Supplementary Material for Inflation Dynamics and Time-Varying Volatility: New Evidence and an Ss Interpretation

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1 Supplementary Appendix 1: Analytical Model

Firms face a dynamic control problem with a single state variable z. At each point in continuous time, the ideal value of z is zero and departures from zero entail a flow cost of bz^2 and firms discount payoffs at rate r. When not adjusting, z follows a Brownian motion. Assume for now that there is no drift so that $dz = \sigma dw$, where dw is the increment to the standard Wiener process. (Note that for the study of aggregate shocks, it is convenient to break shocks into some idiosyncratic and some aggregate component $\sigma = \sigma_I + \sigma_A$, which is without loss of generality in partial equilibrium). Firms can adjust the value of z, subject to a fixed cost k. This environment has been well-studied, and it gives rise to a simple, symmetric Ss rule with a closed form solution. In particular,

Proposition 1 *Firms' optimal policy is to not adjust when* $|z| < S$ *where* $S = \left(\frac{6k\sigma^2}{b}\right)^{1/4}$*, and to adjust* z *to zero when* $|z| \geq S$*. In addition, the frequency of adjustment is given by* $\left(\frac{b}{6k}\right)^{1/2}\sigma$ *and the standard deviation of price changes is given by* S*.*

Proof. A version of this problem was first solved by Barro (1972), and the simplified version solved here is first described in Dixit (1991). As the solution method is now

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standard, I only sketch the basic outline. This firm problem yields a Bellman equation of the form $rV(z) = \frac{1}{2}\sigma^2 V''(z) + bz^2$ when not adjusting together with value matching conditions at the adjustment threshold $V(S) - V(0) = k$ and smooth pasting condition $V'(S) = 0$. The solution to this differential yields a value for the threshold that (for small k) is given by $S = \left(\frac{6k\sigma^2}{b}\right)^{1/4}$. The expected time to hit the boundary of the inaction region starting from zero is given by $T = \frac{S^2}{\sigma^2}$, so that the average number of price adjustments is given by $1/T = \left(\frac{b}{6k}\right)^{1/2}\sigma$. Finally, the symmetry of the problem delivers a standard deviation equal to $\sqrt{\frac{1}{2}(S-0)^2 + \frac{1}{2}(-S-0)^2} = S$. ■

Proposition 2 *In an environment with zero inflation, small aggregate shocks to z do not change either the frequency or standard deviation of price changes. In an environment with positive trend inflation, small aggregate increases in z raise frequency and decrease the standard deviation of price changes.*

Proof. For the first part of the proof, see Alvarez and Lippi (2013). They show that when $dz = -\mu t + \sigma dw$ then $\frac{\partial freq}{\partial \mu}|_{\mu=0} = \frac{\partial var}{\partial \mu}|_{\mu=0} \equiv 0$. At the zero inflation steady-state, marginal increases in μ do not affect the frequency or variance of price changes. Since the model is partial equilibrium, changes in μ are the same as any other aggregate shock to all firms' z. Now consider an environment with positive trend inflation. First note that with strictly positive trend inflation firms' optimal price-setting policies imply $P_{up} > P_{down}$. Furthermore, since firms' optimal adjustment thresholds already account for aggregate shocks to z ($\sigma = \sigma_I + \sigma_A$), these thresholds will not change in response to aggregate shocks, and only the distribution of z within these thresholds will change. Let the optimal thresholds and return point when adjusting be given by $\widetilde{L}, \widetilde{U}, C$. By subtracting C from both thresholds neither the frequency of adjustment nor variance of price changes will be affected, so without loss of generality we can renormalize the optimal policy to be given by L, U with firms raising prices by L when they reach the left threshold and lowering prices by U when they reach the right threshold. Given the stochastic process for z and adjustment thresholds L, U Stokey (2009) Chapter 5 provides formulas for the ergodic distribution of z as well as the relative probability of price increases and the implied frequency of adjustment. In particular, formula 5.16 gives probabilities:

$$\begin{aligned} P_{down} &= \frac{1 - e^{\alpha L}}{e^{\alpha U} - e^{\alpha L}} \\ P_{up} &= \frac{e^{\alpha U} - 1}{e^{\alpha U} - e^{\alpha L}} \end{aligned}$$

and formula 5.23 gives the frequency:

$$freq = -\frac{\alpha\sigma^2}{2} \frac{\left[e^{\alpha U} - e^{\alpha L}\right]}{(U - L) - e^{\alpha L}U + e^{\alpha U}L},$$

with $\alpha = 2\mu/\sigma^2$. Using formula 5.16 it is straightforward to calculate that the variance of price changes is given by

$$var = (1 - P_{up}) P_{up} [U - L].$$

We are then interested in how the frequency and variance of price changes respond to aggregate shocks to z. Positive shocks to z are equivalent to negative shocks to μ, and vice versa, so we can calculate the response of the frequency and variance to a first moment shock by taking their derivative with respect to μ. (Here it is again worth noting that these derivatives will be taken holding U, L constant, which will be true for aggregate shocks but not for permanent changes in μ since permanent changes in μ will shift the optimal policy). Taking these derivatives, it is straightforward to show that $\frac{\partial P_{up}}{\partial \mu} > 0$. Since the variance is maximized at $P_{up} = 1/2$ (which will hold when $\mu = 0$), the variance of price changes is falling in μ. Taking the derivative of frequency yields that $\frac{\partial freq}{\partial \mu} > 0$ if $e^{\alpha U} + e^{\alpha L} > 2$. Furthermore, $e^{\alpha U} + e^{\alpha L} > 2 \iff P_{up} > P_{down}$, which again will hold in an environment with positive inflation. Thus, we get that frequency and variance move in opposite directions in response to first moment shocks.

Finally, it is of interest to compute how frequency and variance move in response to steady-state changes in μ rather than to shocks to μ. (That is, allowing for optimal policy to change in response to changes in μ). To my knowledge, closed form solutions to the system of differential equations that determine the optimal thresholds do not exist for $\mu \neq 0$. However, it is straightforward to solve the system of differential equations numerically. Although I do not have a proof, these results suggest that the above proposition also holds when U, L change in response to μ. That is, starting from positive inflation, increases to trend inflation raise the frequency of adjustment and lower the variance of price changes. ■

2 Supplementary Appendix 2: Computational Procedure

2.1 Computing the model

Let p be a firm's nominal price after adjustment, P be the price level, ω be the disutility of labor, C be aggregate real demand, z be a firm's productivity and θ be the elasticity of substitution. Then current real profits are given by[1]

$$\begin{aligned} \pi(p, z; \chi, a) &= \left(\frac{p}{P} - \frac{\omega C}{az}\right)\left(\frac{p}{P}\right)^{-\theta} C \\ &= \left(\frac{p/S}{P/S} - \frac{\omega C}{az}\right)\left(\frac{p/S}{P/S}\right)^{-\theta} C \end{aligned}$$

[1]Note that the household labor supply problem implies that the real wage is equal to ωC.

Now, note that by assumption $S = PC$. In general, the price level will depend on the current value of the aggregate shocks and the joint distribution of idiosyncratic firm states, but I conjecture that

$$\log \frac{P}{S} = \gamma_0 + \gamma_1 \log a + [\gamma_2 + \gamma_3 \log a]\, \chi,$$

with the mean price flexible price gap: $\chi_1 \equiv \log \frac{P_{-1}}{S} + \log a$. This implies that

$$C = \frac{S}{P} = e^{-(\gamma_0 + \gamma_1 \log a + [\gamma_2 + \gamma_3 \log a]\chi_1)}.$$

Substituting into the profit function and using p/S as the idiosyncratic price state, we can write real profits as

$$\pi\left(p/S, z; \chi_1\right) = \left(p/S - \frac{\omega}{az}\right)\left(p/S\right)^{-\theta} e^{(\gamma_0 + \gamma_1 \log a + [\gamma_2 + \gamma_3 \log a]\chi_1)(\theta - 2)}.$$

Finally, it is straightforward to calculate transition rules for these variables. Since S follows a random walk in logs we get

$$\log \frac{p'}{S'} = \log \frac{p}{S} - \left(\mu + \varepsilon^s\right).$$

By assumption,

$$\log z' = \begin{cases} \rho_z \log z + d_t \sigma_z \varepsilon^z \text{ with probability } p^z \\ \log z \text{ with probability 1-}p^z. \end{cases}$$

and

$$\log a' = \rho_a \log a + \sigma_a \varepsilon^a,$$

and

$$\chi_1' = \gamma_0 + \gamma_1 \log a + [\gamma_2 + \gamma_3 \log a]\, \chi_1 - \left(\mu + \varepsilon^s\right) + \log a'.$$

In addition, each period firms draw iid cost of adjustment f with probability p^f and 0 with probability $1 - p^f$. Thus, we can write the firm i's value function as

$$V\left(\frac{p_{-1}}{S}, z, f; \chi_1, a\right) = \max\left[V^N\left(\frac{p_{-1}}{S}, z; \chi_1, a\right), V^A(z, f; \chi_1, a)\right],$$

with

$$\begin{aligned} V^N\left(\log \frac{p_{-1}}{S}, \log z; \chi_1, \log a\right) &= \pi\left(\frac{p_{-1}}{S}, z; \chi_1, a\right) \\ &\quad + E_{\varepsilon^z, \varepsilon^a, \varepsilon^s, f'}\, QV \begin{pmatrix} \log \frac{p_{-1}}{S} - \left(\mu + \varepsilon^s\right), \rho_z \log z + d\left(a\right) \sigma_z \varepsilon^z, f'; \\ \gamma_0 + \gamma_1 \log a + [\gamma_2 + \gamma_3 \log a]\, \chi_1 \\ - \left(\mu + \varepsilon^s\right) + \rho_a \log a + \sigma_a \varepsilon^a, \\ \rho_a \log a + \sigma_a \varepsilon^a \end{pmatrix} \end{aligned}$$

and

$$V^{A}(\log z, f; \chi_1, \log a) = -f\omega e^{-(\gamma_0+\gamma_1 \log a+[\gamma_2+\gamma_3 \log a]\chi_1)} + \max_{p/S}\left[\pi\left(\frac{p}{S}, z; \chi_1\right) + E_{\varepsilon^z,\varepsilon^a,\varepsilon^s,f'} QV\begin{pmatrix} \log\frac{p}{S} - (\mu+\varepsilon_M), \rho_z \log z + d(a)\sigma_z\varepsilon^z, \\ f'; \gamma_0+\gamma_1\log a + [\gamma_2+\gamma_3\log a]\chi_1 \\ -(\mu+\varepsilon^s) + \rho_a \log a + \sigma_a\varepsilon^a, \\ \rho_a \log a + \sigma_a \varepsilon^a \end{pmatrix}\right]$$

where $Q = \beta\frac{e^{-(\gamma_0+\gamma_1 \log a+[\gamma_2+\gamma_3 \log a]\chi_1)}}{e^{-(\gamma_0+\gamma_1 \log a'+\{\gamma_2+\gamma_3 \log a'\}\{\gamma_0+\gamma_1 \log a+[\gamma_2+\gamma_3 \log a]\chi_1-(\mu+\varepsilon^s)+\log a'\})}}$ is the stochastic discount factor and $\omega e^{-(\gamma_0+\gamma_1 \log a+[\gamma_2+\gamma_3 \log a]\chi_1)}$ is the real wage.

Given this recursive representation, I then solve the problem using value function iteration on a grid. Knotek and Terry (2008) argues that discretizing fixed adjustment cost models has robustness advantages versus collocation or other interpolation methods. Nevertheless, earlier versions of my model were solved using cubic spline interpolation and the results were unchanged. The random variables are discretized using the method of Tauchen (1986). In the benchmark analysis I used 171 grid points for the pricing grid, 21 grid points for the idiosyncratic productivity grid, 14 grid points for the χ_1 grid and 5 grid points for the aggregate productivity grid. Although not a state, expectations must be computed for ε^s, and it was discretized using 7 grid points. Results were unchanged when more grid points were added.

Once the model is solved for a given conjecture for $\boldsymbol{\gamma}$, a panel of 5000 firms[2] is simulated for 14,400 months[3] with a 100 month burnin. The law of motion

$$\log\frac{P}{S} = \gamma_0 + \gamma_1 \log a + [\gamma_2 + \gamma_3 \log a]\chi_1$$

is then updated by regressing these variables on the simulated data. The solution and simulation is then repeated until convergence. In the benchmark analysis, the standard for convergence is a less than 1% change in any of the $\boldsymbol{\gamma}$ coefficients across iterations. Higher standards of convergence did not change the qualitative results.

In addition, at the best fit parameters, I recomputed a version of the model with significantly greater precision and more thoroughly tested the accuracy of aggregate transition rules. Using the method proposed by Den Haan (2010), I computed the maximum error between the conjectured and simulated law of motion over 10,000 periods. Even over this extremely long time frame the maximum difference between aggregate variables computed using only simulation and those computed only using the conjectured law of motion is less than 0.1%, and the average error is much lower. Results suggest that forecasting errors can be made arbitrarily small by increasing grid sizes and simulations. Finally, errors in the forecasting equation are unrelated to output and to volatility in the model. None of the qualitative conclusions of the model are changed when precision is increased from the benchmark analysis.

[2] I investigated panels of up to 500,000 firms. Results were unchanged.

[3] 14,400 is 50 replications of the length of the empirical sample window.

While this version of the paper calibrates the parameters of the model, previous versions of this paper estimated simpler versions of the quantitative model more formally and all of the qualitative conclusions were similar. I explored identification of the micro parameters fairly extensively in this simpler model. The relative size of increases and decreases identifies the elasticity of substitution, the fraction increases identifies the persistence of productivity, the average size and frequency identify the standard deviation of productivity and the size of fixed costs. As mentioned in the body of the text, in the version of the model with random fixed costs and leptokurtic shocks I use data on the frequency of small price changes and the kurtosis of price changes to identify these additional parameters.

References

Alvarez, F. and F. Lippi (2013). Price setting with menu cost for multi-product firms. *Econometrica*.

Barro, R. (1972). A theory of monopolistic price adjustment. *Review of Economic Studies 39*(1).

Den Haan, W. J. (2010). Assessing the accuracy of the aggregate law of motion in models with heterogeneous agents. *Journal of Economic Dynamics and Control 34*(1).

Dixit, A. (1991). Analytic approximations in models of hysteresis. *Review of Economic Studies 58*(1).

Knotek, E. S. K. and S. Terry (2008). Alternative methods of solving state-dependent pricing models. *Federal Reserve Bank of Kansas City Working Paper No. 08-10*.

Stokey, N. L. (2009). *The Economics of Inaction: Stochastic Control Models with Fixed Costs*. Princeton University Press.

Tauchen, G. (1986). Finite state markov-chain approximations to univariate and vector autoregressions. *Economics Letters 20*(2).