\partial_\phi V(C_t, \phi_t)$ measures the marginal change in continuation value coming from a marginal change in beliefs. The variable $\xi_t$ is related to this marginal of beliefs, and therefore can be defined as the information rent of the manager.

**Proposition 4.**

$$\xi_t = \phi_t (1 - \phi_t) \partial_\phi V(C_t, \phi_t).$$ \hspace{1cm} (10)

Moreover,

$$\xi_t = \mathbb{E} \left[ \int_t^\infty e^{-\delta(s-t)} \eta_s \beta_s \phi_s (1 - \phi_s) ds \bigg| \mathcal{F}_t \right],$$ \hspace{1cm} (11)

and in any incentive-compatible contract, $\xi_t \geq 0$.

I can now give an intuitive interpretation of the incentive-compatibility condition (8). First, from equation (7), we know that incentives $\beta_t$ correspond to the total volatility of the continuation value $V_t$. Second, combining (4) and (10), we see that $\eta_t \xi_t = \eta_t \phi_t (1 - \phi_t)$. 
\( \phi_t ) \partial_{\phi} V( \mathcal{C}_t, \phi_t ) \) represents the volatility of the continuation value that originates from the changes in beliefs. Therefore, we can interpret the quantity

\[
\beta_t - \eta_t \xi_t = \beta_t - \eta_t \phi_t (1 - \phi_t) \partial_{\phi} V( \mathcal{C}_t, \phi_t )
\]
as the volatility of the manager’s continuation value that originates from changes in the continuation contract while keeping beliefs fixed.

Seen from this perspective, the incentive-compatibility condition (8) is extremely intuitive. Equation (8) states that, to provide incentives to the manager, the advisor cannot rely on changes in beliefs to punish him for bad performance. Although changes in beliefs do affect the volatility of the continuation value along the equilibrium path, they cannot be exploited to prevent off-equilibrium deviations. The reason is that the manager’s deviations do not affect his beliefs. After a deviation \( m_t \), the manager’s true expected future utility declines by \( (\beta_t - \eta_t \xi_t) \frac{m_t}{\sigma} \), and not by \( \beta_t \frac{m_t}{\sigma} \), as the advisor incorrectly thinks. Therefore, in an incentive-compatible contract, the quantity \( \beta_t - \eta_t \xi_t \) is what matters for incentive provisions, and must be such that

\[
0 = \arg \max_{m \geq 0} \left( u(C_t + m \lambda K_t) - (\beta_t - \eta_t \xi_t) \frac{m}{\sigma} \right).
\]

In other words, the incentive-compatibility condition asserts that, to provide incentives to the manager, the advisor must adjust the continuation contract so that, keeping beliefs constant, the incentives of the manager exceed the private benefit of shirking.

With learning, the advisor faces additional costs in providing incentives to the manager, as indicated by the result in Proposition 4 that the information rent \( \xi_t \) is positive. According to equation (8), for given consumption \( C_t \), capital \( K_t \), and experimentation \( \eta_t \), a manager with uncertain skill will have to bear an additional quantity \( \eta_t \xi_t \) of risk relative to a manager with known skill. Because the manager is risk averse, the advisor will have to compensate him for this additional risk with a higher expected compensation, thus increasing the cost of the contract.

Any contract implies an information rent for the manager, and a larger information rent implies higher costs for the advisor. Therefore, from an ex ante perspective, the advisor prefers to design a contract that implies a small information rent. To do so, she has to commit to reduce incentives, \( \beta_t \), and experimentation, \( \eta_t \), in the future because, as equation (11) shows, the manager’s information rent \( \xi_t \) is a present value of these quantities. However, from an ex post perspective, lowering incentives and experimentation may not be optimal. Consequently, in order to achieve ex ante optimality, the advisor needs to
fully commit to the terms of an initial contract. In sections 4 and 5, I study the optimal contract with full commitment and discuss how the trade-off between ex ante and ex post optimality affects the flow-performance relationship and managerial compensation.

3.5 Verifying Incentive Compatibility

I conclude this section by presenting a condition that can be used to verify the incentive compatibility of a contract. Proposition 3 offers a condition that prevents the manager from engaging in a one-shot deviation. Although this condition is necessary for incentive compatibility, alone it does not guarantee full incentive compatibility. With learning, any hidden shirking will cause a persistent wedge between the manager’s and the other player’s beliefs. Given this wedge, the condition in Proposition 3 may not be sufficient to prevent shirking. Even if (8) holds, the manager’s best response to the contract may still involve a dynamic shirking strategy.

Previous literature has long recognized the challenge that state variables, like beliefs, pose in the design of an optimal contract. The common approach is to solve for an optimal contract by imposing the necessary condition (8) only. This approach is the so called relaxed-problem approach. Then, one should verify whether the contract so obtained satisfies a sufficient incentive-compatibility condition. This strategy is the one undertaken by He (2012) and Di Tella and Sannikov (2018) for private savings, by Prat and Jovanovic (2014), DeMarzo and Sannikov (2017), He et al. (2017), and Cisternas (2018) for learning, and Williams (2011) for generic state variables. I take the same approach and use the following proposition to verify whether the solution to the relaxed problem is actually incentive compatible.

**Proposition 5.** If

\[ u'(C_t)K_t\lambda + \eta_t\xi \leq \beta_t \]

and

\[ \omega_t \geq \eta_t(1 - 2\phi_t)\xi_t, \]

then the contract is incentive compatible.

This proposition contains a sufficient condition for incentive-compatibility. In general, incentive-compatible optimal contracts must satisfy (8), but not necessarily (12). In solving for an optimal contract, I adopt the first-order approach and impose the necessary condition (8) as a constraint on the contract. It can be later verified whether such a contract satisfies condition (12).
To interpret the sufficient condition for incentive compatibility, it is useful to refer to the proof of this proposition in Appendix C, where I show that \( \omega_t - \eta_t(1 - 2\phi_t)\xi_t \) is proportional to the volatility of \( \partial_\phi V(\zeta_t, \phi_t) \). Proposition 5 therefore states that if a contract prevents instantaneous deviations and it reduces the marginal value of beliefs after a negative shock, then it is a fully incentive-compatible contract.

This result has some intuitive appeal. If the contract lowers the marginal value of the manager’s beliefs \( \partial_\phi V(\zeta_t, \phi_t) \) after a bad shock then, following a deviation, the manager would suffer a decrease not only in his continuation value, but also in his information rent. The manager loses part of the option to “impress” other players in the future, thus lowering the value of his informational advantage. Together, this condition and condition (8) are sufficient to induce the manager to always exert full effort.

Equation (12) is likely to hold in an optimal contract. Looking at Proposition 4, we see \( \xi_t \) depends on future incentives \( \beta_t \) and experimentation \( \eta_t \). After a good shock, the advisor has incentives to increase future incentives and future experimentation because of the following reasons. First, a good shock increases expected returns and the advisor will likely take advantage of them by increasing capital under management \( K_t \) and, by (8), incentives \( \beta_t \). Second, experimentation becomes more profitable, so the advisor will likely increase experimentation \( \eta_t \) as well. We therefore have reasonable economic motivations to believe that in the optimal contract, \( \xi_t \) increases after good performance.

4 Optimal Contract

Given the necessary incentive-compatibility condition in Proposition 3, I adopt a first-order approach to solve for the optimal contract under full commitment. According to the first-order approach, I solve for the optimal contract in Definition 4 as a recursive problem subject to the incentive-compatibility condition (8) at every point in time. The advisor is fully committed to the manager’s continuation value \( V_t \) and to his information rent \( \xi_t \), which therefore constitute the recursive state variables of the problem together with beliefs \( \phi_t \)\(^9\). With full commitment, the advisor always honors her past promises in terms of continuation value and information rent, and the laws of motion (7) and (9) represent promise-keeping constraints for the advisor. To provide incentives, the advisor

\(^{9}\)Although full commitment from both the manager and the advisor is certainly a strong assumption, this formulation of the problem offers a key benchmark for alternative specifications. With full commitment, the advisor implements an allocation that yields the best outcome given the frictions of the model. After relaxing the full-commitment assumption in Appendix A, I highlight which results hold independently of the contracting assumptions and which results depend instead on the particular contractual form that is implemented.
specifies how future continuation values and information rents evolve on the basis of performance. She therefore selects incentives \( \beta_t \) and volatility \( \omega_t \) optimally.

Formally, the optimal contract is characterized as a solution to the following optimization problem:

\[
J^*(V_0, \xi_0, \phi_0) = \inf_{(C_t, K_t, \beta_t, \omega_t, \eta_t)_{t \geq 0}} E \left[ \int_0^\infty e^{-rt} \left( \frac{C_t}{q(\eta_t)} - K_t \mu(\eta_t, \phi_t, 0) \right) dt \right] \bigg| \mathcal{F}_0
\]

s.t. \( C_t - \rho_t k_{\rho} + \eta_t \xi_t \leq \beta_t \quad \forall \ t \geq 0 \)

\[
dV_t = \left( \delta V_t - \frac{C_1^{1-\rho}}{1-\rho} \right) dt + \beta_t dW^c_t
\]

\[
d\xi_t = (\delta \xi_t - \eta_t \beta_t \phi_t(1 - \phi_t)) dt + \omega_t dW^c_t
\]

\[
d\phi_t = \eta_t \phi_t(1 - \phi_t) dW^c_t.
\]

Problem (13) is clearly challenging to solve analytically. However, we can characterize some of its properties. For this purpose, consider its associated Hamilton-Jacobi-Bellman (HJB) equation,

\[
rJ^*(V, \xi, \phi) =
\inf_{(C, K, \beta, \omega, \eta, \phi) \in \mathcal{U}_t} \left\{ \frac{C}{q(\eta)} - (\alpha + \sigma \eta \phi)K + J^*_V(V, \xi, \phi) \left( \delta V - \frac{C_1^{1-\rho}}{1-\rho} \right) + J^*_\xi(V, \xi, \phi) (\delta \xi - \eta \beta \phi(1 - \phi)) + \frac{1}{2} J^*_{VV}(V, \xi, \phi) \beta^2 + \frac{1}{2} J^*_{\xi \xi}(V, \xi, \phi) \omega^2 + \frac{1}{2} J^*_{\phi \phi}(V, \xi, \phi) \eta^2 \phi^2(1 - \phi)^2 + J^*_{V \xi}(V, \xi, \phi) \beta \omega + J^*_{V \phi}(V, \xi, \phi) \beta \eta \phi(1 - \phi) + J^*_{\xi \phi}(V, \xi, \phi) \omega \eta \phi(1 - \phi) \right\}.
\]

First, we can verify that, in any optimal contract, the incentive-compatibility condition holds with equality, that is,

\[
K_t = \frac{\beta_t - \eta_t \xi_t}{\lambda \sigma} C_t^{\rho}.
\]

Because excess returns and fees are always positive, the advisor increases the level of assets under management as much as the incentive-compatibility constraint permits. Investors do not provide any capital beyond this level at the current fees because if the incentive-compatibility condition is violated, the manager will shirk and expected returns will not cover the fees investors pay.

Second, the HJB equation (14) is subject to a boundary condition \( J^*(V, 0, \phi) = J^0(V) \),
where the function \( J^0(V) \) is the cost function for the advisor when the manager lacks skill, that is, when \( h = 0 \) is common knowledge.\(^{10}\) This boundary condition is motivated by the promise-keeping constraint on the information rent \( \xi \). If the advisor promises zero information rent to the manager, the only way she can keep this promise is by providing no incentives \( \beta \), and hence no capital \( K \), and/or by stopping experimentation \( \eta \) forever. The advisor clearly prefers to stop experimentation only, because she can still obtain fees equal to the baseline excess return \( \alpha \) by providing capital to the manager.

We cannot obtain analogous boundary conditions for \( J^*(V, \xi, 0) \) and \( J^*(V, \xi, 1) \). The contract never reaches these boundaries in finite time, because beliefs \( \phi \) have no drift and their volatility vanishes as they approach 0 and 1. These singular points, however, do not constitute a problem. We can indeed think of the HJB equation (14) as a state constraint problem whereby beliefs are constrained between 0 and 1. Katsoulakis (1994) and Alvarez et al. (1997) show that state constraints effectively replace boundary conditions in determining the solution of partial differential equations.

Finally, we can derive an endogenous bound for the information rent \( \xi \). Note that at time 0, the advisor has no initial commitment to any information rent. As illustrated in Definition 4, the optimal contract is initially defined only in terms of the initial promised value of the manager, \( V_0 \), and initial beliefs, \( \phi_0 \). Therefore, we can think of the advisor as first setting an initial information rent \( \xi_0 \), and then solving problem (13) given the chosen \( \xi_0 \). The advisor chooses an initial information rent that minimizes her costs. If \( J^*_\xi(V, \xi, \phi) \) is convex in \( \xi \) and a global minimum exists, the advisor will pick \( \xi_0 \) such that

\[
J^*_\xi(V_0, \xi_0, \phi_0) = 0.
\]

We can further characterize the behavior of the information rent and its marginal cost \( J^*_\xi \) in the optimal contract. For any pair of continuation value \( V \) and beliefs \( \phi \), define \( \tilde{\xi}(V, \phi) \) as the information rent that minimizes costs for the advisor. I formally prove the following proposition in Appendix D. However, one could also make some convexity and differentiability assumptions on \( J^*(V, \xi, \phi) \) and directly use the envelope theorem on (14) as in DeMarzo and Sannikov (2017) to derive the following result.

**Proposition 6.** In the optimal contract, two properties hold:

\(^{10}\) \( J^0(V) \) satisfies the HJB equation

\[
r J^0(V) = \inf_{C, \beta} \left\{ C - \alpha \frac{\beta}{\sigma \lambda} C^\rho + J^0_V(V) \left( \delta V - \frac{C^{1-\rho}}{1-\rho} \right) + \frac{1}{2} J^0_{VV}(V) \beta^2 \right\}.
\]
I. The marginal cost of information rent is always non-positive,

\[ J_\xi(V_t, \xi_t, \phi_t) \leq 0 \quad t \geq 0 \]

II. For all \( t \geq 0 \),

\[ \xi_t \leq \bar{\xi}(V_t, \phi_t). \]

Information rents in contracting models tend to be bounded, and propositions analogous to 6 have been derived in DeMarzo and Sannikov (2017) and He et al. (2017). Large information rents expose the manager to risks that are irrelevant for incentive provision. Consequently, the advisor avoids promising excessive information rents.

Even after deriving the properties that I have discussed so far, numerically solving the HJB equation (14) is far from easy, if not far from feasible. Proposition 6 provides the reason. In the optimal contract, the information rent is bounded by \( \xi(V, \phi) \). For values of information rent far above this bound, optimal contracts might not exist, or they might violate regularity properties required for the validity of a dynamic programming approach. To solve for the optimal contract, we need to know the bound \( \bar{\xi}(V, \phi) \). However, to derive this bound, we need to know the optimal contract. We could attempt to numerically solve (14) by sequentially guessing and verifying the bound function \( \bar{\xi}(V, \phi) \). This attempt would be extremely computationally inefficient, if not unfeasible. I take a different approach.

Proposition 6 itself offers a hint to solve the model efficiently. In the optimal contract, the marginal cost \( J_\xi(V_t, \xi_t, \phi_t) \) is bounded above by zero, thus always satisfying the endogenous bound on the information rent. I then formulate a problem in which the marginal cost \( J_\xi(V_t, \xi_t, \phi_t) \) replaces the information rent \( \xi \) as a state variable. This problem is connected to the initial one through a duality relation. In the remainder of this section, I introduce the dual problem of (13), derive its properties, and show that the dual problem offers an efficient way to characterize the contract.

4.1 The Dual Problem

I now introduce the dual problem of (13). Informally speaking, the purpose of the dual problem is to replace the information rent, \( \xi_t \), with its marginal value, \( J_\xi(V_t, \xi_t, \phi_t) \), as a state variable.
Define the multiplier

\[ Y_t = e^{t(r-\delta)} \left[ - \left( \int_0^t e^{s(\delta-r)} \mu(\eta_t, \phi_t, 0) \frac{\eta_t}{\lambda \sigma} C^\rho_t \, ds \right) + Y_0 \right], \]

where \( Y_0 \in \mathbb{R} \), and consider the following problem:

\[
G^*(V_0, Y_0, \phi_0) = \inf_{(C_t, \beta_t, \mu_t, \eta_t), \forall t \geq 0} \mathbb{E} \left[ \int_0^\infty e^{-\gamma t} \left( \frac{C_t}{q(\eta_t)} - \mu(\eta_t, \phi_t, 0) \frac{\beta_t}{\lambda \sigma} C^\rho_t - Y_t \eta_t \beta_t \phi_t (1 - \phi_t) \right) \, dt \mid \mathcal{F}_0 \right]
\]

s.t.

\[
\begin{align*}
&dV_t = (\delta V_t - u(C_t))dt + \beta_t dW^e_t \\
&dY_t = (r - \delta)Y_t \mu_t \eta_t \phi_t (1 - \phi_t)dt + \eta_t \phi_t dW^e_t \\
&d\phi_t = \eta_t \phi_t (1 - \phi_t) dW^e_t.
\end{align*}
\]  

(16)

I call (16) the dual problem, as opposed to problem (13), which, hereafter, I call the primal problem.

The dual problem in (16) has an intuitive appeal. Consider the objective function. The term \( \frac{C_t}{q(\eta_t)} - \mu(\eta_t, \phi_t, 0) \frac{\beta_t}{\lambda \sigma} C^\rho_t \) represents the flow cost of an advisor who runs a fund with no information-rent problem. We can think of this hypothetical fund as one in which the hidden action \( m_t \) is observable but not contractible. In this case, the incentive-compatibility condition is the same as in Lemma 3, because shirking does not induce any belief distortion or any information rent for the manager. The term \( -Y_t \eta_t \beta_t \phi_t (1 - \phi_t) \) represents a penalty for an advisor who sets large incentives \( \beta_t \) or large experimentation rate \( \eta_t \). The severity of the penalty is measured by the multiplier \( Y_t \). The advisor faces a larger penalty for incentives and experimentation if the multiplier \( Y_t \) is more negative. In the dual problem (16), the advisor replaces the information rent of the manager with the multiplier \( Y \) as a relevant state variable.

Having introduced the dual problem, I can finally show the connection with the primal problem (13) and illustrate how the optimal contract can be derived as a solution to the dual problem (16).

**Proposition 7.** The optimal contract is the solution to the dual problem (16) with \( Y_0 = 0 \). The primal cost function \( J^* \) and the dual cost function \( G^* \) are related by

\[
J^*(V_t, \xi_t, \phi_t) = \sup_{Y \leq 0} \{ G^*(V_t, Y, \phi_t) + Y \xi_t \}. \tag{17}
\]

At any time \( t \), the optimal contract implies an information rent \( \xi_t = -G^*_t(V_t, Y_t, \phi_t) \). Moreover,
\[ Y_t = J_{\xi}^*(V_t, \xi_t, \phi_t), \text{ and } Y_t \leq 0 \text{ for all } t \geq 0. \]

Proposition 7 is extremely powerful. It states that any optimal contract can be obtained as a solution to the dual problem if the initial multiplier is chosen appropriately. Moreover, by combining Proposition 7 with Proposition 6, we obtain a recursive characterization of the optimal contract through a dual formulation that automatically satisfies the endogenous bounds on the information rent.

This proposition is key to overcome the computational challenges that I discussed before introducing the dual problem. Looking at the law of motion of the multiplier \( Y_t \) in (16), we see that, for any choice of \((C_t, \beta_t, \eta_t)_{t \geq 0}\), \( Y_t \) always be non-positive as long as \( Y_0 \leq 0 \). Because \( Y_t = J_{\xi}^*(V_t, \xi_t, \phi_t) \), if I restrict the state space to non-positive value of the multiplier \( Y \), then the marginal cost of information rent is guaranteed to remain non-positive. Therefore, standard numerical methods are sufficient to obtain a solution for the optimal contract.

Before numerically solving the dual problem, I exploit the homogeneity of the problem and introduce some notation that will simplify the numerical computations and the discussion of the results. On the basis of Lemma 10 in Appendix D, I can simplify the numerical burden of solving for problem (16) by reducing the number of state variables from three to two. In particular, we can write the dual cost function as

\[ G^*(V, Y, \phi) = \hat{v}^* (y, \phi) , \]

where

\[ \hat{v} = ((1 - \rho)V)^{\frac{1}{1-\rho}} \]

is the consumption equivalent of the manager’s continuation value, and where

\[ y = (1 - \phi)\hat{v}^{-\rho}Y \]

is a scaled version of multiplier \( Y \). We can then define the scaled control variables

\[ c_t = \frac{C_t}{\hat{v}_t}, \quad k_t = \frac{K_t}{\hat{v}_t}, \quad \text{and} \quad \hat{\beta}_t = \frac{\beta_t}{(1 - \rho)V_t}, \]

and derive the law of motion of continuation value \( \hat{v} \) and multiplier \( y \) through Ito’s lemma, thus obtaining

\[ \frac{d\hat{v}_t}{\hat{v}_t} = \left( \frac{\delta}{1-\rho} - \frac{c^{1-\rho}_t}{1-\rho} + \frac{1}{2} \rho \hat{\beta}_t^2 \right) dt + \hat{\beta}_t \hat{v}_t dW_t^e \] (18)
\[ dy_t = -(1 - \phi_t)\mu(\eta_t, \phi_t, 0) \frac{\eta_t}{\lambda \sigma} c_t \, dt \]
\[ + y_t \left( r - \frac{\delta}{1 - \rho} + \rho \frac{c_t^{1 - \rho}}{1 - \rho} + \frac{1}{2} \rho \beta_t^2 + \rho \eta_t \hat{\phi}_t \right) \, dt - y_t [\rho \beta_t + \eta_t \phi_t] \, dW^C_t. \] (19)

In conclusion, instead of solving for \( G^*(V, Y, \phi) \) as a function of three state variables, I solve the following HJB equation, which characterizes \( g^*(y, \phi) \) as a function of the multiplier \( y \) and belief \( \phi \):

\[
rg^*(y, \phi) = \inf_{c, \beta, \eta} \left\{ \frac{c}{q(\eta)} - (\alpha + \sigma \eta \phi) \hat{\beta} \frac{c^\rho}{\sigma \lambda} - y \hat{\beta} \eta \phi 
\right. \\
+ g^*(y, \phi) \left( \frac{\delta}{1 - \rho} - \frac{c^{1 - \rho}}{1 - \rho} + \frac{1}{2} \rho \beta^2 \right) \\
+ g^*_y(y, \phi) \left[ -(1 - \phi_t)\mu(\eta_t, \phi_t, 0) \frac{\eta_t}{\lambda \sigma} c_t \right. \\
\left. + y \left( r - \frac{\delta}{1 - \rho} + \rho \frac{c_t^{1 - \rho}}{1 - \rho} + \frac{1}{2} \rho \beta_t^2 + \rho \eta_t \hat{\phi}_t \right) \right] \\
- g^*_y(y, \phi) y [\rho \beta + \eta \phi] \frac{\beta}{\rho} + g^*_\phi(y, \phi) \eta (1 - \phi) \beta \\
+ \frac{1}{2} g^*_{yy}(y, \phi) y^2 [\rho \beta + \eta \phi]^2 + \frac{1}{2} g^*_{\phi\phi}(y, \phi) \eta^2 \phi^2 (1 - \phi)^2 \\
- g^*_y(y, \phi) y [\rho \beta + \eta \phi] \eta (1 - \phi) \right\}. \] (20)

In the next section, I numerically solve the HJB equation (20) and characterize the implications of the optimal contract for the fund flows and the manager’s compensation. In interpreting the results, readers may refer to the HJB equation (20), to the dynamics of promised value (18), and to the dynamics of the multiplier \( y \) (19). However, in my interpretation of the results, I mostly rely on economic intuition and make only minimal references to the specific details of these equations.

5 Results and Discussion

I numerically solve the partial differential equation (20) by using a finite difference method. To solve for the optimal contract, I use Proposition 7 and restrict the state space to negative values of the multiplier \( y \). Because at \( y = 0 \) the drift of multiplier \( y \) is negative and its volatility is 0, I do not need to impose any restrictions on the control variables to satisfy
Table 1: Model Parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Notation</th>
<th>Value</th>
<th>Source/Target</th>
</tr>
</thead>
<tbody>
<tr>
<td>Interest rate</td>
<td>$r$</td>
<td>2.6%</td>
<td>Average real rate from 2000</td>
</tr>
<tr>
<td>Baseline excess return</td>
<td>$\alpha$</td>
<td>0.1%</td>
<td>Minimum fee in Pastor et al. (2015)</td>
</tr>
<tr>
<td>Baseline volatility</td>
<td>$\sigma$</td>
<td>18%</td>
<td>High vol funds, Pastor et al. (2015)</td>
</tr>
<tr>
<td>Discount rate of manager</td>
<td>$\delta$</td>
<td>5%</td>
<td>Di Tella and Sannikov (2018), DeMarzo et al. (2012)</td>
</tr>
<tr>
<td>Risk aversion / $\rho$</td>
<td></td>
<td>1/3</td>
<td>Di Tella and Sannikov (2018)</td>
</tr>
<tr>
<td>Bound on learning</td>
<td>$\tilde{\eta}$</td>
<td>1%</td>
<td>High alpha funds, Fama and French (2010)</td>
</tr>
<tr>
<td>Cost of learning</td>
<td>$\bar{\eta}$</td>
<td>0.15</td>
<td></td>
</tr>
<tr>
<td>Curvature of learning cost</td>
<td>$d$</td>
<td>2.1</td>
<td>Convexity of cumulative flows</td>
</tr>
<tr>
<td>Gains from shirking</td>
<td>$\lambda$</td>
<td>0.85</td>
<td>Flow-performance slope</td>
</tr>
</tbody>
</table>

the state constraint $y \leq 0$.

5.1 Calibration

To select the parameters of the model, I rely on three main strategies. If a parameter can be directly observed, I use empirical observations of that parameter value. If a parameter cannot be directly observed, but has a strong connection to an outcome of my model that can be observed in the data, I select values of the parameter that yield outcomes that match the data. Finally, I discipline unobservable preference parameters by selecting values previously adopted in the contracting literature. Table 1 summarizes the parameter choice.

I set the interest rate $r$ equal to the average real rate starting from the year 2000. To select a value for the baseline excess return, $\alpha$, I consider the sample of mutual funds that I select in section 6. As in Pastor et al. (2015), I exclude all funds that charge annual fees below 0.1%, because they are unlikely to be actively managed. I therefore take this value to represent the minimum excess return that a fund is able to provide.

Due to the homogeneity of the model, the volatility of returns must be large enough to guarantee the existence of a finite solution for the contracting problem. Intuitively, if the volatility of returns is too low, the advisor could easily detect shirking. She could therefore easily provide incentives to the manager and could exploit the constant returns to scale of the technology to gain unbounded revenues. By adding decreasing returns in actively managed assets, as in Berk and Green (2004), I could avoid the possibility of unbounded returns and would therefore have additional degrees of freedom in the choice of parameters. Unfortunately, decreasing returns would complicate the problem substantially, because the homogeneity property of the dual cost function $G^*$ would fail. I therefore choose a value of the volatility $\sigma$ of 18%, which is close to, but a little larger
Figure 1: Slope of the flow-performance relationship and capital in the optimal contract. The parameters of the model are as in Table 1.

than, the highest percentile of the standard deviation of abnormal returns in Pastor et al. (2015), which is 16.4% on an annualized basis.

I choose the manager’s preference parameters to match choices made in previous contracting literature. I set the discount rate of the manager, $\delta$, equal to 5% as in DeMarzo et al. (2012) and Di Tella and Sannikov (2018), while for the inverse of the manager’s IES, I select $1/3$ as in Di Tella and Sannikov (2018).

I assume the experimentation cost is a power function of experimentation, that is, $q(e) = 1 - \bar{q}e^d$ for some parameters $\bar{q}$ and $d$. I set $d$ to 2.1 to obtain sufficient convexity of cumulative flows in cumulative performance. I then consider the distribution of four-factor gross alpha in Fama and French (2010). They find that the 90th percentile of the distribution is 1.3%. I take this number to represent the excess returns of highly-skilled managers. I therefore set $\bar{q} = 0.15$ and $\bar{\eta} = 1\%$ so that a fund with $\phi = 1$ and multiplier $y = 0$ generates expected of 1.3%, and so that the constraint $\eta \leq \bar{\eta}$ does not bind.

Finally, I set the gains from shirking, $\lambda$, to 0.85. In this way, the slope of the flow-performance relationship for funds with $\phi = 0.5$ and with multiplier $y \in [-0.1, 0]$ approximately matches the value of 17% estimated in Table 4.
Figure 2: Convex relation between cumulative flows and cumulative returns. The solid blue curve represents the cumulative flows as a function of cumulative performance. The dashed red line is the tangent line at 0. Curves are shifted to represent the flows relative to a fund that has a zero cumulative performance. Performance and flows are computed over one year. In 2a, fees are assumed to be fixed and flows include changes in actively managed capital, as well as changes in the holdings of a passive, agency-free index. In 2b, fees match expected returns, and flows include only changes in actively managed capital. Figures are drawn for initial beliefs $\phi_0 = 0.75$ and initial multiplier $y_0 = 0$. The parameters of the model are as in Table 1.

5.2 Flows and Performance

I define the slope of the flow-performance relationship, $\epsilon_K$, as the percentage change in assets under management for a 1% return, that is,

$$\epsilon_K = \frac{dK_t/K_t}{dR_t}.$$

Using the fact that in the optimal contract, $K_t = \hat{\nu}_t k(y_t, \phi_t)$ for an optimal capital-to-value ratio $k(y_t, \phi_t)$, the slope of the flow-performance relationship can be measured as

$$\epsilon_K(y, \phi) = \frac{1}{\sigma} \left( \frac{\sigma_k(y, \phi)}{k(y, \phi)} + \hat{\beta}(y, \phi) \right),$$

where $\sigma_k(y, \phi)$ is the volatility of $k(y, \phi)$.

As Figure 1a shows, the slope of the flow-performance relationship $\epsilon_K$ is positive, meaning that assets under management increase after good performance. This result is consistent with several empirical results.

The prediction that sets my model apart from previous theoretical literature, such as Berk and Green (2004), is that the slope of the flow-performance relationship increases...
after good performance. We can see this result in Figure 1a, where the slope of the flow-performance relationship, $\epsilon_K$, increases when the posterior $\phi$ and the multiplier $y$ increase. These two state variables have positive volatility (recall equations (4) and (19), and recall that $y \leq 0$). Therefore, the slope of the flow-performance relationship $\epsilon_K$ increases after good performance. This prediction is new to the literature. Because no other theoretical or empirical paper has studied how the flow-performance relationship depends on the history of a fund’s returns, I test this prediction in Section 6 using mutual fund data.

Because the slope of the flow-performance relationship $\epsilon_K$ increases in past performance, cumulative flows are a convex function of cumulative performance. As an illustration, Figure 2 shows the one-year flow into a fund with $y = 0$ and $\phi = 0.75$ as a function of the one-year cumulative return. To make the example more empirically relevant, in Figure 2a, I assume the fund charges a fixed fee and collects additional capital to be invested in a costless, agency-free index. We can clearly observe that the net flows into the fund are a convex function of the cumulative returns. As good performance accumulates over time, additional positive returns have a stronger impact on additional flows. As a result, we observe convexity on a yearly basis. In Figure 2b, I allow the fund to change fees in order to match its expected returns. The flows in Figure 2b thus represent changes in the actively managed assets. Even flows of actively managed capital increase more than linearly with cumulative performance.

**Economic Motivation.** Why should the slope of the flow-performance relationship increase after good performance? In general, the result relies on two properties of the model: (i) Assets under management, $K_t$, are positively related to the promised value of the manager, $\hat{v}_t$; (ii) the growth rate of the promised value becomes more volatile when $\phi$ and $y$ increase.

Thanks to the functional assumptions in the model, the relation between assets under management and promised value takes a particularly simple form, $K_t = k_t \hat{v}_t$. However, this relation can be more general. We need two assumptions to obtain a positive relation between assets under management and promised value. First, we need to assume that assets under management enter in the incentive-compatibility constraint and that, to increase assets, the advisor needs to expose the manager to more risk. Second, we need to assume the manager has decreasing absolute risk aversion.

According to the first assumption, if the advisor wants to increase assets, she also has to increase the risk exposure of a risk-averse manager. According to the second assumption, if the manager is promised larger future consumption, he will be more risk tolerant. As a result, the advisor optimally offers more risk and capital to a manager with higher
promised value and, hence, higher risk tolerance. This result justifies why assets under management, $K_t$, are positively related to the promised value of the manager, $\hat{v}_t$. An analogous result holds in the dynamic contracting models in Biais et al. (2010) and DeMarzo and Fishman (2007). These papers show that investments (and disinvestments) at firm level depend on the agent’s promised value. Similar to my model, when the agent has a larger promised value, the principal can easily incentivize him to exert more effort and manage a larger firm.

Given this relation between assets and promised value, the percentage change in assets under management for a 1% return is related to the percentage change in the manager’s promised value for a 1% return, which coincides with $\hat{\beta}/\sigma$ (see equation (18)). I then need to discuss why the growth rate of the promised value becomes more volatile when $\phi$ and $y$ increase. To understand why, let us fix a level of risk tolerance for the manager by fixing a promised value $\hat{v}$. In deciding how much risk $\hat{\beta}$ to assign to the manager, the advisor faces a trade-off. The advisor would like to increase the manager’s risk $\hat{\beta}$ in order to increase capital $k$ and fee revenues. However, doing so is costly for two reasons. First, the advisor has to compensate the risk-averse manager for the additional risk by promising more future consumption. Second, the advisor is committed to low information rents through the multiplier $y$ which penalizes the provision of strong incentives to the manager.

When beliefs $\phi$ and multiplier $y$ increase, the trade-off tilts in favor of a higher volatility of the manager’s promised value, $\hat{\beta}$. When beliefs $\phi$ are higher, expected returns and fees, $\mu(\eta, \phi, 0)$, are also higher, so that the benefits of size $k$ and incentives $\hat{\beta}$ increase. As for the cost of the manager’s incentives, they decline through two channels. First, because the manager is expected to be more productive (beliefs $\phi$ increase) and the advisor less constrained (the multiplier $y$ increases), the advisor faces lower costs in delivering future consumption promises. Therefore, the advisor can more cheaply compensate the manager for any additional risk that he has to bear. Second, the commitment of the advisor to low information rents is less binding (the multiplier $y$ increases). Because the benefits of size increase and the cost of incentives decrease, when $\phi$ and $y$ increase the advisor increases the volatility of the manager’s promised value, $\hat{\beta}$, together with $k$. This result is illustrated in Figure 1b.

The mechanism that I have discussed so far highlights the dynamic connection between capital flows and managerial incentives. Whereas learning links past performance to expected returns in a standard way, the optimal contract links the slope of the flow-performance relationship to expected returns because of a dynamic trade-off between the costs and the benefits of managerial incentives.
5.3 Pay and Performance

Similar to capital, I define the pay-performance sensitivity as the percentage change in compensation for a 1% increase in returns, that is,

$$\epsilon_{\tilde{C}} = \frac{d\tilde{C}_t}{d\tilde{R}_t}. $$

Because in the optimal contract, compensation takes the form $\tilde{C}_t = \hat{v}_t \tilde{c}(y_t, \phi_t)$ for an optimal compensation-to-value ratio $\tilde{c}(y, \phi) = c(y, \phi)/q(y, \phi)$, pay-performance sensitivity takes the form

$$\epsilon_{\tilde{c}}(y, \phi) = \frac{1}{\sigma} \left( \frac{\sigma_{\tilde{c}}(y, \phi)}{\tilde{c}(y, \phi)} + \tilde{\beta}(y, \phi) \right),$$

where $\sigma_{\tilde{c}}(y, \phi)$ is the volatility of $\tilde{c}(y, \phi)$.

From Figure 3a, we can observe that the contract implies a bonus for good performance. Because the value of the pay-performance sensitivity, $\epsilon_{\tilde{c}}$, is always positive, compensation increases when the manager realizes good returns. Moreover, because the pay-performance sensitivity $\epsilon_{\tilde{c}}$ is increasing in both beliefs $\phi$ and the multiplier $y$, compensation will appear convex in cumulative performance. This result is consistent with the widespread use of convex compensation schemes in the money management industry. For example, Ma et al. (2019) document that mutual fund managers’ compensation is often composed of a base salary plus a bonus for good performance. The mechanism driving the convexity of cumulative compensation with respect to cumulative performance is
identical to the one behind the convexity of cumulative capital flows.

In addition to the pay-performance sensitivity, we can study whether performance-based compensation is back-loaded or front-loaded, by looking at how current compensation changes relative to future promises, that is, by looking at how $C/\hat{v}$ changes with performance. If $C/\hat{v}$ decreases with beliefs $\phi$ and multiplier $y$, we say the performance-based compensation is back-loaded. In this case, after good performance, the advisor increases future promised consumption more than current compensation, thus deferring compensation to the future. Similarly, if $C/\hat{v}$ increases with beliefs $\phi$ and multiplier $y$, we say the performance-based compensation is front-loaded.

In the optimal contract, the advisor back-loads the performance-based compensation of the manager, as shown in Figure 3b. This result is consistent with common practices in the mutual fund industry that effectively postpone the delivery of compensation to the manager. For example, Ma et al. (2019) show that more than 30% of mutual fund managers are subject to deferred compensation schemes. At the same time, they illustrate that the vast majority of the pay-for-performance schemes rely on the average return over multiple years (on average, three years) in order to determine the bonus. This latest feature effectively implies that, following good performance, the manager can expect an increase in compensation for the next few years, thus effectively back-loading his compensation.

**Economic Motivation.** Due to the agency frictions, the risk-averse manager faces a risky compensation scheme that is based on his performance. After good performance, the advisor rewards the manager with higher compensation, whereas after bad performance, the manager is punished with lower compensation. However, the riskiness of the compensation must be limited in an optimal contract, because the manager is risk averse. With the exception of highly productive managers, who are subject to very steep incentives, managers face a compensation path that is smoother than returns. This result is consistent with the low point estimates of the pay-performance sensitivity in Ibert et al. (2018).

Similar to the slope of the flow-performance relationship, the pay-performance sensitivity increases after good performance. The motivation is analogous to the one that I have presented for the slope of the flow-performance relationship. Compensation is related to the manager’s promised value. This time, however, this relation is based on the manager’s desire to smooth consumption. After good performance, the advisor expects higher returns from the manager, and therefore desires to allocate more capital to the manager. However, to ensure incentive compatibility with a larger amount of assets un-
der management, the advisor needs to expose the manager to a riskier promised value. Given that compensation is related to promised value, a riskier promised value for the manager translates into riskier compensation.

To understand why the advisor back-loads the manager’s performance-based compensation, let us consider the trade-off that compensation involves. To study this trade-off, fix a level of experimentation $\eta$ and, hence, a level of cost $q(\eta)$, so that we can think of compensation and consumption as proportional. By decreasing the consumption-to-promised-value ratio, $c$, the advisor increases the growth rate of the manager’s promised value (see (18)). A higher promised value involves a trade-off. On the one hand, the advisor faces higher costs from a larger promised value, because she has to deliver larger future compensation. On the other hand, the advisor benefits from a larger promised value, because if the manager has a large promised utility, he can tolerate high levels of risk. Therefore, the advisor can exploit this higher risk tolerance to expose the manager to more risk and increase assets under management and fee revenues.

After a good return, the trade-off tilts in favor of a larger continuation value and thus of a lower consumption-to-promised-value ratio $c$. On the one hand, the advisor can now deliver any given promised utility at a lower cost because, after a good shock, the manager is expected to be more productive (beliefs $\phi$ increase) and the commitment of the advisor to low information rents is less binding (multiplier $y$ increases). On the other hand, because the advisor expects higher returns from the manager, she has stronger incentives to increase the manager’s assets under management. Hence, the advisor desires to increase the risk tolerance of the manager through a higher promised value. Consequently, after good performance, the advisor reduces the ratio of consumption to promised utility in order increase the growth rate of the promised value. By doing so, the advisor back-loads the performance-based compensation of the manager.

5.4 Long-Term Implications and the Dynamics of Multiplier $y$

A full-commitment contract bears implications for flows, incentives, and performance in the long run. While beliefs $\phi$ are martingales, the multiplier $y$ drifts down over time, as shown in Figure 4a. The negative drift of $y$ is a robust outcome that I find across all the parameterizations that I have explored. To understand the implications of a negative drift of $y$ in terms of flows, incentives, and performance, recall that we can interpret $-y$ as a penalty for incentives and learning. If the multiplier $y$ becomes more negative, the advisor faces a large penalty for incentives, $\hat{\beta}$, and for learning, $\eta$. Consequently, by giving
a negative drift to $y$, the advisor is committing herself to reduce incentives and expected performance over time. 

By reducing incentives $\hat{\beta}$ over time, the advisor reduces the slope of the flow-performance relationship and the pay-performance sensitivity. Figures 1a and 3a show that for increasingly more negative values of the multiplier $y$, flows and pay react increasingly less to performance. Moreover, for very negative values of the multiplier $y$, the slope of the flow-performance relationship is barely affected by past performance, thus suggesting the history dependence of the slope is, on average, substantially smaller for managers with long tenure. I test this prediction in section 6.3.

By reducing experimentation $\eta$ over time, the advisor also reduces the productivity of the manager. In Figure 5, we see that for very negative values of the multiplier $y$, the advisor requires low experimentation $\eta$ from the manager, which results in lower expected returns. Almazan et al. (2004) show that older fund managers are subject to more investment constraints, which, effectively, may limit the informativeness of their returns. As long as age and tenure are correlated, my model could provide one additional framework to interpret these empirical findings. Consistently with an optimal contract, more experienced managers may simply be required to undertake less sophisticated investments and to be subject to lower risk.

**ECONOMIC MOTIVATION.** To understand why the advisor wants to reduce the incentives and the experimentation of the manager over time, let us consider the incentive constraint (8) and the manager’s information rent. The advisor faces a trade-off between
(a) Experimentation

(b) Expected excess return

Figure 5: Experimentation and performance in the optimal contract. The parameters of the model are as in Table 1.

ex ante and ex post efficiency. From an ex ante perspective, the advisor wants to minimize the manager’s information rent. The information rent is costly, because it represents a risk that the advisor cannot exploit to provide incentives to the risk-averse manager. However, from equation (11), the manager’s information rent corresponds to the present value of future incentives and experimentation. If the advisor designs a contract with a small information rent, she has to commit to future experimentation \( \eta \) and incentives \( \hat{\beta} \) that are inefficiently low from an ex post perspective. The commitment of the advisor to these ex post inefficient incentives and experimentation is captured by the multiplier \( y \). By giving a negative drift to the multiplier, the advisor increases the penalty for incentives and experimentation over time. To sum up, in designing the ex ante optimal contract, the advisor commits to decreasing incentives and learning over time through the negative drift of multiplier \( y \). By taking ex post inefficient actions, she can reduce the ex ante information rent of the manager, and thus relax the incentive-compatibility constraint (8).

Full commitment by the advisor is crucial for this result. After any period of time, the advisor is tempted to renegotiate the contract, leaving the manager with an unchanged continuation value, but increasing his information rent to the point at which the multiplier \( y \) is equal to 0. In Appendix A, I study contracts in which advisor does not commit to an information rent. As expected, incentives \( \hat{\beta} \) and experimentation \( \eta \) will simply be a function of beliefs \( \phi \), because no other state variable enforces ex post inefficient choices of incentives and experimentation. However, all the other predictions of the model about the flow-performance relationship and the manager’s compensation continue to robustly
hold, regardless on the commitment of the advisor.

In Appendix B, I provide a simple two-period contracting model with learning. This model provides an additional example of the trade-off between ex ante and ex post optimality that the principal faces. This two-period model is substantially different from the dynamic model of the paper. However, I can analytically show that the advisor commits to low experimentation ex post in order to achieve ex ante optimality. The two-period model hence provides further support for the results of this section and allows us to frame these results as general outcomes of contracting models with learning.

6 Empirical Tests

As shown in the previous section, the model generates a slope of the flow-performance relationship that varies over time in response to the manager’s past performance. The model highlights the dynamic nature of the flow-performance relationship and its connection to the manager’s incentives.

In this section, I test the predictions of the model. From the discussion in section 5.2, we should empirically observe that, everything else being equal, funds with better past performance display a steeper slope in their flow-performance relationship. Moreover, based on section 5.4, we should observe that past performance has a weaker effect on the steepness of the flow-performance relationship if the manager has longer tenure.

I test the model’s prediction in mutual fund data. I focus on US mutual funds investing in US equity, which provide a sample of money managers with very liquid and ample investment opportunities. In principle, the general economic mechanism of the model could apply to other money managers such as private equity funds, hedge funds, and bond funds. However, previous studies point out that these intermediaries face capacity constraints or undertake illiquid investments. These frictions on the assets side of the intermediary may mechanically introduce concavity in the flow-performance relationship.11 Because my model, for tractability reasons, abstracts from size constraints and illiquidity, the sample of US equity mutual fund constitutes the natural testing ground of my theory.

Kaplan and Schoar (2005) find that the relation between flows and performance is concave in private equity funds. They explain this result by observing that private equity funds may have access to a limited number of deals, and that the human capital of general partners cannot be easily scaled. Getmansky et al. (2004) illustrate that hedge funds’ returns are highly serially correlated, and they show that a likely cause for the serial correlation is the hedge funds’ exposure to illiquid securities. Goldstein et al. (2017) show that the relation between flows and performance is concave in corporate bond mutual funds. As a possible explanation, they suggest that because of the illiquidity of the corporate bond market, investors face strategic complementarities when fleeing a fund with bad performance (Chen et al., 2010).
6.1 Data and Variables of Interest

The sample of mutual funds comes from the Center for Research in Security Prices (CRSP). I consider monthly observations from December 1999 to December 2018.

I include only actively managed funds that invest in US stocks, thus excluding index funds, ETFs, and bond and commodity funds. Because Elton et al. (2001) document an upward bias in the performance of small funds, I include funds starting from the date at which their assets under management exceed 15 million in 2011 dollars, as in Pastor et al. (2015). Finally, I restrict the sample to funds that are open to new investors.\(^{12}\)

CRSP reports net monthly returns and annual expense ratios for every share class. I compute the gross returns of every share class by adding \(\frac{1}{12}\) of the expense ratio to the net monthly return. I then compute fund-level gross returns and expense ratios by taking a weighted average of these quantities across each fund’s share classes, where the weights are given by the total net asset value of each share class. Pastor et al. (2015) observe that actively managed funds are unlikely to charge less than a 0.1% annual fee. Thus, following their procedure, I exclude observations whose fees are below the 0.1% threshold, since these observations might represent index funds or data entry mistakes. I also exclude observations whenever fees exceed 10% per year.

The model predicts that the manager’s tenure matters for the dynamic behavior of the flow-performance relationship. To test this prediction, I compute the tenure of the manager at the fund management company. CRSP provides the tenure of managers at each fund, which is not the relevant measure of tenure according to my model. However, I can use this information, together with managers’ and companies’ identifiers, to construct a measure of the length of the contractual relationship between managers and fund companies.\(^{13}\) Funds may have multiple managers. When I study the relation between tenure and the flow-performance relationship, I consider the tenure of the most senior manager.

The model yields predictions about the relation between flows of capital, current performance, and past performance. I measure the net flow of capital into fund \(i\) at time \(t + 1\) as

\[
F_{it+1} = \frac{K_{it+1} - K_{it}}{K_{it}} - R_{it+1}^N, \tag{22}
\]

\(^{12}\)By excluding small funds and funds closed to new investors, I avoid the incubation bias identified by Evans (2010). Moreover, in my empirical analysis I control for fund flows over the previous 12 months, thus effectively excluding funds of less than one year of age.

\(^{13}\)For each manager-company pair, I define the manager start date as the date at which the manager first started working for the management company. I obtain this date by looking at when the manager-company pair first appears in the dataset and at CRSP-reported start dates of the manager in any of the company’s funds. This algorithm allows me to keep track of managers within a management company. However, it does not allow me to track transfers of managers across companies. After this procedure, I can compute the tenure of the manager in a management company.
where $K_{it}$ are the assets under management of fund $i$ at time $t$, and $R^N_{it+1}$ is the net return that fund $i$ delivers to investors from time $t$ to time $t+1$.

To measure performance, first I compute the fund’s benchmark return as the average gross return of funds with the same investment objective of the fund. Hunter et al. (2014) show this benchmark return accounts for commonalities across similar active strategies. Then I obtain the fund’s performance as the gross return of the fund in excess of its style benchmark, that is,

$$
\tilde{R}_{it} = R_{it} - \bar{R}_{s(i)t},
$$

(23)

where $R_{it}$ is the gross return of fund $i$ from $t-1$ to $t$, $s(i)$ is the style of fund $i$ (as described by the CRSP objective classification), and $\bar{R}_{st}$ is the average gross return of all funds with the same investment objective $s$. Khorana et al. (2007) and Spiegel and Zhang (2013) use analogous measures of funds’ objective-adjusted performance.

I measure the past performance of a fund manager by considering the average performance of the fund over the previous $l$ months (provided that the manager of the fund did not change in those $l$ months),

$$
\text{PastPerf}_{i[t-l,t-1]} = \frac{1}{l} \sum_{j=1}^{t} \tilde{R}_{it-j}.
$$

(24)

Finally, to improve clarity in my subsequent discussion, I introduce a separate notation for the cumulative performance of the manager,

$$
\text{CumPerf}_{i[t-l,t]} = \text{PastPerf}_{i[t-l,t]}.
$$

I refer to cumulative performance $\text{CumPerf}_{i[t-l,t]}$ at month $t$ when I consider a measure of performance up to and including the current month $t$, whereas I refer to past performance $\text{PastPerf}_{i[t-l,t-1]}$ at month $t$ when I consider a measure of performance that does not include month $t$.

Berk and van Binsbergen (2015) and Pastor et al. (2015) document that the CRSP database contain data-entry mistakes, some of them representing large outliers in the distribution of flows and returns. To avoid the risk that my results are driven by outliers, I remove the tails of the distributions of capital flows and gross returns, keeping only observations within the 1st and 99th percentiles.

In Table 2, I report the summary statistics of the data used in my empirical analysis. The final sample contains 3,610 funds, 5,769 fund-manager observations, and 313 months. Table 3 contains the description of the variables used in the empirical analysis. In this sec-
Table 2: Summary statistics. The sample contains 3,903 funds, 7,635 fund-manager pairs, and 229 months.

<table>
<thead>
<tr>
<th>Variable</th>
<th>N</th>
<th>Mean</th>
<th>St. Dev.</th>
<th>Min</th>
<th>Pctl(25)</th>
<th>Median</th>
<th>Pctl(75)</th>
<th>Max</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F_{it+1}$</td>
<td>283,111</td>
<td>-0.001</td>
<td>0.038</td>
<td>-0.173</td>
<td>-0.015</td>
<td>-0.005</td>
<td>0.007</td>
<td>0.304</td>
</tr>
<tr>
<td>$\tilde{R}_{it}$</td>
<td>283,111</td>
<td>0.00004</td>
<td>0.019</td>
<td>-0.237</td>
<td>-0.009</td>
<td>0.000</td>
<td>0.009</td>
<td>0.193</td>
</tr>
<tr>
<td>Tenure$_{it}$ (years)</td>
<td>283,111</td>
<td>11.671</td>
<td>8.292</td>
<td>0.085</td>
<td>5.490</td>
<td>9.753</td>
<td>16.005</td>
<td>62.038</td>
</tr>
<tr>
<td>$K_{it}$ (USD mln)</td>
<td>283,111</td>
<td>1,209</td>
<td>3,947</td>
<td>0.1</td>
<td>61</td>
<td>228</td>
<td>880</td>
<td>135,373</td>
</tr>
<tr>
<td>ExpRatio$_{it}$ (% per year)</td>
<td>282,966</td>
<td>1.235</td>
<td>0.498</td>
<td>0.100</td>
<td>0.950</td>
<td>1.180</td>
<td>1.455</td>
<td>9.950</td>
</tr>
<tr>
<td>FundAge$_{it}$ (years)</td>
<td>283,095</td>
<td>15.493</td>
<td>13.576</td>
<td>0.493</td>
<td>6.499</td>
<td>12.008</td>
<td>19.553</td>
<td>94.526</td>
</tr>
</tbody>
</table>

Table 3: Definition of variables

<table>
<thead>
<tr>
<th>Variable</th>
<th>Definition</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F_{it+1}$</td>
<td>Net flow</td>
<td>Growth rate of AUM from month $t$ to month $t+1$ minus net return</td>
</tr>
<tr>
<td>$\tilde{R}_{it}$</td>
<td>(Current) performance</td>
<td>Gross return of fund $i$ in excess of its style benchmark</td>
</tr>
<tr>
<td>$R_{st}$</td>
<td>Style benchmark</td>
<td>Equally weighted average gross return of all funds with style $s$</td>
</tr>
<tr>
<td>PastPerf$_{it}$[t−l,t−1]</td>
<td>Past performance</td>
<td>Manager’s average performance from month $t-l$ to month $t-1$</td>
</tr>
<tr>
<td>CumPerf$_{it}$[t−l,t]</td>
<td>Cumulative performance</td>
<td>Manager’s average performance from month $t-l$ to month $t$</td>
</tr>
<tr>
<td>$X_{it}$</td>
<td>Controls</td>
<td>12 lags of net flows into the fund, log of fund size, expense ratio, log of fund age, and log of the manager’s tenure</td>
</tr>
<tr>
<td>$ι_{FMgr}$</td>
<td>Fund-manager fixed effect</td>
<td>Fixed effect for each fund-manager pair</td>
</tr>
<tr>
<td>$ι_{SMon}$</td>
<td>Style-month fixed effect</td>
<td>Monthly fixed effects for funds with the same investment objective</td>
</tr>
</tbody>
</table>

As a baseline test, I estimate the regression equation

$$F_{it+1} = a_1 \tilde{R}_{it} + a_2 \text{PastPerf}_{it[t-l,t-1]} \tilde{R}_{it} + a_3 \text{PastPerf}_{it[t-l,t-1]} \tilde{R}_{it}^2 + a_4 (\text{PastPerf}_{it[t-l,t-1]})^2 + a_5 \tilde{R}_{it}^2 + c'X_{it} + t_{it}^{FMgr} + t_{it}^{SMon} + u_t,$$  \hspace{1cm} (25)

and test whether $a_2 > 0$. A positive coefficient on the term that capture the interac-

6.2 Prediction 1: History Dependence of the Flow-Performance Relationship

The first prediction of the model can be summarized as follows.

tion between returns and past performance, PastPerf\textsubscript{[t−l,t−1]}\textbar R\textsubscript{it}, indicates that the flow-performance slope is an increasing function of past performance. Whereas the model predicts that \( a_1 \) is also positive, testing for \( a_1 > 0 \) alone cannot be taken as evidence for my theory. Other theories, for example, Berk and Green (2004) and Lynch and Musto (2003), predict a positive coefficient \( a_1 \). Controls \( X_{it} \) include 12 lags of monthly net flows into the fund, the logarithm of fund size, its expense ratio, the logarithm of the fund’s age, and the logarithm of the manager’s tenure. \( \iota_{it}^{FMgr} \) is fixed effect for the fund-manager pair and \( \iota_{it}^{SMon} \) is style-month fixed effect. Standard errors are double-clustered at the month and at the fund level. Because I measure past performance at the fund-manager level, for an observation to be included in the estimation we need a history of at least \( l + 1 \) months of performance for each fund-manager pair.

Results. Columns (1) and (2) of Table 4 provide estimates of model (25), where the past performance of the manager is computed over the previous six months, that is \( l = 6 \).

The data support the model’s prediction that the slope of the flow-performance relationship positively depends on the history of performance, as reflected by the positive coefficients on the PastPerf\textsubscript{[t−6,t−1]}\textbar R\textsubscript{it} term. One may suspect that a positive coefficient on PastPerf\textsubscript{[t−6,t−1]}\textbar R\textsubscript{it} captures the convexity of flows in the cumulative performance up to month \( t \), that is, convexity in CumPerf\textsubscript{[t−6,t−1]}\textbar R\textsubscript{it}. In columns (3) and (4), I replace the square of past performance with the square of cumulative performance up to and including month \( t \). Because of the correlation between PastPerf\textsubscript{[t−6,t−1]}\textbar R\textsubscript{it} and (CumPerf\textsubscript{[t−6,t−1]}\textbar R\textsubscript{it})\textsuperscript{2}, the coefficient on the interaction term PastPerf\textsubscript{[t−6,t−1]}\textbar R\textsubscript{it} decreases. However, it remains positive and statistically significant. This result highlights that the flow-performance slope increases with past performance even after controlling for the convexity of flows with respect to cumulative performance up to the current month.

In columns (5) and (6), I address two possible concerns. First, one may suspect that a positive coefficient on the interaction term PastPerf\textsubscript{[t−6,t−1]}\textbar R\textsubscript{it} captures a convexity in cumulative performance that the quadratic term (CumPerf\textsubscript{[t−6,t−1]}\textbar R\textsubscript{it})\textsuperscript{2} does not fully capture. Second, one may suspect that a positive coefficient on PastPerf\textsubscript{[t−6,t−1]}\textbar R\textsubscript{it} actually indicates that flows become more sensitive to past performance after good current performance. In columns (5) and (6), I show that my results are not affected by these concerns: When current performance is positive, there is only weak statistical evidence that flows are more sensitive to past performance. On the contrary, when past performance is positive, there is strong statistical evidence that flows become more sensitive to current performance. These results support the prediction of the model that good past performance increases the sensitivity of flows to current performance.
Table 4: Effect of past performance on the slope of the flow-performance relationship. The history dependence of the slope of the flow-performance relationship is measured by the coefficient on $\text{PastPerf}_{i[t-6,t-1]} \cdot \tilde{R}_{it}$. $\tilde{R}_{it}$ measures current performance and is calculated, for each month $t$, as the gross return of fund $i$ in excess of the equally weighted average gross return of all funds with the same style. PastPerf$_{i[t-6,t-1]}$ measures the past performance of the manager and is calculated as the average excess return over the style benchmark in the six months from $t-6$ to $t-1$. The dependent variable, $F_{it+1}$, measures the net flow of capital and is calculated as the growth rate of assets under management from month $t$ to month $t+1$ minus the net return over the same period. CumPerf$_{i[t-6,t]}$ is the average performance of the manager over the style benchmark in the months from $t-6$ to $t$. $\mathbb{I}[-]$ is the indicator function. Controls include 12 lags of monthly net flows into the fund, the log of fund size, its expense ratio, the log of fund age, and the log of the manager’s tenure. Standard errors are in parentheses and they are double-clustered at the fund and at the month level.

<table>
<thead>
<tr>
<th></th>
<th>$F_{it+1}$ (Net Flow)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(1)</td>
</tr>
<tr>
<td>$\tilde{R}_{it}$</td>
<td>0.172***</td>
</tr>
<tr>
<td></td>
<td>(0.010)</td>
</tr>
<tr>
<td>PastPerf$<em>{i[t-6,t-1]} \cdot \tilde{R}</em>{it}$</td>
<td>2.562***</td>
</tr>
<tr>
<td></td>
<td>(0.670)</td>
</tr>
<tr>
<td>$\tilde{R}<em>{it} \cdot \mathbb{I}[\text{PastPerf}</em>{i[t-6,t-1]} &gt; 0]$</td>
<td>0.067***</td>
</tr>
<tr>
<td></td>
<td>(0.015)</td>
</tr>
<tr>
<td>PastPerf$<em>{i[t-6,t-1]} \cdot \mathbb{I}[\tilde{R}</em>{it} &gt; 0]$</td>
<td>0.052*</td>
</tr>
<tr>
<td></td>
<td>(0.031)</td>
</tr>
<tr>
<td>PastPerf$_{i[t-6,t-1]}$</td>
<td>0.354***</td>
</tr>
<tr>
<td></td>
<td>(0.027)</td>
</tr>
<tr>
<td>$(\text{PastPerf}_{i[t-6,t-1]})^2$</td>
<td>0.566</td>
</tr>
<tr>
<td></td>
<td>(0.944)</td>
</tr>
<tr>
<td>$(\text{CumPerf}_{i[t-6,t]})^2$</td>
<td>0.770</td>
</tr>
<tr>
<td></td>
<td>(1.285)</td>
</tr>
<tr>
<td>$\mathbb{I}[\tilde{R}_{it} &gt; 0]$</td>
<td>0.000</td>
</tr>
<tr>
<td></td>
<td>(0.000)</td>
</tr>
<tr>
<td>$\mathbb{I}[\text{PastPerf}_{i[t-6,t-1]} &gt; 0]$</td>
<td>0.002***</td>
</tr>
<tr>
<td></td>
<td>(0.000)</td>
</tr>
<tr>
<td>$\tilde{R}_{it}^2$</td>
<td>0.571***</td>
</tr>
<tr>
<td></td>
<td>(0.128)</td>
</tr>
<tr>
<td>Controls</td>
<td>Yes</td>
</tr>
<tr>
<td>Style-Month FE</td>
<td>Yes</td>
</tr>
<tr>
<td>Fund-Manager FE</td>
<td>No</td>
</tr>
<tr>
<td>Observations</td>
<td>186,192</td>
</tr>
<tr>
<td>$R^2$</td>
<td>0.347</td>
</tr>
</tbody>
</table>

Notes: $^*p \leq .10$; $^{**}p \leq .05$; $^{***}p \leq .01$
Figure 6: History dependence of the relation between flows and performance. Figures (a) and (b) show how flows change with current performance and how the change depends on past performance. I sort funds into deciles based on their current performance and into halves based on their past performance. Past performance is the average excess return over the style benchmark in the previous 6 months. I then run regression of flows on dummies for the deciles of current performance, dummies for the halves of past performance, and interactions between the two sets of dummies. As controls, I include dummy variables for cumulative performance CumPerf_{t-6,t-1} sorted into deciles, 12 lagged flows, the logarithm of fund age, the logarithm of the manager’s tenure, the logarithm of lagged assets under management, fund fees, fund fixed effects, and month fixed effects. The shaded areas represent 95% confidence intervals for the change in the effect of current performance on flows when past performance increases above the median. Confidence intervals are constructed by double-clustering standard errors at the month and at the fund level.

In Figure (a), I plot the effect of current good performance (that is, performance relative to the first decile) on flows, while, in Figure (b), I plot the effect of current bad performance (that is, performance relative to the tenth decile) on flows.

In Figure 6, I use a non-parametric approach to show that flows become more sensitive to current performance after better performance in the past. In Figure 6a, for every month I sort past performance PastPerf_{t-6,t-1} into halves (above and below median), and I sort current performance \( \bar{R}_{it} \) into deciles. I then regress flows on dummies for the deciles of current performance, dummies for the halves of past performance, and interactions between the two sets of dummies. I also sort cumulative performance CumPerf_{t-6,t} into deciles and add dummy variables for cumulative performance deciles as controls. I include the same controls \( X_{it} \) of equation (25), fund fixed effects, and month fixed effects. Standard errors are double-clustered at the month and at the fund level.

Figure 6a plots the incremental effect of current performance on flows (relative to the first decile of current performance), and it shows that such incremental effect is larger for above-median past performers. For example, the blue dot at decile 10 represents the
increase in flows for a fund with top current performance, relative to a fund with bad current performance, conditional on past performance being above the median. The shaded blue area represents the 95% confidence interval for the incremental effect of past performance. If the effect of current performance for below-median past performers (the triangle) lies outside the shaded area, then we reject, at a 95% confidence level, the hypothesis that the relation between flows and current performance is the same for above-median past performers (the dot) and below-median past performers (the triangle).

Someone may suspect that the history-dependence of the flow-performance relationship is due to investors’ inertia. For example, investors may wait to observe good performance both in the past and in the current month before investing in a fund. This hypothesis would imply that, after good performance in the past, flows are less sensitive to bad performance. Figure 6b illustrates the opposite happens in the data. The figure plots the effect of current bad performance (that is, performance relative to the tenth decile of current performance) on flows, and it highlights that flows are more sensitive to bad performance for funds that experienced above-median performance in the past. Consistently with my model, after good performance in the past, flows are more sensitive not only to current good performance, but also to current bad performance. We can therefore reject the hypothesis that the history-dependence of the flow-performance relationship is due to investors’ inertia.

### 6.3 Prediction 2: Flows, Performance, and Managers’ Tenure

The model predicts a relation between the manager’s tenure and the history dependence of the flow-performance slope. The prediction can be stated as follows.

**Prediction 2.** The slope of the flow-performance relationship increases less with past performance if the manager has longer tenure.

Two mechanisms drive this prediction. The first mechanism is the optimal contract itself. As discussed in section 5, in the long run, the advisor will constrain the manager’s experimentation and incentives up to the point where the slope of the flow-performance relationship will no longer change with past performance. The second mechanism is the convergence of beliefs. If the manager’s skill is a fixed (unknown) parameter, then, in the long run, beliefs will converge in probability to the true value, and a negligible amount of learning will take place.
I test the second prediction of the model by using the following regression:

\[
F_{it+1} = a_0 + a_1 \tilde{R}_{it} + a_2 \text{PastPerf}_{[t-1, t-1]} \tilde{R}_{it} + a_3 \text{PastPerf}_{[t-1, t-1]} + a_4 (\text{PastPerf}_{[t-1, t]})^2 + a_5 \tilde{R}_{it}^2 \\
+ a_3 ^\text{TenureQuintile}_{it} \left( a_0 T_j + a_1 ^\text{TenureQuintile}_{it} \tilde{R}_{it} + a_2 \text{PastPerf}_{[t-1, t-1]} \tilde{R}_{it} + a_3 \text{PastPerf}_{[t-1, t-1]} + a_4 (\text{PastPerf}_{[t-1, t]})^2 + a_5 \tilde{R}_{it}^2 \right) \\
+ a_3 ^\text{AgeQuintile}_{it} \left( a_0 A_j + a_1 ^\text{AgeQuintile}_{it} \tilde{R}_{it} + a_2 \text{PastPerf}_{[t-1, t-1]} \tilde{R}_{it} + a_3 \text{PastPerf}_{[t-1, t-1]} + a_4 (\text{PastPerf}_{[t-1, t]})^2 + a_5 \tilde{R}_{it}^2 \right) \\
+ a_3 ^\text{SizeQuintile}_{it} \left( a_0 S_j + a_1 ^\text{SizeQuintile}_{it} \tilde{R}_{it} + a_2 \text{PastPerf}_{[t-1, t-1]} \tilde{R}_{it} + a_3 \text{PastPerf}_{[t-1, t-1]} + a_4 (\text{PastPerf}_{[t-1, t]})^2 + a_5 \tilde{R}_{it}^2 \right) \\
+ c' X_{it} + l_{it}^{FMgr} + l_{it}^{SMon} + u_t.
\]

TenureQuintile\(_{it}^j = 1\) if, in month \(t\), the tenure of the manager of fund \(i\) belongs to the \(j\)th quintile of the distribution of managerial tenure in month \(t\), otherwise TenureQuintile\(_{it}^j = 0\). Similarly, AgeQuintile\(_{it}^j = 1\) if, in month \(t\), the age of fund \(i\) belongs to the \(j\)th quintile of the distribution of fund age in month \(t\), otherwise AgeQuintile\(_{it}^j = 0\). SizeQuintile\(_{it}^j = 1\) if, in month \(t\), the size of fund \(i\) belongs to the \(j\)th quintile of the distribution of fund size in month \(t\), otherwise SizeQuintile\(_{it}^j = 0\).

I verify Prediction 2 by testing whether \(a_2 ^T_j < 0\) for \(j = 2, \ldots, 5\). As in regression (25), controls \(X_{it}\) include 12 lags of monthly net flows into the fund, the logarithm of fund size, its expense ratio, the logarithm of fund age, and the logarithm of the manager’s tenure. I also control for fund-manager fixed effects, \(l_{it}^{FMgr}\), and style-month fixed effects, \(l_{it}^{SMon}\). Standard errors are double-clustered at the month and at the fund level.

**RESULTS.** Figure 7a shows that slope of the flow-performance relationship increases less with past performance if the manager has longer tenure. The dots at quintiles 2 to 5 are the estimates of \(a_2 ^T_j\) for \(j = 2, \ldots, 5\), and the vertical lines represent 90% confidence intervals for the incremental effect of managerial tenure. Because the confidence intervals do not contain the zero for quintiles 3 to 5 (graphically, the red vertical lines do not intersect the horizontal dashed line), we reject the hypothesis that \(a_2 ^T_j \geq 0\) at a 95% confidence level for \(j = 3, \ldots, 5\). In order words, the slope of the flow-performance relationship
Figure 7: Effect of managerial tenure and fund age on the history dependence of the relation between flows and current performance. Past performance is computed over 6 months. I run regression

\[ F_{it+1} = a_0 + a_1 \tilde{R}_{it} + a_2 \text{PastPerf}_{it|t-6,t-1} \tilde{R}_{it} + a_3 \text{PastPerf}_{it|t-6,t-1} + a_4 (\text{PastPerf}_{it|t-6,t})^2 + a_5 \tilde{R}_{it}^2 \]

\[ + \sum_{j=2}^{5} \text{TenureQuintile}_{it}^j \left( a_{0j} + a_{1j} \tilde{R}_{it} + a_{2j} \text{PastPerf}_{it|t-6,t-1} \tilde{R}_{it} + a_{3j} \text{PastPerf}_{it|t-6,t-1} + a_{4j} (\text{PastPerf}_{it|t-6,t})^2 + a_{5j} \tilde{R}_{it}^2 \right) \]

\[ + \sum_{j=2}^{5} \text{AgeQuintile}_{it}^j \left( a_{0j} + a_{1j} \tilde{R}_{it} + a_{2j} \text{PastPerf}_{it|t-6,t-1} \tilde{R}_{it} + a_{3j} \text{PastPerf}_{it|t-6,t-1} + a_{4j} (\text{PastPerf}_{it|t-6,t})^2 + a_{5j} \tilde{R}_{it}^2 \right) \]

\[ + \sum_{j=2}^{5} \text{SizeQuintile}_{it}^j \left( a_{0j} + a_{1j} \tilde{R}_{it} + a_{2j} \text{PastPerf}_{it|t-6,t-1} \tilde{R}_{it} + a_{3j} \text{PastPerf}_{it|t-6,t-1} + a_{4j} (\text{PastPerf}_{it|t-6,t})^2 + a_{5j} \tilde{R}_{it}^2 \right) \]

\[ + c' X_{it} + \iota_{it}^{FMgr} + \iota_{it}^{SMon} + u_t, \]

where TenureQuintile\(^j_{it} = 1\) if, in month \(t\), the tenure of the manager of fund \(i\) belongs to the \(j^{th}\) quintile of the distribution of managerial tenure in month \(t\); AgeQuintile\(^j_{it} = 1\) if, in month \(t\), the age of fund \(i\) belongs to the \(j^{th}\) quintile of the distribution of fund age in month \(t\); SizeQuintile\(^j_{it} = 1\) if, in month \(t\), the size of fund \(i\) belongs to the \(j^{th}\) quintile of the distribution of fund size in month \(t\). Table 3 contains the description of all the other variables used in the regression. Standard errors are double-clustered at the month and at the fund level.

In Figure (a), the dots in the figure represents estimated coefficients \(a_{Tj}^j\)'s. The vertical lines represent 90% confidence intervals for the incremental effect of tenure on the flow-performance slope relative to the first quintile: If the vertical red line at quintile \(j\) does not cross the dashed horizontal line, then we reject the hypothesis that \(a_{Tj}^j \geq 0\) at a 95% confidence level.

Figure (b) is the analogous of Figure (a) for the effect of fund age on the history-dependence of the flow-performance relationship.
increases less with past performance for funds with more senior managers.

Figure 7b is analogous to Figure 7a, but plots the incremental effect of fund age on the history dependence of the flow-performance relationship. No clear correlation exits between the fund’s age and the extent to which the flow-performance slope depends on the history of performance. Compared to previous studies, my model and my empirical analysis establish a connection between managerial tenure and the history dependence of the flow-performance slope, rather than between fund age and the value of the slope. Therefore, my results complement, but do not overlap with, the results of Chevalier and Ellison (1997), who show that the flow-performance relationship is less steep in older funds. Similarly, the theoretical predictions of my model are distinct from those of Berk and Green (2004), who explain why older funds have a flatter flow-performance relationship.

7 Conclusions

In this paper, I study how mutual fund flows respond to performance when portfolio managers face optimal incentive contracts under moral hazard and learning. I show that both flows and managerial compensation increase more than linearly in cumulative past performance, consistent with empirical evidence.

I develop a dynamic model that explicitly takes into account the two challenges that money management firms face: raising capital from investors and providing incentives to portfolio managers. I illustrate the connection between the two challenges and the optimal strategy that money management firms should undertake. I show that the empirical patterns in capital flows and managerial compensation are consistent with this strategy.

The model highlights the dynamic nature of the relation between flows and performance. In particular, the model offers novel testable predictions on the dynamics of the flow-performance relationship. First, after a history of good performance, flows react more strongly to current performance. Second, the flow-performance relationship depends less on the history of performance for managers with longer tenure. I test these predictions in mutual fund data and provide empirical support for the model.

By focusing on contracting frictions inside the money management industry, the model offers a basis for further research on financial intermediaries. Since I provided a partial equilibrium model, a natural extension would be to explore the asset-pricing implications of optimal money management contracts. Building on this extension, we can ask how monetary policy in the form of quantitative easing affects incentives in the money management industry and to what extent the intermediary sector facilitates or hinders the transmission of monetary policy to the real economy. Finally, we can modify the
model and make it suitable to study optimal contracts for other types of money managers. For example, by considering illiquid investments and Poisson shocks, we could explain common patterns in the hedge fund industry, such as the use of lock-up provisions and high-water marks.
A ALTERNATIVE CONTRACTUAL ENVIRONMENTS

The design and implementation of the optimal contract requires the full commitment of the advisor to the initial promises. The ex ante optimal contract implies actions that are ex post inefficient. For example, the advisor would like to renegotiate the contract at any time to reset $Y = 0$. Similarly, a contract with full commitment is not robust to competition between advisors or to the assumption that the manager could leave the advisor to open his own fund. In this section, I explore these two alternative scenarios. In the first one, I consider a renegotiation-proof (or state-contingent) contract. In the second one, I explore the incentives that a market of atomistic investors can provide to a manager without the full commitment of an advisor.

I first present these two contractual environment formally, then show their equivalence and discuss their outcomes.

A.1 RENEGOTIATION-PROOF CONTRACT

To understand the intuition behind a renegotiation-proof contract, imagine that multiple advisors are competing for the same manager. Suppose that one of them offers the contract in section 4 to the manager. After some time has passed, the advisor will be committed to undertaking ex post inefficient actions, captured by a strictly negative multiplier $Y_t < 0$. Let $\hat{\xi}(V_t, Y_t, \phi_t)$ be the information rent implied by this contract at time $t$. At that point, the advisor is willing to transfer wealth $J(V_t, \hat{\xi}(V_t, Y_t, \phi_t), \phi_t)$ to another advisor who will then employ the manager by promising the same continuation value $V_t$, but with a re-set information rent $\hat{\xi}(V_t, 0, \phi_t)$. At these terms, the first advisor would be indifferent between keeping the manager and transferring him. The manager is also indifferent, because he obtains the same continuation value with the old and the new advisor. However, the new advisor makes a strictly positive gain. She obtains a wealth transfer of $J(V_t, \hat{\xi}(V_t, Y_t, \phi_t), \phi_t)$ from the first advisor, but she will have to bear costs $J(V_t, \hat{\xi}(V_t, 0, \phi_t), \phi_t)$ to employ the manager, where $J(V_t, \hat{\xi}(V_t, 0, \phi_t), \phi_t) < J(V_t, \hat{\xi}(V_t, Y_t, \phi_t), \phi_t)$.

Therefore, if the manager can be transferred across funds and an advisor cannot commit to retain the manager, the contract of section 4 would not be credible. The terms of the contract would be continuously renegotiated, making it impossible to implement an ex ante optimal contract that requires ex post inefficient choices.

I therefore develop a notion of renegotiation-proof contracts that offer a credible alternative to optimal contracts when the advisor cannot fully commit to the terms of the contract. The advisor may lack commitment because the manager can be transferred
across funds, as discussed above. Alternatively, the advisor may lack commitment because, after writing an initial contract, the advisor and the manager could mutually agree to change the terms of the contract ex post, as in the “commitment and renegotiation” model of Laffont and Tirole (1990).

Define the following set:

\[ I(V, \phi, t) = \left\{ C_t : C_t \text{ is IC, } E \left[ \int_t^\infty e^{-R(s-t)} a(C_s + m_s \lambda K_s) \, ds \mid F_t, C_t \right] = V, E[h \mid F_t] = \phi \right\} , \]

which represents the set of all contracts that are incentive compatible (as in Definition 2) and that provide the manager with continuation value \( V \) starting from beliefs \( \phi \).

To provide a definition of renegotiation-proof contract, I first define \( J(C) \) as the costs for the advisor that offers contract \( C \), and I define \( O(C, t) \) as the time-\( t \) continuation contract implied by \( C \). Building on the definition in Strulovici (2011), I define a (weakly) renegotiation-proof contract as follows.

**Definition 5 (Weakly Renegotiation-Proof Contract).** \( C \in J(V_0, \phi_0, 0) \) is weakly renegotiation proof if, for all \( t \geq 0 \) and \( t' \geq 0 \) such that \( V_t = V_{t'} \) and \( \phi_t = \phi_{t'} \), \( J(O(C, t)) = J(O(C, t')) \).

In a renegotiation-proof contract, the advisor must be unable to renegotiate the terms of the contract in order to leave the manager indifferent and reduce costs for herself. In this environment, the advisor still fully commits to a promised value for the manager, but she is unable to commit to the manager’s information rent. Therefore, the nature of the commitment in this model is similar to the “commitment and renegotiation” assumption in Laffont and Tirole (1990): Although the advisor and the manager can write a long-term contract, they can later agree to alter the terms of the initial contract if doing so is mutually beneficial.

Even when the advisor cannot commit to an information rent for the manager, enforcing full effort remains optimal. In other words, Proposition 2 remains valid because it does not rely on any particular assumption about the advisor’s commitment. Moreover, (3) also remains valid because it characterizes incentive compatibility in any generic principal-agent setting.

For any given contract, the manager has an information rent given by equation (11). However, unlike in the optimal contract with full commitment, the advisor does not take into account how the contract affects the agent’s information rent. The advisor will instead take the process for the information rent as given and design a contract that is optimal given this process. This behavior reflects the fact that the manager and the advisor can mutually agree to change the terms of the contract in the future and modify the im-
plied information rent.

We can refine the search for a renegotiation-proof contract by looking at the set of Markovian contracts, under the assumption that the optimal renegotiation-proof contract is unique given an initial promised value and a posterior.

**Definition 6.** A contract $C \in \mathcal{I}(V_0, \phi_0, 0)$ is Markovian if, for all $t \geq 0$ and $t' \geq 0$ such that $V_t = V_{t'}$ and $\phi_t = \phi_{t'}$, $O(C, t) = O(C, t')$.

**Lemma 4.** Any Markovian contract is weakly renegotiation proof. If the optimal renegotiation-proof contract is unique, then it is Markovian.

In this paper, I do not explore the issue of multiple optimal contracts. I simply focus on the optimal Markovian contract, which, given the previous lemma, is a renegotiation-proof contract. Consequently, given expression (11) for the manager’s information rent and the Markovian structure of the contract, we also conclude that the manager’s information rent is Markovian in the manager’s continuation value and beliefs. Using the homogeneity of the problem, I can characterize an optimal renegotiation-proof contract as follows.

**Proposition 8.** As before, let $\hat{v}_t = ((1 - \rho)V_t)^{1-\rho}$. In an optimal renegotiation-proof contract, we have that $C_t = \hat{v}_t c_R(\phi_t), \beta_t = (1 - \rho)V_t \hat{\beta}_R(\phi_t)$ and $\eta_t = \eta_R(\phi_t)$, where $c_R(\phi), \hat{\beta}_R(\phi)$ and $\eta_R(\phi)$ are the optimal controls in the HJB equation:

$$rJ_R(\phi) = \min_{c, \beta, \epsilon} \left\{ \frac{c}{q(\eta)} - \mu(\eta, \phi, 0) \frac{\hat{\beta} - \eta z_R(\phi)}{\sigma \lambda} c' + J_R(\phi) \left( \frac{\delta}{1 - \rho} - \frac{c^{1-\rho}}{1 - \rho} + \frac{1}{2} \rho \hat{\beta}^2 \right) \right. $$

$$\left. + \hat{\beta} \eta \phi (1 - \phi) J_R'(\phi) + \frac{1}{2} \eta^2 \phi^2 (1 - \phi)^2 J_R''(\phi) \right\}. \quad (27)$$

The function $z_R$ solves the differential equation:

$$z_R(\phi)c_R(\phi)^{1-\rho} - \hat{\beta}_R(\phi)\eta_R(\phi)\phi(1 - \phi) - (1 - \rho)\hat{\beta}_R(\phi)\eta_R(\phi)\phi(1 - \phi)z'_R(\phi)$$

$$= \frac{1}{2} \eta_R(\phi)^2 \phi^2 (1 - \phi)^2 z''_R(\phi). \quad (28)$$

The cost function for the advisor is $\hat{v}_t J_R(\phi_t)$ and the manager’s information rent is $\xi_t = (1 - \rho)V_t z_R(\phi_t)$. 51
A.2 Market-Based Incentives

I now consider the case of a manager that directly collects money from investors. No explicit contracts are written; however, the manager can maintain a stake in the fund in order to ensure incentive compatibility. To maintain comparison with the problem that I have studied so far, I assume the manager’s consumption is observable by the market, together with the amount of assets under management, the manager’s stake, and his experimentation. Consequently, these quantities must be set as functions of the returns observed by the market, because the market could punish observed deviations by leaving the manager in autarky.

In this framework, the manager possesses some wealth, which he allocates between a risk-free asset and the fund he runs. He raises capital from investors, who supply capital perfectly elastically at the risk-free rate $r$. Therefore, the manager can collect fees on the investor’s capital for the difference between the expected return of his fund and the risk-free rate.

Absent an advisor who designs an explicit incentive contract, the manager must resort to other implicit incentives schemes to mitigate moral hazard problem. In particular, the manager holds a stake in the fund. If the manager had none of his wealth invested in the fund, he would be tempted to shirk to obtain private benefits. In this situation, rational investors would not be willing to provide capital. If, instead, the manager has some of his own wealth invested in the fund, then investors, to some extent, trust the manager with their own money. The stake of the manager should be interpreted in a broad sense. One way the manager can hold a stake in the fund is by being an investor in the fund. Alternatively, the manager may simply receive cash payments that depend on the performance of the fund and that are not immediately consumed. Such an example is a manager who charges symmetric (fulcrum) fees for performance.

As in the principal-agent formulation, Proposition (2) continues to hold and the manager will choose a strategy that will credibly enforce full effort. The proof of Proposition (2) holds by simply reinterpreting the advisor’s cost function as the wealth of the manager. However, the incentive-compatibility condition needs to be slightly reformulated to fit the new contractual environment.

The incentive-compatibility condition remains essentially unchanged: Changes in the continuation value of the manager, keeping the manager’s beliefs fixed, should exceed the marginal utility of shirking, as in equation (8). However, because no advisor is committing to a promised value for the manager, the implementation of this incentive-compatibility condition has to rely on the dynamics of the manager’s wealth and beliefs.

In equilibrium, the manager’s continuation value is given by a function $V(A, \phi)$, which
depends on the manager’s wealth $A$ and the equilibrium beliefs about his skill $\phi$. The manager’s wealth evolves as

$$dA_t = \left( rA_t - \frac{C_t}{q(\eta_t)} + \mu(\eta_t, \phi_t, 0)K_t \right) dt + \Theta_t \sigma dW^c_t,$$

where $\Theta$ is the manager’s stake. The manager collects fees on the capital he raised from investors. Fees account for a part $\mu(\phi_t, \eta_t, 0)(K_t - \Theta_t)$ of the manager’s instantaneous cash flow. The remaining part, $\mu(\phi_t, \eta_t, 0)\Theta_t$, is the expected excess return from his stake in the fund. In an incentive-compatible contract, fees and expected returns coincide and the stake $\Theta_t$ does not affect expected cash flows, but only the volatility of cash flows.

I therefore rewrite the incentive-compatibility condition (8) by using the Markovian characterization of the manager’s continuation value.

**Lemma 5.** If the market-based allocation is incentive compatible,

$$V_A(A_t, \phi_t)\Theta_t \sigma + V_\phi(A_t, \phi_t)\eta_t \phi_t (1 - \phi_t) \geq u'(C_t)K_t \lambda \sigma + \eta_t \xi_t,$$

where $\xi_t$ evolves as

$$d\xi_t = [\delta \xi_t - \eta_t \phi_t (1 - \phi_t)(V_A(A_t, \phi_t)\Theta_t \nu(\eta_t) + V_\phi(A_t, \phi_t)\eta_t \phi_t (1 - \phi_t))] dt + \delta_t dW_t.$$

This incentive-compatibility condition includes a role for the career concerns of the manager, which were initially studied by Fama (1980) and Holmström (1999). The incentive-compatibility condition (29) captures the result in Gibbons and Murphy (1992) that direct incentives and career concerns add up in determining the total incentives of the manager. In my model, direct incentives are represented by the value of the manager’s stake, $V_A(A_t, \phi_t)\Theta_t \sigma$. Career concerns are represented by the difference $V_\phi(A_t, \phi_t)\eta_t \phi_t (1 - \phi_t) - \eta_t \xi_t$. To see why, recall that $V_\phi(A_t, \phi_t)$ is the marginal value of equilibrium beliefs, which are common among the market and the manager. From the discussion in section 3.4, the information rent $\xi_t$ is the product between $\phi_t (1 - \phi_t)$ and the marginal value of the manager’s private beliefs. Therefore, the difference

$$V_\phi(A_t, \phi_t) - \frac{\xi_t}{\phi_t (1 - \phi_t)}$$

represents the marginal value of the market’s beliefs. The manager prefers the market’s beliefs to be high in order to collect higher fees. The incentive-compatibility condition
(29) can then be written as

\[ V_A(A_t, \phi_t) \Theta_t \sigma + \eta_t \phi_t (1 - \phi_t) \left( V_\phi(A_t, \phi_t) - \frac{\xi_t}{\phi_t (1 - \phi_t)} \right) \geq u'(C_t) K_t \lambda \sigma, \]

which highlights that both direct incentives and career concerns play a role in providing incentives to the manager.

Because the manager and the investors interact in a spot market, the implicit market-based contract must also be renegotiation proof. The manager’s information rent will therefore be a function of his wealth and equilibrium beliefs,

\[ \xi_t = \xi(A_t, \phi_t). \]

Given the functional form of the utility function, the manager’s continuation value and information rent are homogeneous in wealth, that is, \( V(A, \phi) = A^{1-\rho} v_M(\phi) \) and \( \xi(A, \phi) = A^{1-\rho} z_M(\phi) \). Therefore, the incentive-compatibility constraint can be written as

\[ (1 - \rho) v_M(\phi_t) \hat{\Theta}_t \sigma + v_M'(\phi_t) \eta_t \phi_t (1 - \phi_t) \geq (1 - \rho) c_t^{-\rho} k_t \lambda \sigma + \eta_t z_M(\phi_t), \]

where I use \( \hat{\Theta}_t = \frac{\Theta_t}{A_t} \), \( c_t = \frac{C_t}{A_t} \) and \( k_t = \frac{K_t}{A_t} \).

This contracting environment can thus be characterized as follows.

**Proposition 9.** In an optimal market-based contract, \( C_t = A_t c_M(\phi_t), \Theta_t = A_t \hat{\Theta}_M(\phi_t) \) and \( \eta_t = \eta_M(\phi_t) \), where \( c_M(\phi), \hat{\Theta}_M(\phi) \) and \( \eta_M(\phi) \) are the optimal controls in the HJB equation

\[
\delta v_M(\phi) = \max_{\phi, \hat{\Theta}, \eta, k} \left\{ c^{1-\rho} + v_M(\phi)(1 - \rho) \left[ r - \frac{c}{q(\eta)} + \mu(\eta, \phi, 0) k - \frac{1}{2} \rho \hat{\Theta}^2 \sigma^2 \right] \right. \\
+ (1 - \rho) \sigma \eta \hat{\Theta} (1 - \phi) v_M'(\phi) + \frac{1}{2} \eta^2 \phi^2 (1 - \phi)^2 v''_M(\phi) \right\} \\
\text{s.t.} \ (1 - \rho) v_M(\phi) \hat{\Theta} \sigma + v'_M(\phi) \eta_t \phi_t (1 - \phi) = (1 - \rho) c^{-\rho} k \lambda \sigma + \eta \dot{z}_M(\phi). \quad (30)
\]

The function \( z_M \) solves the differential equation

\[
\left[ \delta - (1 - \rho) \left( r - \frac{c_M(\phi)}{q(\eta_M(\phi))} + \mu(\eta_M(\phi), \phi, 0) k_M(\phi) - \frac{1}{2} \rho \hat{\Theta}(\phi)^2 \sigma^2 \right) \right] z_M(\phi) = \\
(1 - \rho) \eta_M(\phi) \phi (1 - \phi) \hat{\Theta}(\phi) \sigma v_M(\phi) + \eta_M(\phi)^2 \phi^2 (1 - \phi)^2 v'_M(\phi) \\
+ (1 - \rho) \eta_M(\phi) \phi (1 - \phi) \hat{\Theta}(\phi) \sigma z'_M(\phi) + \frac{1}{2} \eta_M(\phi)^2 \phi^2 (1 - \phi)^2 z''_M(\phi). \quad (32)
\]
The value function for the manager is \( A^{1-\rho} v(\phi_t) \) and his information rent is \( \xi_t = A^{1-\rho} z_M(\phi_t) \).

A.3 RESULTS

I introduced the two models of this section separately in order to highlight the difference between the two contracting environments. However, the two models are, as the next proposition shows, outcome equivalent. As long as the manager has the same commitment power of the advisor and as long as the market can observe the same actions that the advisor can observe, a model with market-based incentives is equivalent to a principal-agent model. In particular, the two models imply the same social welfare and the same outcomes in terms of flows and compensation.

**Proposition 10.** The renegotiation-proof contract and the market-based contract are equivalent.

In Appendix E, I provide a verification argument based on the recursive representations of the models in Propositions 8 and 9. However, we can immediately observe that the renegotiation-proof contracting problem coincides with the problem of a manager who wants to find the minimal wealth that could support a given lifetime utility with market-based incentives.

We can now study the flow-performance relationship and the compensation scheme in these contractual environments. As the figures in this section show, the results are qualitatively identical to the full-commitment model with only one exception.\(^{14}\) Because the advisor or the manager is unable to commit to ex post inefficient actions, the flow-performance relationship and the pay-performance sensitivity depend only on the current beliefs \( \phi \). In the full-commitment model, the advisor could commit to reduce future information rents through a multiplier \( y \) with a negative drift. As a result, over time, the advisor would change the manager’s incentives and experimentation, even for the same value of beliefs \( \phi \). This channel of non-stationarity is absent when the advisor cannot commit to an information rent for the manager.

As before, the flow-performance relationship is positive and increasing in beliefs \( \phi \) (Figure 8a). A good return is associated not only with a positive flow of capital into the fund, but also with an increase in the slope of the flow-performance relationship, because beliefs \( \phi \) increase after good performance. The economic motivation is identical to the one I discussed in section 5.2 and is related to the willingness of the advisor (or the manager) to increase capital, relative to the manager’s risk tolerance, when beliefs are higher (Figure 8b).

\(^{14}\)I numerically solve for the models using the same parameters of section 5.
The relation between compensation and performance also matches the full-commitment contract. Compensation increases with returns, and the pay-performance sensitivity increases with past performance (Figure 9a), which then results in a convex relation between cumulative pay and cumulative performance. Moreover, performance-based compensation is again back-loaded after positive returns (Figure 9b). Interestingly, the manager does not need an advisor to defer his compensation and increase his stake in the fund. This result is consistent with anecdotal evidence that hedge fund managers tend to invest most of their performance fees in the hedge fund itself (Agarwal et al., 2009).

Although the outcomes of the two models are identical, the implementation is different. In the contract designed by the advisor, the advisor adjusts the explicit contractual terms of the manager to ensure incentive compatibility (Figure 10a). In the market-based allocation, the manager adjusts his implicit incentives by holding a larger stake in the fund (Figure 10b). In the optimal mechanisms, these two implementations will provide the manager with identical incentives and, consequently, result in an identical flow-performance relationship and compensation scheme.

![Figure 8: Slope of the flow-performance relationship and capital in the optimal renegotiation-proof contract. The parameters of the model are as in Table 1.](image-url)
Figure 9: Pay-performance sensitivity and compensation in the optimal renegotiation-proof contract. The parameters of the model are as in Table 1.

Figure 10: Incentives in the optimal renegotiation-proof contract and in the market-based contract. The parameters of the models are as in Table 1.
B Static Model

In this appendix, I consider a two-period contracting model in order to provide an analytically tractable example of how, when moral hazard interacts with uncertainty, a principal optimally designs a contract that features reduced experimentation in the future.

To improve tractability, the assumptions on the agent’s preferences are different from the main model. However, rather than being a shortcoming of this example, this difference further goes to the generality of my result.

Before studying the two-period model, consider a one-period model to establish a benchmark. Suppose returns are normally distributed and given by

\[ R \sim N(\alpha + \eta h, \sigma^2) - m, \]

where \( m \) is the agent’s hidden action, which gives him a private benefit \( \lambda m \), and \( h \in \{0, 1\} \) is the agent’s unknown skill. The principal is risk neutral, whereas the agent has CARA utility with absolute risk aversion \( \gamma \). Because returns are normal, the agent’s utility can be expressed in the mean-variance form. I focus on linear contracts that enforce \( m = 0 \). In these contracts, the agent’s final consumption is given by

\[ V = \Delta + C, \]

where

\[ \Delta = \beta(R - (\alpha + \eta \phi_0)) \]

captures the risk exposure of the agent, and \( C \) represents his expected consumption.

The principal controls experimentation through \( \eta \). I assume that \( \eta \in \{0, 1\} \) and that it involves no costs. To ensure incentive compatibility, we must impose \( \beta \geq \lambda \). Then, the optimal contract solves

\[
\begin{align*}
\max & \mathbb{E}[R] - C \\
\text{s.t.} & \beta \geq \lambda \\
& C - \frac{\gamma}{2} \beta^2 \geq U_0.
\end{align*}
\]

It is straightforward to verify the following lemma:

**Lemma 6.** In a one-period contract, \( \beta = \lambda \) and \( \eta = \mathbb{1}\{\phi_0 \geq 0\} \).

Now consider a two-period model. Output in period \( t \) is normally distributed and
given by

\[ R_t \sim N(\alpha + \eta_t h, \sigma^2) - m_t, \]

where \( m_t \) is the hidden action of the agent, which gives him a private benefit \( \lambda a_t \), with \( \lambda \in (0, 1) \). To reduce unnecessary complications, I assume \( \eta_0 = 1 \) and \( \phi_0 > 0 \). The principal can choose \( \eta_1 \in \{0, 1\} \). Because \( m_t > 0 \) is inefficient, I focus on contracts that implement \( m_t = 0 \).

Neither the principal or the agent discount the future. The principal is risk neutral, whereas the agent has CARA preferences over the final payout. The final compensation of the agent is given by

\[ V = \Delta_0 + \Delta_1 + C, \]

where \( \Delta_t \) is the performance-based compensation at time \( t \), and \( C \) is a promised expected compensation that is pinned down by a participation constraint.

I focus on linear contracts where \( \Delta_t \) takes the form \( \Delta_t = \beta_t (R_t - (\alpha + \eta_t \phi_t)) \), and where \( \beta_t \) is chosen in order to make the contract incentive compatible. Because the agent is risk averse, in an optimal contract, \( \beta_t \) will be chosen to be as small as possible, as long as it ensures incentive compatibility.

Finally, once the principal and the agent sign a two-period contract, they fully commit to it. In particular, the principal can commit to a future choice of \( \eta_1 \).

It is immediate to notice that \( \beta_1 = \lambda \), because the last stage of the game is analogous to a standard static problem. However, the choice of \( \beta_0 \) depends on future learning. In particular, to ensure incentive compatibility, me must have

\[ \beta_0 \geq \lambda (1 - E[\eta_1 \phi_m(R_0, 0)]), \]

where \( \phi(R_0, m) \) is the principal’s posterior as a function of the time 0 return \( R_0 \) and the time 0 hidden action \( m \).

By Bayes’ Law, the posterior is given by

\[ \phi_1 = (1 - \bar{v})\phi_0 + \bar{v}(R_0 - \alpha), \]

so that the constraint on \( \beta_0 \) becomes

\[ \beta_0 \geq \lambda + \lambda \bar{v} E[\eta_1]. \]

This inequality is the counterpart of (8) in the paper, and \( \lambda \bar{v} E[\eta_1] \) represents the agent’s information rent in this two-period model.
Similar to the one-period model, the principal solves

\[
\max E[R_0 + R_1] - C \\
\text{s.t. } \beta_0 \geq \lambda + \lambda \nu E[\eta_1] \\
C - \frac{\gamma}{2}(\beta_0^2 + \lambda^2) \geq U_0.
\]

Because the incentive-compatibility and participation constraints must bind, the problem reduces to finding an \(R_0\)-measurable learning strategy \(\eta_1\) that maximizes

\[
E[\eta_1(\phi_1 - \gamma \lambda^2 \bar{v})] - \frac{\gamma}{2} \lambda^2 \bar{v}^2 (E[\eta_1])^2.
\]

Unlike the one-period model, setting \(\eta_1 = 1\) when expected returns are positive is no longer always optimal.

**Proposition 11.** There exists a \(\tilde{\phi}\) such that \(\eta_1 = 0\) if \(\phi_1 < \tilde{\phi}\), whereas \(\eta_1 = 1\) if \(\phi_1 > \tilde{\phi}\). Moreover, \(\tilde{\phi} > 0\).

*Proof.* Let a state be a return realization \(R\) at time 0. Suppose that there exist a set of states \(RR\) and a set of states \(RR'\) such that the posterior \(\phi_1\) in all states in \(RR\) is lower than the posterior in all states of \(RR'\), and such that also that \(\eta_1 = 1\) in \(RR\), while \(\eta_1 = 0\) in \(RR'\). Let \(\nu\) be the measure associated with the normal density with mean \(\alpha + \phi_0\) and variance \(\sigma^2\). Consider \(n = \min(\nu(\RR), \nu(\RR'))\) and suppose that \(n = \nu(\RR')\). Now consider a new contract with a new experimentation policy \(\eta'\). This policy is such that: (i) if \(R \in rr'\), \(\eta'(rr') = 1\), (ii) if \(R \in rr\), \(\eta(R) = 0\) (iii) if \(R \in RR' \setminus rr'\), \(\eta'(R) = 0\), and (iii) if \(R \in RR \setminus rr\), \(\eta'(R) = 1\). Since \(rr'\) and \(rr\) have the same measure, it follows that \((E[\eta'_1])^2 = (E[\eta_1])^2\).

However, since \(\phi_1(R') \geq \phi_1(R)\) for all \(R' \in rr'\) and \(R \in rr\), under the new policy we have \(E[\eta'_1(\phi_1 - \gamma \lambda^2 \bar{v})] > E[\eta_1(\phi_1 - \gamma \lambda^2 \bar{v})]\). This contradicts \(\eta_1\) being optimal. Given that \(\eta_1\) is binary, it immediately follows that the optimal choice takes the form of a threshold rule. Denote the threshold with \(\bar{\phi}\).

It remains to show that the threshold \(\bar{\phi}\) is positive. To do so, consider the objective function, which can be written as

\[
E \left[ \eta_1 \left( \phi_1 - \gamma \lambda^2 \bar{v} - \frac{\gamma}{2} \lambda^2 \bar{v}^2 E[\eta_1] \right) \right].
\]

The necessary condition for optimality is that

\[
\eta_1 = 0 \text{ if } \phi_1 \leq \gamma \lambda^2 \bar{v} + \frac{\gamma}{2} \lambda^2 \bar{v}^2 E[\eta_1],
\]

60
which then implies that \( \bar{\phi} \geq \gamma \lambda^2 \bar{v} + \frac{3}{2} \lambda^2 \bar{v}^2 \mathbb{E}[\eta_1] > 0. \)

As in the dynamic model of the paper, the principal commits to a reduced amount of future learning in order to improve current incentives. The mechanism in this static model is very clear: Future experimentation \( \eta_1 \) requires stronger incentives \( \beta_0 \). But because exposing the agent to risk is costly, the principal has an incentive to reduce future experimentation below the ex post optimal level in order to improve ex ante incentives.
C PROOFS FOR SECTION 3

Before proceeding to the proofs, I need to introduce some formal notation that I have omitted in the main body of the paper. Let \((\Omega, \mathcal{F}, P)\) be a probability space. \((W_t)_{t \geq 0}\) is a Wiener process on this probability state and the manager’s skill \(h\) is a random variable on \((\Omega, \mathcal{F})\). The path of returns \((R_t)_{t \geq 0}\) is also a random variable \((\Omega, \mathcal{F})\) on \((\Omega, \mathcal{F})\). I denote with \(P^e\) a probability measure over the set of paths of returns for a given contract \(c\). As in section 3.3, \((\mathcal{F}_t)_{t \geq 0}\) is the filtration generated by the path of returns \((R_t)_{t \geq 0}\), possibly augmented by the \(P\)-null sets.

C.1 PROOF OF PROPOSITION 1

Proof. Let \(P^G\) be the probability measure on \((\Omega, \mathcal{F}_\infty)\) conditional on \(h = 1\) and let \(P^B\) be the probability measure on \((\Omega, \mathcal{F}_\infty)\) conditional on \(h = 0\).

Let
\[
W_t^{c,0} = \int_0^t dR_s - (r + \mu(\eta_s, 0, m_s))ds
\]  
(33)
and define the likelihood ratio
\[
X_t = \exp \left\{ \int_0^t \eta_s dW_s^{c,0} - \frac{1}{2} \int_0^t \eta_s^2 ds \right\},
\]
which represents the ratio between the likelihood that the path \((R_s)_{0 \leq s \leq t}\) is generated by a skilled manager \((h = 1)\) and the likelihood that the same path is generated by an unskilled manager \((h = 0)\).

Since players use Bayes’ rule to form beliefs,
\[
\phi_t = \frac{pX_t}{pX_t + (1 - p)}.
\]  
(34)
We can then apply Ito’s lemma and obtain
\[
d\phi_t = -\frac{(1 - p)p^2}{(pX_t + (1 - p))^3} \eta_t X_t^2 dt + \frac{(1 - p)p}{(pX_t + (1 - p))^2} \eta_t X_t dW_t^{c,0}.
\]
Using (33) and (34), we conclude that
\[
d\phi_t = \eta_t (1 - \phi_t) \phi_t \frac{1}{\sigma} (dR_t - (r + \mu(\eta_t, \phi_t, m_t)) dt).
\]
C.2 PROOF OF PROPOSITION 2

I proceed by contradiction. Let $C$ be an optimal contract and let $(m_t)_{t \geq 0}$ be the best response of the manager to the contract. Suppose that $(m_t)_{t \geq 0}$ is strictly larger than zero with positive probability.

Since this contract is $\mathcal{F}_t$-adapted, this means that the contract specifies the time $t$ allocation $(C_t, K_t, \eta_t, m_t)$ as a function of the history of returns $(R_s)_{0 \leq s \leq t}$. For example, compensation at time $t$ can be written as,

$$C_t = C^t((R_s)_{0 \leq s \leq t}),$$

experimentation at time $t$ can be written as

$$\eta_t = \eta^t((R_s)_{0 \leq s \leq t}).$$

and shirking at time $t$ can be written as

$$m_t = m^t((R_s)_{0 \leq s \leq t}).$$

Now consider an alternative contract $\hat{C}$. This contract is designed in the following way. If the history of returns at time $t$ is $(R_s)_{0 \leq s \leq t}$, the contract specifies capital and experimentation at time $t$ as equal to the capital and experimentation that contract $C$ specifies after history $(R_s - \int_0^s \hat{m}_u du)_{0 \leq s \leq t}$, where

$$\hat{m}_t = m^t \left( (R_s - \int_0^s \hat{m}_u du)_{0 \leq s \leq t} \right).$$

For example, experimentation at time $t$ for contract $\hat{C}$ is given by

$$\hat{\eta}_t = \eta^t \left( (R_s - \int_0^s \hat{m}_u du)_{0 \leq s \leq t} \right).$$

However, under contract $\hat{C}$, consumption is specified as

$$\hat{C}_t = C^t \left( (R_s - \int_0^s \hat{m}_u du)_{0 \leq s \leq t} \right) + \lambda \hat{m}_t \hat{K}_t.$$

If the agent never shirks when he’s offered the alternative contract $\hat{C}$, he obtains a consumption process that coincides with the consumption process he obtains by shirking.
in contract \( C \). If the agent chooses a shirking process \( (m_t')_{t \geq 0} \) with contract \( \hat{C} \), he obtains the same consumption process he would have obtained in contract \( C \) from a shirking process \( (m_t + m'_t)_{t \geq 0} \).

Because \( (m_t)_{t \geq 0} \) is the best response of the manager to contract \( C \), that is,

\[
(m_t)_{t \geq 0} \in \arg \max_{(m'_t)_{t \geq 0}} \mathbb{E} \left[ \int_0^\infty e^{-\delta t} u(C_t + m'_t \lambda K_t) \, dt \bigg| \mathcal{F}_0, C \right]
\]

then it must be the case that \( (0)_{t \geq 0} \) is the best response of the manager to contract \( \hat{C} \), that is,

\[
(0)_{t \geq 0} \in \arg \max_{(m'_t)_{t \geq 0}} \mathbb{E} \left[ \int_0^\infty e^{-\delta t} u(\hat{C}_t) \, dt \bigg| \mathcal{F}_0, \hat{C} \right],
\]

since, otherwise, \( (m_t)_{t \geq 0} \) would not be a best response in the original contract.

Under the new contract \( \hat{C} \) the agent receives the same lifetime utility as in contract \( C \). However, the costs for the principal change by

\[
\mathbb{E} \left[ \int_0^\infty e^{-rt} \left( 1 - \frac{\lambda}{q(\bar{\eta})} \right) m_t K_t \, dt \right] \leq 0.
\]

Since \( q(\bar{\eta}) \geq \lambda \), then the principal is now bearing lower costs. This contradicts the assumption that \( C \) is an optimal contract.

### C.3 Proof of Proposition 3

I use the stochastic maximum principle to derive necessary conditions for incentive compatibility, as in (Williams, 2011). Fix an incentive compatible contract with full effort, \( C \). Consider a shirking process \( m = (m_t)_{t \geq 0} \). Let \( P^C \) be the probability measure over path of returns for which \( W_t^C \) is a standard Brownian motion. Let \( P^m \) be a measure under which

\[
W_t^m = W_t^e - \int_0^t \frac{m_s}{\sigma} \, ds
\]

is a standard Brownian motion.

I denote with \( \tau \) a stopping time at which the principal stops experimentation forever, that is, \( \tau = \inf\{t \geq 0 : \eta_s \leq 0, \forall s \geq t\} \). After time \( \tau \), Lemma 3 applies and the skill of the manager becomes irrelevant for returns and incentive provision. In particular, the manager’s continuation value at \( \tau \), \( V_\tau \), does not depend on beliefs about the manager’s skill.

Instead of considering the posterior beliefs \( \phi_t \) as state variable, it is convenient to work
with the log-likelihood ratio $x_t = \log X_t$, where $X_t$ is the likelihood ratio. We can then express beliefs as a function of the log-likelihood ratio,

$$\psi(x) = \frac{pe^x}{1 - p + pe^x}. \tag{35}$$

More precisely, let $x_t$ be the principal’s log-likelihood ratio and let $x_t + \Delta x_t$ be the agent’s log-likelihood ratio. The laws of motion of $x_t$ and $\Delta x_t$ are given by:

$$dx_t = \left(\psi(x_t) - \frac{1}{2}\right) \eta_t^2 dt + \eta_t dW_t^C$$

$$d\Delta x_t = \frac{m_t}{\sigma} \eta dt.$$

The continuation value of the agent, given $\mathcal{C}$ and $m$, can be written as

$$\mathbb{E} \left[ \int_0^\infty \Gamma_m e^{-\delta t} u(C_t + m_t \lambda K_t) \, dt \mid \mathcal{F}_0 \right].$$

where $\Gamma_m \equiv \frac{dm}{dP^C}$ is a density process representing the change of measure for the path of returns induced by the shirking strategy $m$. By Girsanov’s Theorem, $\Gamma_m$ as

$$d\Gamma_t = \left(-\frac{m_t}{\sigma} + (\psi(x_t + \Delta x_t) - \psi(x_t))\right) \Gamma_t dW_t^C.$$

Let $\mathcal{V}$ be the multiplier for $\Gamma_m$ and let $\beta_t$ be the multiplier’s volatility. Similarly, let $\xi_t$ be the multiplier for $\Delta x_t$ with volatility $\omega_t$. The Hamiltonian for the agent’s optimization problem is the following:

$$\Gamma u(C + m\lambda K) + \left(-\frac{m}{\sigma} + (\psi(x + \Delta x) - \psi(x))\right) \Gamma \beta + \frac{m}{\sigma} \eta \xi \tag{36}$$

If $m = 0$ is optimal, it must be the case that $m = 0$ maximizes (36) with $\Gamma_m = 1$ and $\Delta x = 0$, which happens only if

$$u'(C)\lambda K + \frac{\eta \xi}{\sigma} - \frac{\beta}{\sigma} \leq 0.$$

The multipliers $\mathcal{V}$ and $\xi$ solve the following backward stochastic differential equations (BSDEs)

$$d\mathcal{V}_t = (\delta \mathcal{V}_t - u(C_t)) \, dt + \beta_t dW_t^C.$$
\[ d\xi_t = (\delta \xi_t - \phi_t (1 - \phi_t) \eta_t \beta_t) dt + \delta_t dW^e_t, \]  
(37)

with terminal conditions
\[ \mathcal{V}_\tau = V_\tau \]
\[ \xi_\tau = 0. \]

Solving the BDSE for \( \mathcal{V}_t \), we obtain
\[ \mathcal{V}_t = V_t = E \left[ \int_t^\infty e^{-\delta s} u(C_s) \, ds \bigg| \mathcal{F}_t \right]. \]

C.4 Proof of Proposition 4

Proof. To show that \( \xi_t = \phi_t (1 - \phi_t) \partial_x V_t \), it is sufficient to show that \( \xi_t = \partial_x V_t \), where \( x_t \) is the log-likelihood ratio at time \( t \). The fact that \( \partial_x V_t = \phi_t (1 - \phi_t) \partial_x V_t \) follows from equation (35).

Consider
\[ V_t = E \left[ \int_t^\infty e^{-\delta s} u(C_s) \, ds \bigg| \mathcal{F}_t \right]. \]

Given an initial log-likelihood ratio \( x_t \), we can write
\[ (1 - p + pe^{x_t}) V_t = E \left[ \int_t^\infty (1 - p + pe^{\Delta x_s + x_t}) e^{-\delta s} u(C_s) \, ds \bigg| \mathcal{F}_t, h = 0 \right]. \]

Differentiating with respect to \( x_t \) we obtain
\[ pe^{x_t} V_t + (1 - p + pe^{x_t}) \partial_x V_t = E \left[ \int_t^\infty pe^{\Delta x_s + x_t} e^{-\delta s} u(C_s) \, ds \bigg| \mathcal{F}_t, h = 0 \right], \]
(38)

which can be rearranged as
\[ \partial_x V_t = \phi_t (G_t - V_t) \]

where
\[ G_t = E \left[ \int_t^\infty e^{-\delta s} u(c_s) \, ds \bigg| \mathcal{F}_t, h = 1 \right]. \]

Using Girsanov’s theorem and Ito’s lemma, we can derive the law of motion of \( \phi_t (G_t - V_t) \),
\[ d\phi_t (G_t - V_t) = (\delta \phi_t (G_t - V_t) - \eta_t \beta_t \phi_t (1 - \phi_t)) dt + \omega_t dW^e_t, \]
(39)

for some \( \mathcal{F}_t \)-adapted process \( (\omega_t^t)_{t \geq 0} \), with the terminal condition \( \phi_\tau (G_\tau - V_\tau) = 0 \), where \( \tau = \inf \{ t \geq 0 : \eta_s \leq 0, \forall s \geq t \} \). Using the comparison principal for BSDEs (37) and (39)
(Pham, 2009), we conclude that $\xi_t = \phi_t (G_t - V_t) = \partial_t V_t$. Moreover, we can solve the BSDEs and obtain

$$\xi_t = \mathbb{E} \left[ \int_t^\infty e^{-\delta s} \eta_s \beta_s \phi_s (1 - \phi_s) \, ds \middle| \mathcal{F}_t \right].$$

It now remains to prove that, in an incentive compatible contract, $\xi_t \geq 0$. In order to show that $\xi_t$ is positive, consider the BSDE:

$$\hat{\xi}_t = (\delta \hat{\xi}_t - \eta_t^2 \phi_t (1 - \phi_t) \xi_t - \eta_t \phi_t (1 - \phi_t) u'(C_t) \lambda K_t \sigma) \, dt + \hat{\delta} dW^c_t,$$

with terminal condition $\hat{\xi}_\tau = 0$.

By the incentive-compatibility condition (8), $-\eta_t \xi_t - u_c(C_t) K_t \lambda \sigma \geq -\beta_t$. Moreover, $\eta_t \phi_t (1 - \phi_t) \geq 0$. Hence, from the comparison principle for BSDE (Pham, 2009), it follows that $\hat{\xi}_t \leq \xi_t$. Moreover, $\hat{\xi}_t$ can be written in closed form as

$$\hat{\xi}_t = \mathbb{E} \left[ \int_t^\infty e^{-\int_s^t \delta - \eta_u \phi_u (1 - \phi_u) \, du} \eta_s \beta_s \phi_s (1 - \phi_s) u'(C_s) \lambda K_s \sigma \, ds \middle| \mathcal{F}_t \right] \geq 0,$$

from which we conclude that $\xi_t \geq 0$.

\section*{C.5 Proof of Proposition 5}

\textit{Proof.} Let $\tilde{\phi}_t$ be the agent’s posterior at time $t$, while $\phi_t$ is the principal’s. Define

$$\zeta_t = \frac{\xi_t}{\phi_t (1 - \phi_t)},$$

which evolves as

$$d\zeta_t = \left( \delta \zeta_t - \eta_t \beta_t + \eta_t^2 \phi_t (1 - \phi_t) \zeta_t - \eta_t (1 - 2 \phi_t) \omega_{\zeta_t} \right) \, dt + \omega_{\zeta_t} dW^c_t,$$

where

$$\omega_{\zeta_t} = \frac{\omega_t - \xi_t (1 - 2 \phi_t) \eta_t}{\phi_t (1 - \phi_t)}. \tag{40}$$

I want to show that, if the conditions of the proposition are satisfied, then $V_t + (\tilde{\phi}_t - \phi_t) \zeta_t$ is an upper bound on the continuation value of the agent at time $t$. Since $\tilde{\phi}_0 = \phi_0$, this will prove that the agent has no (strictly) better strategy than choosing $m_t = 0$ for all $t \geq 0$.

Let $\tau = \inf\{t \geq 0 : \eta_s \leq 0, \forall s \geq t\}$ be the stopping time at which the principal stops experimentation. For $t \geq \tau$ we have that $\zeta_t = 0$ and standard arguments imply that $V_t$ is an upper bound for the agent’s continuation value (DeMarzo and Sannikov, 2006; Sannikov, 2008).
For \( t < \tau \), consider an arbitrary deviation up to time \( t \) and let

\[
G_t = \int_0^t e^{-\delta s} u(C_s + m_s \lambda K_s) \, ds + e^{-\delta t} \left( V_t + (\tilde{\phi}_t - \phi_t) \zeta_t \right).
\]

It suffices to show that \( G_t \) is a supermartingale for \( t < \tau \). Indeed, in this case

\[
G_t \geq E[G_\tau | \mathcal{F}_t] = E \left[ \int_0^\tau e^{-\delta s} u(C_s + m_s \lambda K_s) \, ds + e^{-\delta \tau} V_\tau \right]_{\mathcal{F}_t},
\]

which then would imply that

\[
V_t + (\tilde{\phi}_t - \phi_t) \zeta_t \geq E \left[ \int_t^\tau e^{-\delta (s-t)} u(C_s + a_s \lambda K_s) \, ds + e^{-\delta \tau} V_\tau \right]_{\mathcal{F}_t}.
\]

In order to show that \( W_t \) is a supermartingale, it is sufficient to prove that the drift of \( dW_t \) is non-positive. Using Ito’s lemma, and after some simplifications, the drift of \( dG_t \) reduces to

\[
e^{-\delta t} \left[ u(C_t + m_t \lambda K_t) - u(C_t) - \beta_t \frac{m_t}{\sigma} + \phi_t (1 - \phi_t) \zeta_t \frac{m_t}{\sigma} - (\tilde{\phi}_t - \phi_t) \omega \zeta_t \frac{m_t}{\sigma} \right].
\]

Recall that \( \zeta_t = \phi_t (1 - \phi_t) \zeta_t \). Combining the assumptions of the proposition with equation (40), we conclude that the drift is maximized for \( m_t = 0 \). For \( m_t = 0 \) the drift of \( dG_t \) is zero. Hence, for an arbitrary deviation \( m_t \), drift of \( G_t \) is non-positive and \( G_t \) is a supermartingale.
D Proofs for Section 4

To prove the two propositions in Section 4, I first develop a series of lemmas to help organize the results. Before presenting the lemmas, let me introduce some notation that will facilitate the exposition. I denote with $C_{V,\xi,\phi}^{P}$ the optimal contract for the primal problem when the initial state is given by $V, \xi$ and $\phi$. I denote with $C_{V,Y,\phi}^{D}$ the optimal contract for the dual problem when the initial state is given by $V, Y$ and $\phi$.

Let $\mathcal{C}$ be an incentive-compatible contract that enforces no shirking, delivers expected lifetime utility $V$ to the agent and that implies an information rent $\xi$. I denote with $J(V, \xi, \phi|\mathcal{C})$ the primal cost function when the principal chooses contract $\mathcal{C}$. Similarly, I denote with $G(V, Y, \phi|\mathcal{C})$ be the dual cost function when the principal chooses contract $\mathcal{C}$.

Then we must have that $J^*(V, \xi, \phi) = J(V, \xi, \phi|C_{V,\xi,\phi}^{P})$ and $G^*(V, Y, \phi) = G(V, Y, \phi|C_{V,Y,\phi}^{D})$.

Any optimal contract for the dual problem implies an information rent. Given a contract $\mathcal{C}$ that specifies $\mathcal{F}_t$-adapted processes for incentives and experimentation, $(\beta_t)_{t \geq 0}$ and $(\eta_t)_{t \geq 0}$, I denote with $\tilde{\xi}(\mathcal{C}, \phi)$ the information rent implied by the contract $\mathcal{C}$ when beliefs are $\phi$, that is,

$$\tilde{\xi}(\mathcal{C}, \phi) = E \left[ \int_0^\infty e^{-\delta t} \eta_t \beta_t (1 - \phi_t) \, dt \bigg| \mathcal{F}_0 \right].$$

I will often consider the information rent implied by an optimal contract for the dual problem $C_{V,Y,\phi}^{D}$. To simplify notation I denote this information rent as $\hat{\xi}(V, Y, \phi) = \tilde{\xi}(C_{V,Y,\phi}^{D}, \phi)$.

Throughout this appendix, I use the following notation for the left and right derivatives of a function. Consider a function of $n$ variables $h(x_1, \ldots, x_n)$. I denote the left derivative with respect to $x_i$ as

$$h_{x_i^-}(x_1, \ldots, x_n) \equiv \lim_{\varepsilon \to 0^-} \frac{h(x_1, \ldots, x_i, \ldots, x_n) - h(x_1, \ldots, x_i - \varepsilon, \ldots, x_n)}{\varepsilon},$$

and I denote the right derivative with respect to $x_i$ as

$$h_{x_i^+}(x_1, \ldots, x_n) \equiv \lim_{\varepsilon \to 0^+} \frac{h(x_1, \ldots, x_i + \varepsilon, \ldots, x_n) - h(x_1, \ldots, x_i, \ldots, x_n)}{\varepsilon}.$$  

Whenever the left and right derivative coincide, I use the usual notation $h_{x_i}(x_1, \ldots, x_n)$ to denote the derivative of $h$ with respect to $x_i$.

**Lemma 7.** Consider an incentive compatible contract $\mathcal{C}$. If $E \left[ \int_0^\infty e^{-rt} \left( \mu(\eta_t, \phi_t, 0) \frac{\eta_t \xi_t}{\lambda \sigma} C_t^{P} \right) \, dt \bigg| \mathcal{F}_0 \right]$
is finite, then

\[ E \left[ \int_0^\infty e^{-rt} (Y_t \eta_t \beta_t \phi_t (1 - \phi_t)) \, dt \right| F_0 \] \]

\[ = Y_0 \tilde{\xi}(C, \phi_0) - E \left[ \int_0^\infty e^{-rt} \left( \mu(\eta_t, \phi_t, 0) \frac{\eta_t \xi_t}{\lambda \sigma} C_t^\rho \right) \, dt \right| F_0 \] 

**Proof.** Let

\[ \tilde{Y}_t = \left[ - \left( \int_0^t e^{(\delta - r)s} \mu(\eta_s, \phi_s, 0) \frac{\eta_s}{\lambda \sigma} C_t^\rho \, ds \right) + \tilde{Y} \right] \]

so that

\[ Y_t = e^{(r - \delta)\tilde{Y}_t} \]

For a finite \( T > 0 \), we can use integration by parts to obtain

\[ E \left[ \int_0^T e^{-rt} (Y_t \eta_t \beta_t \phi_t (1 - \phi_t)) \, dt \right| F_0 \]

\[ = \tilde{Y}_0 E \left[ \int_0^T e^{-\delta s} \eta_s \beta_s \phi_s (1 - \phi_s) \, ds \right| F_0 \]

\[ + E \left[ \int_0^T e^{-\delta t} \left( \int_t^T e^{-\delta(s-t)} \eta_s \beta_s \phi_s (1 - \phi_s) \, ds \right) \, d\tilde{Y}_t \right| F_0 \] 

\[ = \tilde{Y}_0 E \left[ \int_0^T e^{-\delta s} \eta_s \beta_s \phi_s (1 - \phi_s) \, ds \right| F_0 \]

\[ - E \left[ \int_0^T e^{-rt} \left( \int_t^T e^{-\delta(s-t)} \eta_s \beta_s \phi_s (1 - \phi_s) \, ds \right) \mu(\eta_t, \phi_t, 0) \frac{\eta_t}{\lambda \sigma} C_t^\rho \right| F_0 \] 

Using the monotone convergence theorem and the law of iterated expectations we can conclude that

\[ E \left[ \int_0^\infty e^{-rt} (Y_t \eta_t \beta_t \phi_t (1 - \phi_t)) \, dt \right| F_0 \] \]

\[ = \lim_{t \to \infty} E \left[ \int_0^T e^{-rt} (Y_t \eta_t \beta_t \phi_t (1 - \phi_t)) \, dt \right| F_0 \]

\[ = \tilde{Y}_0 \tilde{\xi}(C, \phi_0) - E \left[ \int_0^\infty e^{-rt} \left( \mu(\eta_t, \phi_t, 0) \frac{\eta_t \xi_t}{\lambda \sigma} C_t^\rho \right) \, dt \right| F_0 \] 

Since \( \tilde{Y}_0 = Y_0 \), this concludes the proof. \( \square \)
**Lemma 8.** $G^*(V,Y,\phi)$ is decreasing and concave in $Y$.

**Proof.** Fix $V$ and $\phi$, consider $Y^0$ and $Y^1$ such that $Y^0 \leq Y^1$. Then

$$J^*(V,Y^1,\phi) \leq J(V,Y^1,\phi|C^D_{V,Y^0,\phi}) = J^*(V,Y^0,\phi) + (Y^0 - Y^1)\xi_0 \leq J^*(V,Y^0,\phi),$$

where the first equality follows by applying Lemma 7.

To show convexity, consider $\nu \in [0, 1]$ and define $Y^\nu = \nu Y^0 + (1-\nu)Y^1$. Then

$$\nu J^*(V,Y^0,\phi) + (1-\nu)J^*(V,Y^1,\phi) \leq \nu J(V,Y^0,\phi|C^D_{V,Y^\nu,\phi}) + (1-\nu)J(V,Y^1,\phi|C^D_{V,Y^\nu,\phi}) \leq J^*(V,Y^\nu,\phi).$$

The last equality follows from the fact that, under contract $C^D_{V,Y^\nu,\phi} = ((C^\nu_t, K^\nu_t, \eta^\nu_t, 0))_{t \geq 0}$,

$$Y^\nu_t = e^{(r-\delta)t} \left[-\left(\int_0^t e^{(\delta-\gamma)s}(\alpha + \sigma \eta^\nu_t \phi^\nu_t)\frac{\eta^\nu_t}{\lambda \sigma} C^\nu_t \right) \right] + Y^\nu.$$

**Lemma 9 (Weak Duality).** Let $C$ be an incentive compatible contract that delivers expected lifetime utility $V$ to the agent and that implies an information rent $\xi$. Then

$$J(V,\xi,\phi|C) = G(V,Y,\phi|C) + Y\xi$$

**Proof.** This follows immediately from Lemma (7), since

$$J(V,\xi,\phi|C) = E \left[ \int_0^\infty e^{-rt} \left( \frac{C_t}{q(\eta_t)} - \mu(\eta_t,\phi_t,0)\frac{\eta_t}{\lambda \sigma} C^\rho_t \right) dt \bigg| F_0 \right] = G(V,Y,\phi|C) + E \left[ \int_0^\infty e^{-rt} \left( \mu(\eta_t,\phi_t,0)\frac{\eta_t}{\lambda \sigma} C^\rho_t \right) dt \bigg| F_0 \right] + E \left[ \int_0^\infty e^{-rt} (Y_t \eta_t \beta_t (1-\phi_t)) dt \bigg| F_0 \right] = G(V,Y,\phi|C) + Y\xi.$$

**Lemma 10.** The dual cost function is homogeneous with $G^*(V,Y,\phi) = \hat{v}g^*(y,\phi)$ for a continuous function $g^*(y,\phi)$.

**Proof.** Let

$$\hat{v} = ((1-\rho)V)^{\frac{1}{1-\rho}}$$
be the consumption equivalent of the manager’s continuation value and let
\[ y = (1 - \phi)\hat{v}^{-\rho}Y \]
be scaled version of multiplier \( Y \). Define the scaled control variables
\[ c_t = \frac{C_t}{\hat{v}_t}, \quad k_t = \frac{K_t}{\hat{v}_t}, \quad \hat{\beta}_t = \frac{\beta_t}{(1 - \rho)\hat{v}_t}. \tag{41} \]
The laws of motion of promised value \( \hat{v} \) and multiplier \( y \) can be obtained by applying Ito’s lemma,
\[ \frac{d\hat{v}_t}{\hat{v}_t} = \left( \frac{\delta}{1 - \rho} - \frac{c_t^{1-\rho}}{1 - \rho} + \frac{1}{2} \rho \hat{\beta}_t^2 \right) dt + \hat{\beta}_t \hat{v}_t dW^c_t \tag{42} \]
and
\[ dy_t = -(1 - \phi_t)\mu(\eta_t, \phi_t, 0) \frac{\eta_t}{\lambda \sigma} c_t^\rho dt \]
\[ + y_t \left( r - \frac{\delta}{1 - \rho} + \rho \frac{c_t^{1-\rho}}{1 - \rho} + \frac{1}{2} \rho \hat{\beta}_t^2 + \rho \eta_t \hat{\beta}_t \phi_t \right) dt - y_t \left[ \rho \hat{\beta}_t + \eta_t \phi_t \right] dW^c_t. \tag{43} \]

From equation (42), we obtain
\[ \hat{v}_t = \hat{v}_0 \exp \left\{ \int_0^t \left( \frac{\delta}{1 - \rho} - \frac{c_s^{1-\rho}}{1 - \rho} + \frac{1}{2} \rho \hat{\beta}_s^2 - \frac{1}{2} \hat{\beta}_s^2 \right) ds + \int_0^t \hat{\beta}_s dW^c_s \right\} \tag{44} \]

Using expressions (41) and equation (44), we can write the objective function as
\[ \hat{v}_0 \mathbb{E} \left[ \int_0^\tau e^{-\int_0^t \rho^- \left( \frac{c_s^{1-\rho}}{1 - \rho} + \frac{1}{2} \rho \hat{\beta}_s^2 \right) ds} B_t \left( \frac{c_t}{q(\eta)} - \frac{\beta_t}{\lambda \sigma} c_t^{\rho - 1} \right) dt \mid \mathcal{F}_0 \right] \tag{45} \]
where \( B_t \) is a density process,
\[ B_t = \exp \left\{ \int_0^t \hat{\beta}_s dW^c_s - \frac{1}{2} \int_0^t \hat{\beta}_s^2 ds \right\}. \]

Minimizing (45) is equivalent to minimizing the expectation in (45) subject to beliefs and the law of motion of multiplier \( y \), (43). To see why, notice that any policy that is optimal for initial states \( \hat{v}_0, y_0, \) and \( \phi_0 \) must also be optimal for \( \hat{v}_0', y_0, \) and \( \phi_0 \), with \( \hat{v}_0' \neq \hat{v}_0 \).
Le \( g^*(y, \phi) \) be defined by

\[
g^*(y_0, \phi_0) = \inf_{(c_t, \tilde{\phi}_t, \eta_{t}) \geq 0} \bar{E} \left[ \int_{0}^{T} e^{-f_{0}^{t} - \left( \frac{c_{t}}{1 - \rho} \right) ds} \left( \frac{c_{t}}{q(\eta)} - \mu(\eta_{t}, \phi_{t}, 0) \frac{\tilde{\beta}_{t}}{\lambda} c_{t} - y_t \phi_t \eta_t \beta_t \right) dt \bigg| \mathcal{F}_0 \right]
\]

s.t. \( d\phi_t = \eta_t \phi_t (1 - \phi_t) \hat{\beta} dt + \eta_t \phi_t (1 - \phi_t) d\tilde{W}_t \)

\[
dy_t = -(1 - \phi_t) \mu(\eta_t, \phi_t, 0) \frac{\eta_t}{\lambda} c_t dt + y_t \left( r - \frac{\delta}{1 - \rho} + \rho \frac{c_t^{1 - \rho}}{1 - \rho} + \frac{1}{2} \rho \beta_t^2 + \rho \eta_t \beta_t \phi_t \right) dt \\
\quad - y_t [\rho \beta_t + \eta_t \phi_t] \hat{\beta} dt - y_t [\rho \beta_t + \eta_t \phi_t] d\tilde{W}_t,
\]

(46)

where \( \tilde{W}_t \) is a Brownian motion under the measure \( \bar{Q} \) such that \( \frac{d\bar{Q}}{dp}|_{\mathcal{F}_t} = B_t \). Then \( G^*(V, Y, \phi) = \hat{v} g^*(y, \phi) \) where \( \hat{v} = ((1 - \rho)V)^{\frac{1}{1 - \rho}} \) and \( y = (1 - \phi)\hat{v}^{-\rho}Y \).

\textbf{Lemma 11.} If \( Y < 0 \), \( G^* \) is differentiable with respect to \( Y \).

\textit{Proof.} Let us use Lemma 10 to obtain \( G^*(V, Y, \phi) = \hat{v} g^*(y, \phi) \) for \( y = (1 - \phi)\hat{v}^{-\rho}Y \). Since \( G^* \) is concave in \( Y \), then \( g^* \) is concave in \( y \). Therefore, the right and left derivatives of \( \hat{v} g^* \) with respect to \( y \) always exist in the interior of the domain. Moreover, \( G^*_t(V, Y, \phi) \) exists if and only if \( \hat{v} g^*_y(y, \phi) \) exists. Suppose that there exist a point, \((\hat{v}_0, y_0, \phi_0)\) where \( g^*_y < g^*_y \). Consider a test function \( F \) such that \( F(\hat{v}, y, \phi) - \hat{v} g^*(y, \phi) \) has a local minimum at \((\hat{v}_0, y_0, \phi_0)\) and such that \( F(\hat{v}_0, y_0, \phi_0) - \hat{v}_0 g^*(y_0, \phi_0) = 0 \).

Then consider a \( p \in (g^*_y, g^*_y) \) and take another test function

\[
F_{\varepsilon}(\hat{v}, y, \phi) = F(\hat{v}, y_0, \phi) + \hat{v} p(y - y_0) - \frac{1}{2\varepsilon}(\hat{v}(y - y_0))^2.
\]

For any arbitrary \( \varepsilon > 0 \), \( F_{\varepsilon}(\hat{v}, y, \phi) \geq \hat{v} g^*(y, \phi) \) in a neighborhood of \((\hat{v}_0, y_0, \phi_0)\) and \((\hat{v}_0, y_0, \phi_0)\) is a minimizer of \( F_{\varepsilon}(\hat{v}, y, \phi) - \hat{v} g^*(y, \phi) \). Then the viscosity subsolution property
of $F_\varepsilon$ (Pham, 2009) implies that

$$rg^*(y, \phi) - \inf_{c,\beta,\eta} \left\{ \frac{c}{q(\varepsilon)} - (\alpha + \sigma \eta \phi) \hat{\beta} \frac{c^\sigma}{\sigma \lambda} - y \hat{\beta} \eta \phi ight.$$  

$$+ g^*(y, \phi) \left( \frac{\delta}{1 - \rho} - \frac{c^{1-\rho}}{1 - \rho} + \frac{1}{2} \rho \hat{\beta}^2 \right)$$  

$$+ p \left[ -(1 - \phi_t) \mu(\eta, \phi, 0) \frac{\eta}{\lambda \sigma} c^\rho 
\quad + y \left( r - \frac{\delta}{1 - \rho} + \rho \frac{c^{1-\rho}}{1 - \rho} + \frac{1}{2} \rho \hat{\beta}^2 + \rho \eta \hat{\beta} \phi \right) \right] \right\} \leq 0. \tag{47}$$

$[\rho \hat{\beta} + \eta \phi]^2$ is always strictly positive. To see why, suppose that $\eta = 0$. In this case, since $g^*(y, \phi)$ is decreasing and concave in $y$, we must have that $g^*(y, \phi) - \frac{1}{2} py \geq 0$. Then the first order condition with respect to $\hat{\beta}$ holds and imply that

$$\alpha \frac{c^\rho}{\lambda \sigma} = (g^*(y, \phi) - \frac{1}{2} py) \hat{\beta},$$

which in turn implies that $\hat{\beta} > 0$ because $c$ is also positive, since $g^*(y, \phi) - \rho py \geq 0$ and the marginal utility of consumption is infinite at $c = 0$. Since $[\rho \hat{\beta} + \eta \phi]^2$ is strictly positive and $\varepsilon$ is arbitrary, the inequality in (47) is a contradiction. Hence, $\hat{\xi}^*_g(y, \phi) = \hat{\xi}^*_g(y, \phi)$, from which it immediately follows that $G_{\xi^+}(V, Y, \phi) = G_{\xi^-}(V, Y, \phi)$.

**Lemma 12.** If $Y < 0$, then

$$G_{\xi^+}(V, Y, \phi) = -\hat{\xi}(V, Y, \phi).$$

**Proof.** Consider the left derivative

$$G_{\xi^-}(V, Y, \phi) = \lim_{\epsilon \to 0^+} \frac{G^*(V, Y, \phi) - G^*(V, Y - \epsilon, \phi)}{\epsilon}.$$
Since $G^*(V, Y - \epsilon, \phi) < G(V, Y - \epsilon, \phi|C_{V,Y,\phi})$, then

$$G_{Y-}^*(V, Y, \phi) \geq \lim_{\epsilon \to 0^+} \frac{G^*(V, Y, \phi) - G(V, Y - \epsilon, \phi|C_{V,Y,\phi})}{\epsilon} = -\mathbb{E} \left[ \int_0^T e^{-\delta s}\beta_s\phi_s(1 - \phi_s) \, ds \middle| \mathcal{F}_0 \right] = -\hat{\xi}(V, Y, \phi)$$

Similarly, consider the right derivative

$$G_{Y+}^*(V, Y, \phi) = \lim_{\epsilon \to 0^+} \frac{G^*(V, Y + \epsilon, \phi) - G^*(V, Y, \phi)}{\epsilon}.$$

Then

$$G_{Y+}^*(V, Y, \phi) \leq \lim_{\epsilon \to 0^+} \frac{G(V, Y + \epsilon, \phi|C_{V,Y,\phi}) - G^*(V, Y, \phi)}{\epsilon} = -\mathbb{E} \left[ \int_0^T e^{-\delta s}\beta_s\phi_s(1 - \phi_s) \, ds \middle| \mathcal{F}_0 \right] = -\hat{\xi}(V, Y, \phi).$$

Hence

$$G_{Y+}^*(V, Y, \phi) \leq -\hat{\xi}(V, Y, \phi) \leq G_{Y-}^*(V, Y, \phi).$$

Since $G^*$ is differentiable with respect to $Y$ when $Y < 0$, we conclude that $G_{Y+}^*(V, Y, \phi) = -\hat{\xi}(V, Y, \phi)$.

**Lemma 13.** $\hat{\xi}(V, 0, \phi) = \lim_{\epsilon \to 0^+} \hat{\xi}(V, -\epsilon, \phi)$

**Proof.** Let $\xi_0 = \hat{\xi}(V_0, 0, \phi_0)$. Since $Y_t < 0$ for all $t > 0$, this means that $\xi_t = \hat{\xi}(V_t, Y_t, \phi_t) \leq \lim_{\epsilon \to 0^+} \hat{\xi}(V, -\epsilon, \phi)$. Define $\xi^{-\epsilon} \equiv \lim_{\epsilon \to 0^+} \hat{\xi}(V, -\epsilon, \phi)$. Suppose that $D \equiv \hat{\xi}(V, 0, \phi) - \xi^{-\epsilon} > 0$.

In general,

$$\xi_0 = \mathbb{E} \left[ \int_0^t e^{-\delta s}\beta_s\phi_s(1 - \phi_s) \, ds \middle| \mathcal{F}_0 \right] + \mathbb{E} \left[ e^{-D_t} \xi_t | \mathcal{F}_t \right]$$

$$\leq \mathbb{E} \left[ \int_0^t e^{-\delta s}\beta_s\phi_s(1 - \phi_s) \, ds \middle| \mathcal{F}_0 \right] + \xi^{-\epsilon},$$

which implies that

$$D \leq \mathbb{E} \left[ \int_0^t e^{-\delta s}\beta_s\phi_s(1 - \phi_s) \, ds \middle| \mathcal{F}_0 \right].$$

Since $t$ is arbitrary, we conclude that $D = 0$. □
**Lemma 14.** For any arbitrary $\xi$ and $Y$,

$$G^*(V,Y,\phi) \leq J^*(V,\xi,\phi) - Y\xi.$$ 

*Proof.* Using Lemma 9, we can see that, for any arbitrary $\xi$ and $Y$,

$$J^*(V,\xi,\phi) - Y\xi = J(V,\xi,\phi|\mathcal{E}_{V,\xi,\phi}) - Y\xi = G(V,Y,\phi|\mathcal{E}_{V,\xi,\phi}) \geq G^*(V,Y,\phi).$$

$\square$

**Lemma 15.** For any arbitrary $Y$,

$$G^*(V,Y,\phi) \geq J^*(V,\hat{\xi}(V,Y,\phi),\phi) - Y\hat{\xi}(V,Y,\phi).$$

*Proof.* Using Lemma 9, we can see that, for any arbitrary $Y$,

$$J^*(V,\hat{\xi}(V,Y,\phi),\phi) \leq J(V,\hat{\xi}(V,Y,\phi),\phi|\mathcal{E}_{V,Y,\phi}) = G^*(V,Y,\phi) + Y\hat{\xi}(V,Y,\phi).$$

$\square$

**Lemma 16.** If $Y = \arg\sup_{Y' \leq 0}\{G^*(V,Y',\phi) + Y'\xi\}$, then

$$\hat{\xi}(V,Y,\phi) = \xi, \text{ if } Y < 0$$

and

$$\hat{\xi}(V,0,\phi) \leq \xi.$$ 

*Proof.* Since $G^*(V,Y',\phi)$ is concave and differentiable with respect to $Y$ for $Y < 0$, the first-order condition holds and

$$-G^*_Y(V,Y,\phi) = \xi$$

if $Y < 0$, and

$$-G^*_Y(V,0,\phi) \leq \xi$$

if $Y = 0$. Using Lemmas 12 and 13, we conclude the proof. $\square$

**Lemma 17.** $J^*(V,\hat{\xi}(V,0,\phi),\phi) \leq J^*(V,\xi,\phi)$ for all $\xi \geq \hat{\xi}(V,0,\phi)$.

*Proof.* If $\xi \geq \hat{\xi}(V,0,\phi)$ then $0 = \arg\sup_{Y' \leq 0}\{G^*(V,Y',\phi) + Y'\xi\}$. Let

$$\bar{\xi} = \arg\min_{\xi \geq \hat{\xi}(V,0,\phi)} \{J^*(V,\xi,\phi)\}.$$
From Lemma 14, it follows that \( G^*(V, 0, \phi) \leq J(V, \xi, \phi) \). But, by Lemma 15, we must also have \( G^*(V, 0, \phi) \geq J^*(V, \xi(V, 0, \phi), \phi) \). Therefore, \( J(V, \xi, \phi) \) is minimized by \( \xi = \xi(V, 0, \phi) \).

**Lemma 18.** Consider \( \xi \leq \hat{\xi}(V, 0, \phi) \) and let \( Y = \arg \sup_{Y' \leq 0} \{ G^*(V, Y', \phi) + Y'\xi \} \). Then \( \mathcal{C}^D_{V,Y,\phi} \) is optimal also for the primal problem for initial states \( V, \xi \) and \( \phi \). Moreover,

\[
J^*(V, \xi, \phi) = \sup_{Y' \leq 0} \{ G^*(V, Y', \phi) + Y'\xi \}.
\]

**Proof.** By Lemma 16, \( \mathcal{C}^D_{V,Y,\phi} \) implies information rent \( \xi \), so it is a candidate for an optimal contract given states \( V, \xi \) and \( \phi \). Then the following holds:

\[
J(V, \xi, \phi|\mathcal{C}^D_{V,Y,\phi}) \geq J^*(V, \xi, \phi)
\]

\[
\geq \sup_{Y' \leq 0} \{ G^*(V, Y', \phi) + Y'\xi \} = G^*(V, Y, \phi) + Y\xi
\]

\[
= G^*(V, Y, \phi) + Y\hat{\xi}(V, Y, \phi).
\]

The first inequality holds because \( \mathcal{C}^D_{V,Y,\phi} \) cannot be better than the optimal contract, the second inequality holds because of Lemma 14. The following equality holds because \( Y = \arg \sup_{Y' \leq 0} \{ G^*(V, Y', \phi) + Y'\xi \} \). Finally, the last equality holds because of Lemma 16.

By Lemma 19, \( J^*(V, \xi, \phi|\mathcal{C}^D_{V,Y,\phi}) = G^*(V, Y, \phi) + Y\hat{\xi}(V, Y, \phi) \). Hence, all the inequalities hold with equality, \( \mathcal{C}^D_{V,Y,\phi} \) is optimal for the primal problem and

\[
J^*(V, \xi, \phi) = \sup_{Y' \leq 0} \{ G^*(V, Y', \phi) + Y'\xi \}.
\]

**Lemma 19.** \( J^*(V, \xi, \phi) \) is differentiable, decreasing and convex in \( \xi \) for \( \xi \leq \hat{\xi}(V, 0, \phi) \).

**Proof.** When \( \xi \leq \hat{\xi}(V, 0, \phi) \), the solution to the maximization problem in (48) is interior and the envelope theorem holds (Milgrom and Segal, 2002). Therefore \( J^*_\xi(V, \xi, \phi) \) exists and it is equal to \( Y \), where \( Y = \arg \sup_{Y' \leq 0} \{ G^*(V, Y', \phi) + Y'\xi \} \). Since \( Y \leq 0 \), \( J^*(V, \xi, \phi) \) is decreasing in \( \xi \) for \( \xi \leq \hat{\xi}(V, 0, \phi) \).

To prove convexity consider \( \xi^0 \leq \hat{\xi}(V, 0, \phi) \) and \( \xi^1 \leq \hat{\xi}(V, 0, \phi) \) and let \( \xi^\nu = (1 - \nu)\xi^0 + \nu\xi^1 \).
\( \nu \xi^1 \). Using the concavity of \( G^*(V, Y, 0) \) and Lemma 18 we obtain

\[
J^*(V, \xi^\nu, \phi) = \sup_{Y' \leq 0} \{ G^*(V, Y', \phi) + Y'\xi^\nu \}
\leq (1 - \nu) \sup_{Y' \leq 0} \{ G^*(V, Y', \phi) + Y'\xi^0 \} + \nu \sup_{Y' \leq 0} \{ G^*(V, Y', \phi) + Y'\xi^1 \}
= (1 - \nu) J^*(V, \xi^0, \phi) + \nu J^*(V, \xi^1, \phi).
\]

\[\square\]

**Lemma 20.** \( \hat{\xi}(V, 0, \phi) \) is a global minimum of \( J^*(V, \xi, \phi) \) with respect to \( \xi \).

**Proof.** From Lemma 17, \( \hat{\xi}(V, 0, \phi) \) is a global minimum for \( \xi \geq \hat{\xi}(V, 0, \phi) \). From Lemma 19, \( \hat{\xi}(V, 0, \phi) \) is a global minimum also for \( \xi \leq \hat{\xi}(V, 0, \phi) \). Therefore, \( \xi \geq \hat{\xi}(V, 0, \phi) \) is a global minimum \( J^*(V, \xi, \phi) \) with respect to \( \xi \). \[\square\]

**D.1 Proof of Propositions 6 and 7**

**Proof.** Consider an initial promised value for the agent, \( V_0 \), and initial beliefs \( \phi_0 \). Since \( \hat{\xi}(V_0, 0, \phi_0) \) is a global minimum for \( J^*(V_0, \xi, \phi_0) \) with respect to \( \xi \) (by Lemma 20), the principal sets \( \xi_0 = \hat{\xi}(V_0, 0, \phi_0) \). By Lemma 18, \( c_{V_0, 0, \phi_0}^D \) is the optimal contract for the principal at time zero, and equation (48) holds. At any time \( t Y_t \leq 0 \), and combining Lemmas 16, 18, and 19, we obtain that \( \xi_t = \hat{\xi}(V_t, Y_t, \phi_t) = -G^*_Y(V_t, Y_t, \phi_t) \) and that \( Y_t = J^*_\xi(V_t, \xi_t, \phi_t) \). We then obtain Proposition 7.

Let \( \bar{\xi}(V_t, \phi_t) = \hat{\xi}(V_t, 0, \phi_t) \). Considering that \( J^*_\xi(V_t, \xi_t, \phi_t) = Y_t \leq 0 \) and that \( \xi_t = -G^*_Y(V_t, Y_t, \phi_t) \leq -G^*_Y(V_t, 0, \phi_t) \leq \hat{\xi}(V_t, 0, \phi_t) \) (by the concavity of \( G^* \) and by Lemma 13), we obtain Proposition 6. \[\square\]
E Proofs for Appendix A

E.1 Proof of Lemma 4

Proof. The first part of the Lemma follows directly from the definition of Markovian contract and weakly renegotiation-proof contract.

Suppose that the optimal renegotiation-proof contract is unique. Consider stopping times \( t \geq 0 \) and \( t' \geq 0 \) such that \( V_t = V_{t'} \) and \( \phi_t = \phi_{t'} \). By uniqueness of the contract, it follows that \( \mathcal{O}(\mathcal{C}, t) = \mathcal{O}(\mathcal{C}, t') \). To see why, if \( \mathcal{O}(\mathcal{C}, t) \neq \mathcal{O}(\mathcal{C}, t') \) then contract \( \mathcal{C}' \) such that \( \mathcal{O}(\mathcal{C}', s) = \mathcal{O}(\mathcal{C}, s) \) for all \( s \neq t \) and \( \mathcal{O}(\mathcal{C}', t) = \mathcal{O}(\mathcal{C}, t') \) would be incentive-compatible given \((V_0, \phi_0)\), weakly renegotiation-proof and payoff-equivalent to \( \mathcal{C} \), thus contradicting the uniqueness of the optimal renegotiation-proof contract.

\( \square \)

E.2 Proof of Proposition 8

Proof. Using the expression (11) for \( \xi_t \), we obtain

\[
\xi_t = \mathbb{E} \left[ \int_t^\infty e^{-\delta(s-t)} \beta_s \eta_s \phi_s (1 - \phi_s) \, ds \bigg| \mathcal{F}_t \right]
= \mathbb{E} \left[ \int_t^\infty e^{-\delta(s-t)} (1 - \rho) V_s \beta_s \eta_s \phi_s (1 - \phi_s) \, ds \bigg| \mathcal{F}_t \right]
= \mathbb{E} \left[ \int_t^\infty e^{-\delta(s-t)} \hat{\beta}_s \eta_s \phi_s (1 - \phi_s) \, ds \bigg| \mathcal{F}_t \right]
= \hat{\beta}_0^{1 - \rho} \mathbb{E} \left[ \int_t^\infty e^{-\delta(s-t)} \left( \frac{\hat{v}_s}{\hat{v}_t} \right)^{1 - \rho} \hat{\beta}_s \eta_s \phi_s (1 - \phi_s) \, ds \bigg| \mathcal{F}_t \right]
= (1 - \rho) V_t z_t,
\]

where

\[
z_t = \mathbb{E} \left[ \int_t^\infty e^{\int_t^s (-\rho u) \, du} \hat{B}_{ts} \beta_s \eta_s \phi_s (1 - \phi_s) \, ds \bigg| \mathcal{F}_t \right], \tag{49}
\]

for a density process \( \hat{B}_{ts} \) such that

\[
\hat{B}_{ts} = \exp \left\{ \int_t^s (1 - \rho) \beta_u dW_u \ - \frac{1}{2} \int_t^s (1 - \rho)^2 \beta_u^2 \, du \right\}.
\]
The objective function can be written as

\[
\hat{v}_0 \mathbb{E} \left[ \int_0^T e^{-\int_0^t r - \left( \frac{\rho}{1-\rho} - \frac{\delta_{1-\rho}}{1-\rho} + \frac{1}{2} \rho \hat{\beta}^2 \right) ds} B_t \left( \frac{c_t}{q(\eta)} - \mu(\eta_t, \phi_t, 0) \frac{\hat{\beta}_t - \eta_t z_t}{\lambda \sigma^2} c_t^2 \right) dt \left| \mathcal{F}_0 \right. \right],
\]

(50)

where \( B_t \) is a density process,

\[
B_t = \exp \left\{ \int_0^t \hat{\beta}_s dW^c_s - \frac{1}{2} \int_0^t \hat{\beta}^2 ds \right\}.
\]

Minimizing equation (50) is equivalent to minimizing the expectation in (50) subject to the law of motion of beliefs (under the probability measure implied by the density process \( B_t \)) and where \( z_t \) is given by (49). To see why, notice that if a strategy \((c_t)_{t \geq 0}, (\hat{\beta}_t)_{t \geq 0}, (\eta_t)_{t \geq 0}\) is optimal for \( \hat{v}_0 \) and \( \phi \), then it must also be optimal for \( \hat{v}'_0 \) and \( \phi \), even if \( \hat{v}'_0 \neq \hat{v}_0 \). Equation (27) in Proposition 8 is the associated HJB equation.

Because the strategy \((c_t)_{t \geq 0}, (\hat{\beta}_t)_{t \geq 0}, (\eta_t)_{t \geq 0}\) depends only on beliefs \( \phi \), then the information rent implied by the contract is given by \((1 - \rho)V_t z_R(\phi_t)\) where

\[
z_R(\phi_t) = \mathbb{E} \left[ \int_0^\infty e^{\int_t^\infty (-c(\phi_s)^{1-\rho}) \ d\tilde{W}_s} \hat{\beta}(\phi_s) \eta(\phi_s) \phi_s (1 - \phi_s) ds \left| \mathcal{F}_t \right. \right],
\]

(51)

Equation (28) in Proposition 8 is therefore obtained in the following way. According to equation (51), the drift of \( z_R(\phi_t) \) (under the probability measure implied by the density process \( \tilde{B}_t \)) is

\[
c_R(\phi_t)^{1-\rho} z_R(\phi_t) - \hat{\beta}_R(\phi_t) \eta_R(\phi_t) \phi_t (1 - \phi_t).
\]

The drift of \( z_R(\phi_t) \) (under the probability measure implied by the density process \( \tilde{B}_t \)) can also be obtained by using Ito’s lemma:

\[
(1 - \rho) \hat{\beta}_R(\phi) \eta_R(\phi) \phi (1 - \phi) z_R'(\phi) + \frac{1}{2} \eta_R(\phi)^2 \phi^2 (1 - \phi)^2 z_R''(\phi).
\]

Equating the two drifts, we obtain the differential equation (28) that characterizes \( z_R \).
E.3 Proof of Proposition 9

Proof. The proof is analogous to the proof of Proposition 8 and it is therefore omitted. We just need to use $A_t$, instead of $\hat{v}_t$, as the scaling process. We also need to consider that

$$\beta_t = V_\lambda(A_t, \phi_t)\Theta_t\sigma + V_{\theta}(A_t, \phi_t)\eta_t(1 - \phi_t)$$

when deriving the differential equation for the information rent.

E.4 Proof of Proposition 10

Proof. Consider the HJB equation (30) in Proposition 9. I will show that $v_T(\phi) = J_R(\phi)^{-1-\rho}$ with $c_M(\phi) = c_R(\phi)J_R(\phi)$, $\eta_M(\phi) = \eta_R(\phi)$, and $\hat{\theta}(\phi)$ such that $v_M(\phi)\hat{\theta}\sigma + \frac{\nu_M(\phi)}{1-\rho}\eta(\phi)\phi(1 - \phi) = \hat{\beta}_R(\phi)J_R(\phi)^{-1-\rho}$ is a solution to that HJB equation (30) with $z_M(\phi) = \left(\frac{J_R(\phi)^{1-\rho}}{1-\rho}\right)^{-1}z_R(\phi)$.

Using $v_M(\phi) = J_R(\phi)^{-1-\rho}$ in the HJB equation (30), we obtain

$$\frac{\delta}{1 - \rho} = \max_{c, \Theta, \eta, k} \left\{ \frac{c^{1-\rho}}{1 - \rho} J_R(\phi)^{-1-\rho} \left[ r - \frac{c}{q(\eta)} + \mu(\eta, \phi, 0)k - \frac{1}{2} \rho \hat{\theta}\sigma^2 \right] \right.$$

$$\left. - (1 - \rho)\eta\hat{\theta}\sigma(1 - \phi)J_R(\phi)^{-1}J'_R(\phi) \right.$$

$$\left. - \frac{1}{2}(\rho - 2)\eta^2\phi^2(1 - \phi)^2J_R(\phi)^{-2}J'_R(\phi)^2 - \frac{1}{2} \eta^2\phi^2(1 - \phi)^2J_R(\phi)^{-2}J''_R(\phi) \right\}$$

s.t.

$$s.t. \ (1 - \rho)v_M(\phi)\hat{\theta}\sigma + v'_M(\phi)\eta_t(1 - \phi) = (1 - \rho)c^{-\rho}k\lambda\sigma + \eta z_M(\phi).$$

Substituting for $\hat{\beta}(\phi)$, the incentive-compatibility constraint becomes

$$\hat{\beta} = J_R(\phi)^{1-\rho}c^{-\rho}k\lambda\sigma + \eta J_R(\phi)^{1-\rho}z_M(\phi) \left(\frac{1}{1 - \rho}\right),$$

while

$$\hat{\theta}\sigma = \hat{\beta} + \phi(1 - \phi)J_R(\phi)^{-1}J'_R(\phi).$$

(52)
After some simplifications, the HJB becomes

\[
\frac{\delta}{1 - \rho} = \max_{c, \beta, \eta, k} \left\{ \frac{c^{1 - \rho}}{1 - \rho} J_R(\phi)^{1 - \rho} + \left[ r - \frac{c}{q(\eta)} + \mu(\eta, \phi, 0)k \right] - \frac{1}{2} \rho \dot{\beta}^2 - \phi(1 - \phi) \eta \dot{\beta} J_R(\phi)^{-1} J_R'(\phi) - \frac{1}{2} \phi^2 (1 - \phi)^2 \eta^2 J_R(\phi)^{-1} J_R''(\phi) \right\}
\]

s.t. \( \dot{\beta} = J_R(\phi)^{1 - \rho} c - \rho k \lambda \sigma + \eta J_R(\phi)^{1 - \rho} \frac{z_M(\phi)}{1 - \rho} \).

Consider now \( \tilde{c} = c J_R(\phi), \tilde{k} = J_R(\phi) \) and \( z_R(\phi) = J_R(\phi)^{1 - \rho} \frac{z_M(\phi)}{1 - \rho} \). The HJB can thus be written as

\[
\frac{\delta}{1 - \rho} J_R(\phi) = \max_{\tilde{c}, \beta, \eta, k} \left\{ \frac{\tilde{c}^{1 - \rho}}{1 - \rho} + \left[ r - \frac{\tilde{c}}{q(\eta)} + \mu(\eta, \phi, 0)k \right] - \frac{1}{2} \rho \dot{\beta}^2 J_R(\phi) \right\}
\]

s.t. \( \dot{\beta} = \tilde{c} - \rho k \lambda \sigma + \eta z_R(\phi) \).

This equation is equivalent to

\[
\begin{align*}
\min_{\tilde{c}, \beta, \eta} \left\{ \tilde{c} - \eta z_R(\phi) \beta - \eta z_R(\phi) \sigma \lambda \beta - \eta \dot{z}_R(\phi) \right\} + \frac{\rho}{1 - \rho} \left( \delta - \frac{\tilde{c}^{1 - \rho}}{1 - \rho} + \frac{1}{2} \rho \dot{\beta}^2 \right) J_R'(\phi) + \eta \dot{\beta} \eta (1 - \phi) J_R''(\phi) + \frac{1}{2} \eta^2 \phi^2 (1 - \phi)^2 J_R''(\phi) \right\}.
\end{align*}
\]

Once we verify that \( z_R(\phi) = J_R(\phi)^{1 - \rho} \frac{z_M(\phi)}{1 - \rho} \), then the equation that we have just derived coincides with the HJB equation (27) in Proposition 8. In particular, the optimal controls are \( c_R(\phi), \hat{\beta}_R(\phi) \) and \( \eta_R(\phi) \). It therefore remains to verify that \( z_R(\phi) = J_R(\phi)^{1 - \rho} \frac{z_M(\phi)}{1 - \rho} \).

Combining the HJB equation (30) with the ODE (32), we obtain

\[
\begin{align*}
\left[ -\frac{c_M(\phi)^{1 - \rho}}{v_M(\phi)} - (1 - \rho) \sigma \eta \hat{\phi}(1 - \phi) v_M'(\phi) - \frac{1}{2} \eta^2 \phi^2 (1 - \phi)^2 v_M''(\phi) \right] z_M(\phi) &= \left( 1 - \rho \right) \eta M(\phi) \phi(1 - \phi) \hat{\phi}(\phi) \sigma v_M(\phi) + \eta M(\phi) \phi^2 (1 - \phi)^2 v_M'(\phi) + (1 - \rho) \eta M(\phi) \phi(1 - \phi) \hat{\phi}(\phi) \sigma z_M'(\phi) + \frac{1}{2} \eta M(\phi) \phi^2 (1 - \phi)^2 z_M''(\phi).
\end{align*}
\]

We can then use substitutions and simplifications analogous to the ones we used for
the HJB equation (27), and obtain

\[ z_R(\phi)c_R(\phi)^{1-\rho} - \hat{\beta}_R(\phi)\eta_R(\phi)\phi(1 - \phi) - (1 - \rho)\hat{\beta}_R(\phi)\eta_R(\phi)\phi(1 - \phi)z'_R(\phi) \\
= \frac{1}{2}\eta_R(\phi)^2 \phi^2(1 - \phi)^2 z''_R(\phi). \]

We have therefore verified that \( v_M(\phi) = J_R(\phi)^{1-\rho}, c_M(\phi) = c_R(\phi)J_R(\phi), \eta_M(\phi) = \eta_R(\phi), \hat{\Theta}_M(\phi)\sigma = \hat{\beta}_T(\phi) + \phi(1 - \phi)J_R(\phi)^{-1}J'_R(\phi), \) and \( z_M(\phi) = \left(\frac{J_R(\phi)^{1-\rho}}{1-\rho}\right)^{-1} z_R(\phi). \)
Figure 11: History dependence of the relation between flows and performance. Past performance is computed over 12 months. Figures (a) and (b) show how flows change with current performance and how the change depends on past performance. I sort funds into deciles based on their current performance and into halves based on their past performance. Past performance is the average excess return over the style benchmark in the previous 12 months. I then run regression of flows on dummies for the deciles of current performance, dummies for the halves of past performance, and interactions between the two sets of dummies. As controls, I include dummy variables for cumulative performance $\text{CumPerf}_{i[t-6,t]}$ sorted into deciles, 12 lagged flows, the logarithm of fund age, the logarithm of the manager’s tenure, the logarithm of lagged assets under management, fund fees, fund fixed effects, and month fixed effects. The shaded areas represent 95% confidence intervals for the change in the effect of current performance on flows when past performance increases above the median. Confidence intervals are constructed by double-clustering standard errors at the month and at the fund level.

In Figure (a), I plot the effect of current good performance (that is, performance relative to the first decile) on flows, while, in Figure (b), I plot the effect of current bad performance (that is, performance relative to the tenth decile) on flows.
Table 5: Effect of past performance on the slope of the flow-performance relationship. Past performance is computed over 12 months. The history dependence of the slope of the flow-performance relationship is measured by the coefficient on PastPerf$_{i, [t-12, t-1]} \cdot \tilde{R}_{it}$. PastPerf$_{i, [t-12, t-1]}$ measures the past performance of the manager and is calculated as the average excess return over the style benchmark in the twelve months from $t - 12$ to $t - 1$. The dependent variable, $F_{it+1}$, measures the net flow of capital and is calculated as the growth rate of assets under management from month $t$ to month $t + 1$ minus the net return over the same period. CumPerf$_{i, [t-12, t]}$ is the average performance of the manager over the style benchmark in the months from $t - 12$ to $t$. $\mathbb{I}[-\cdot]$ is the indicator function. Controls include 12 lags of monthly net flows into the fund, the log of fund size, its expense ratio, the log of fund age, and the log of the manager’s tenure. Standard errors are in parentheses and they are double-clustered at the fund and at the month level.

<table>
<thead>
<tr>
<th></th>
<th>$F_{it+1}$ (Net Flow)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(1)</td>
</tr>
<tr>
<td>$\tilde{R}_{it}$</td>
<td>0.184***</td>
</tr>
<tr>
<td></td>
<td>(0.010)</td>
</tr>
<tr>
<td>PastPerf$<em>{i, [t-12, t-1]} \cdot \tilde{R}</em>{it}$</td>
<td>3.768***</td>
</tr>
<tr>
<td></td>
<td>(1.141)</td>
</tr>
<tr>
<td>$\tilde{R}<em>{it} \cdot \mathbb{I}[\text{PastPerf}</em>{i, [t-12, t-1]} &gt; 0]$</td>
<td>0.062***</td>
</tr>
<tr>
<td></td>
<td>(0.019)</td>
</tr>
<tr>
<td>PastPerf$<em>{i, [t-12, t-1]} \cdot \mathbb{I}[\tilde{R}</em>{it} &gt; 0]$</td>
<td>0.041</td>
</tr>
<tr>
<td></td>
<td>(0.045)</td>
</tr>
<tr>
<td>PastPerf$_{i, [t-12, t-1]}$</td>
<td>0.432***</td>
</tr>
<tr>
<td></td>
<td>(0.035)</td>
</tr>
<tr>
<td>$(\text{PastPerf}_{i, [t-12, t-1]})^2$</td>
<td>0.427</td>
</tr>
<tr>
<td></td>
<td>(2.017)</td>
</tr>
<tr>
<td>$(\text{CumPerf}_{i, [t-12, t]})^2$</td>
<td>0.502</td>
</tr>
<tr>
<td></td>
<td>(2.368)</td>
</tr>
<tr>
<td>$\mathbb{I}[\tilde{R}_{it} &gt; 0]$</td>
<td>-0.000</td>
</tr>
<tr>
<td></td>
<td>(0.000)</td>
</tr>
<tr>
<td>$\mathbb{I}[\text{PastPerf}_{i, [t-12, t-1]} &gt; 0]$</td>
<td>0.002***</td>
</tr>
<tr>
<td></td>
<td>(0.000)</td>
</tr>
<tr>
<td>$\tilde{R}_{it}^2$</td>
<td>0.620***</td>
</tr>
<tr>
<td></td>
<td>(0.162)</td>
</tr>
</tbody>
</table>

Control Variables: Yes Yes Yes Yes Yes Yes
Style-Month FE: Yes Yes Yes Yes Yes Yes
Fund-Manager FE: No Yes No Yes No Yes
$R^2$: 0.351 0.383 0.351 0.383 0.351 0.383

Notes: *p ≤ .10; **p ≤ .05; ***p ≤ .01
Figure 12: Effect of managerial tenure and fund age on the history dependence of the relation between flows and current performance. Past performance is computed over 12 months. I run regression

\[ F_{it+1} = a_0 + a_1 \tilde{R}_{it} + a_2 \text{PastPerf}_{it[t-12,t-1]} \tilde{R}_{it} + a_3 \text{PastPerf}_{it[t-12,t-1]} + a_4 (\text{PastPerf}_{it[t-12,t-1]}^2) + a_5 \tilde{R}_{it}^2 \]

\[ + \sum_{j=2}^{5} \text{TenureQuintile}_{it}^j \left( a_0^j + a_1^j \tilde{R}_{it} + a_2^j \text{PastPerf}_{it[t-12,t-1]} \tilde{R}_{it} + a_3^j \text{PastPerf}_{it[t-12,t-1]} + a_4^j (\text{PastPerf}_{it[t-12,t-1]}^2) + a_5^j \tilde{R}_{it}^2 \right) \]

\[ + \sum_{j=2}^{5} \text{AgeQuintile}_{it}^j \left( a_0^j + a_1^j \tilde{R}_{it} + a_2^j \text{PastPerf}_{it[t-12,t-1]} \tilde{R}_{it} + a_3^j \text{PastPerf}_{it[t-12,t-1]} + a_4^j (\text{PastPerf}_{it[t-12,t-1]}^2) + a_5^j \tilde{R}_{it}^2 \right) \]

\[ + \sum_{j=2}^{5} \text{SizeQuintile}_{it}^j \left( a_0^j + a_1^j \tilde{R}_{it} + a_2^j \text{PastPerf}_{it[t-12,t-1]} \tilde{R}_{it} + a_3^j \text{PastPerf}_{it[t-12,t-1]} + a_4^j (\text{PastPerf}_{it[t-12,t-1]}^2) + a_5^j \tilde{R}_{it}^2 \right) \]

\[ + \gamma X_{it} + \epsilon_{it}^{FMgr} + \epsilon_{it}^{SMon} + u_t, \]

where TenureQuintile\(_{it}^j\) = 1 if, in month \(t\), the tenure of the manager of fund \(i\) belongs to the \(j^{th}\) quintile of the distribution of managerial tenure in month \(t\); AgeQuintile\(_{it}^j\) = 1 if, in month \(t\), the age of fund \(i\) belongs to the \(j^{th}\) quintile of the distribution of fund age in month \(t\); SizeQuintile\(_{it}^j\) = 1 if, in month \(t\), the size of fund \(i\) belongs to the \(j^{th}\) quintile of the distribution of fund size in month \(t\). Table 3 contains the description of all the other variables used in the regression. Standard errors are double-clustered at the month and at the fund level.

In Figure (a), the dots in the figure represents estimated coefficients \(a_j^{Tj}\)'s. The vertical lines represent 90% confidence intervals for the incremental effect of tenure on the flow-performance slope relative to the first quintile: If the vertical red line at quintile \(j\) does not cross the dashed horizontal line, then we reject the hypothesis that \(a_j^{Tj} \geq 0\) at a 95% confidence level.

Figure (b) is the analogous of Figure (a) for the effect of fund age on the history-dependence of the flow-performance relationship.
REFERENCES


