12. Rethinking Mathematical Necessity

> We had been trying to make sense of the role of convention in *a priori* knowledge. Now the very distinction between *a priori* and empirical begins to waver and dissolve, at least as a distinction between sentences. (It could of course still hold as a distinction between factors in one's adoption of a sentence, but both factors might be operative everywhere.[1]

A consequence of Quine's celebrated critique of the analytic-synthetic distinction—a consequence drawn by Quine himself—is that the existence of mathematical entities is to be justified in the way in which one justifies the postulation of theoretical entities in physics. As Quine himself once put it,[2]

> Certain things we want to say in science may compel us to admit into the range of variables of quantification not only physical objects but also classes and relations of them; also numbers, functions, and other objects of pure mathematics. For mathematics—not uninterpreted mathematics, but genuine set theory, logic, number theory, algebra of real and complex numbers, and so on—is best looked upon as an integral part of science, on a par with the physics, economics, etc. in which mathematics is said to receive its applications.

As I read this and similar passages in Quine's writings, the message seems to be that in the last analysis it is the utility of statements about mathematical entities for the prediction of sensory stimuli that justifies belief in their existence. The existence of numbers or sets becomes a hypothesis on Quine's view, one not dissimilar in kind from the existence of electrons, even if far, far better entrenched.

It follows from this view that certain questions that can be raised about the existence of physical entities can also be raised about the existence of mathematical entities—questions of indispensability and questions of parsimony, in particular. These views of Quine's are views that I shared ever since I was a student (for a year) at Harvard in 1948–49, but, I must confess, they are views that I now want to criticize. First, however, I want to present a very different line of thinking—one which goes back to Kant and to Frege. This line is one that, I believe, Carnap hoped to detranscendentalize; and in Carnap's hands it turned into linguistic conventionalism. My strategy in this essay will be to suggest that there is a different way of stripping away the transcendental baggage while retaining what (I hope) is the insight in Kant (and perhaps Frege's)[3] view; a way which has features in common with the philosophy of the later Wittgenstein rather than with that of Carnap.

Kant and Frege

What led me to think again about the Kantian conception of logic was a desire to understand an intuition of Wittgenstein's that I had never shared. For the early Wittgenstein it was somehow clear that logical truths do not really say anything, that they are empty of sense (which is not the same thing as being nonsense), *sinnlos* if not *unsinnig.* (There are places in the *Investigations* in which Wittgenstein, as I read him, confesses that he still feels this inclination, although he does not surrender to it.) *Obviously,* sentences of pure logic are statements with content, I thought; if proved, they are moreover *true* statements, and their negations are *false* statements. But I felt dissatisfaction; dissatisfaction with my own inability to put myself in Wittgenstein's shoes (or his skin) and to even imagine the state of mind in which one would hold that truths of logic are "tautologies," that they are *sinnlos.* It was then that I thought of Kant.

Kant's lectures on logic[4] contain one of his earliest—perhaps the earliest—polemic against what we now call "psychologism." But that is not what interests me here, although it is closely related to it. What interests me here is to be found in *The Critique of Pure Reason* itself, as well in the lectures on logic, and that is the repeated insistence that illogical thought is not, properly speaking, thought at all. Not only does Kant insist on this, both in the lectures on logic and in the *Critique,* but his philosophical arguments in the *Critique* employ this doctrine in different ways. One employment has to do with the issue of thought about noumena. Kant allows that noumena are not in space and time. They are not related as "causes" and "effects." They may not be "things" as we creatures with a rational nature and a sensible nature are forced to conceive "things." But we are not allowed to suppose that they violate laws of logic; not because we have some *positive* knowledge about noumena, but because we know something about thought, and the "thought" that the noumena might not obey the laws of logic is no thought at all, but rather an incoherent play of representations.

This is in striking contrast to Descartes's view that God could have created a world which violated the laws of logic.

But Kant's view goes further than this. A metaphysician who thinks of "logical space" as a Platonic realm of some sort might agree with Kant: logical laws hold not only in "the actual world" but in all the other "possible worlds" as well. On such a view, logical laws are still descriptive; it is just that they describe *all* possible worlds, whereas empirical laws describe only some possible worlds (including the actual one). This opens the possibility of turning Kant's flank, as it were, by claiming that while indeed the laws of logic hold in all possible worlds, *God could have created an altogether different system of possible worlds.*

On my reading of the first *Critique,* there are points in that work at which Kant at least entertains the idea that talk of "noumena" is empty, that the notion of a noumenon has only a kind of formal meaning. But even when he entertains this possibility, Kant never wavers from the view that even *formal* meaning must conform to the laws of logic. It is this that brought home to me the deep difference between an *ontological* conception of logic, a conception of logic as descriptive of some domain of actual and possible entities, and Kant's (and, I believe, Frege's) conception. Logic is not a description of what holds true in "metaphysically possible worlds," to use Kripke's phrase. It is a doctrine of *the form of coherent thought.* Even if I think of what turns out to be a "metaphysically impossible world," my thought would not be a thought at all unless it conforms to logic.

Indeed, logic has no metaphysical presuppositions at all. For to say that thought, in the normative sense of *judgment which is capable of truth,* necessarily conforms to logic is not to say something which a metaphysics has to *explain.* To explain anything *presupposes* logic; for Kant, logic is simply prior to all rational activity.

While I would not claim that Frege endorses this view of Kant, it seems to me that his writing reflects a tension between the pull of the Kantian view and the pull of the view that the laws of logic are simply the most general and most justified views we have. If I am right in this, then the frequently heard statement that for Frege the laws of logic are like "most general laws of nature" is not the whole story. It is true that as *statements* laws of logic are simply quantifications over "all objects"—and all concepts as well—in *Begriffsschrift*. There is in Frege no "metalanguage" in which we could say that the laws of logic are "logically true"; one can only assert them in the one language, *the* language. But at times it seems that their *status*, for Frege as for Kant, is very different from the status of empirical laws. (It was, I think, his dissatisfaction with Frege's waffling on this issue that led the early Wittgenstein to his own version of the Kantian view.)

It was this line of thinking that helped me to understand how one might think that logical laws are *sinnlos* without being a Carnapian conventionalist. Laws of logic are without content, in the Kant-and-possibly-Frege view, insofar as they do not *describe* the way things are or even the way they (metaphysically) *could* be. The ground of their truth is that they are the formal presuppositions of thought (or better, *judgment*). Carnap's conventionalism, as interpreted by Quine in "Truth by Convention" (in *From a Logical Point of View*), was an *explanation* of the origin of logical necessity in human stipulation; but the whole point of the Kantian line is that logical necessity neither requires nor can intelligibly possess any "explanation."

Quine on Analyticity

In a certain sense, Quine's attack on analyticity in "Two Dogmas of Empiricism" (in *From a Logical Point of View*) does not touch the truths of pure logic. These form a special class, a class characterized by the fact that in them only the logical words occur essentially.[5] If we define an "analytic" truth as one which is either (a) a logical truth in the sense just specified, or (b) a truth which comes from a logical truth by substituting synonyms for synonyms, then the resulting notion of analyticity will inherit the unclarity of the notion of *synonymy;* and this unclarity, Quine argued, is fatal to the pretensions of the philosophical notion in question. But if we choose to retain the term "analytic" for the truths of pure logic, this problem with the notion of synonymy will not stop us.

But what would the point be? The definition of a logical truth as one in which only logical words occur essentially does not imply that logical truths are *necessary*. And Quine's doctrine that "no statement is immune from revision" implies that (in whatever sense of "can" it is

true that we can revise any statement) it is also true that we "can" revise the statements we call "logical laws" if there results some substantial improvement in our ability to predict, or in the simplicity and elegance of our system of science.[6]

This doctrine of the *revisability of logic* would of course be anathema to Kant and to Frege (who says that the discovery that someone rejects a logical law would be the discovery of a hitherto unknown form of madness).

The idea that logic is just an empirical science is so implausible that Quine himself seems hesitant to claim precisely this. There are two respects in which Quine seems to recognize that there is something correct in a more traditional view of logic. In the first place, he suggests that the old distinction between the analytic and the synthetic might point to a sort of continuum, a continuum of unrevisability ("There are statements which we choose to surrender last, if at all"),[7] or of bare behavioristic reluctance to give up, or of "centrality." We were wrong, the suggestion is, in thinking that any statement is absolutely immune from revision, but there are some we would certainly be enormously reluctant to give up, and these include the laws of traditional logic. And in the second place, Quine sometimes suggests that it is part of translation practice to translate others so that they come out believing the same logical laws that we do.[8] Thus revising the laws of logic might come to no more—by our present lights—that *changing the meanings* of the logical particles. It is on the first of these respects that I wish to focus now.

It seems right to me that giving up the analytic-synthetic dichotomy does not mean—that is, *should not mean*—thinking of all our beliefs as empirical. (To think that way is not really to give up the dichotomy, but rather to say that one of the two categories—the analytic—has null extension.) "There are no analytic sentences, only synthetic ones" would be a claim very different from "There is no epistemologically useful analytic-synthetic *distinction* to be drawn." Saying that there is an analytic-synthetic *continuum* (or rather, an *a priori–a posteriori* continuum—since Quine identifies the rejection of the analytic with the rejection of the a priori in his writing)[9] rather than an analytic-synthetic *dichotomy* is a promising direction to go if one wishes to reject the dichotomy as opposed to rejecting the analytic (or the *a priori*). But does the idea of "reluctance to give up" capture what is at stake, what is *right* about the idea that logical truths are quite unlike empirical hypotheses?

Consider the following three sentences:

(1) It is not the case that the Eiffel Tower vanished mysteriously last night and in its place there has appeared a log cabin.
(2) It is not the case that the entire interior of the moon consists of Roquefort cheese.
(3) For all statements p, $\ulcorner -(p \cdot -p) \urcorner$ is true.

It is true that I am much more reluctant to give up (3) than I am to give up (1) or (2). But it is also the case that I find a fundamental difference, a difference in kind, not just a difference in degree,[10] between (3) and (2), and it is this that Quine's account(s) may not have succeeded in capturing. As a first stab, let me express the difference this way: I can imagine finding out that (1) is false, that is, finding out that the Eiffel Tower vanished overnight and that a log cabin now appears where it was. I can even imagine finding out that (2) is false, although the reluctance to trust our senses, or our instruments, would certainly be even greater in the case of (2) than in the case of (1). But I cannot imagine *finding out* that (3) is false.[11]

It Ain't Necessarily So

But is talk about "imagining" so and so not gross psychologism? In some cases it is. Perhaps I could be convinced that certain describable observations would establish to the satisfaction of all reasonable persons that the interior of the moon consists entirely of Roquefort cheese. Perhaps I could be convinced that my feeling that it is "harder to imagine" the falsity of (2) than to imagine the falsity of (1) is just a "psychological fact," a fact about *me*, and not something of methodological significance. But to convince me that it is possible to imagine the falsity of (3) you would have to put an alternative logic in the field;[12] and *that* seems a fact of methodological significance, if there is such a thing as methodological significance at all.

To explain this remark, I would like to review some observations I made many years ago in an essay titled "It Ain't Necessarily So."[13]

In that essay, I argued against the idea that the principles of Euclidean geometry originally represented an empirical hypothesis. To be sure, they were not necessary truths. They were *false;*[14] false considered as a description of the space in which bodies exist and move, "physical space," and one way of showing that a body of statements is not necessary is to show that the statements are not even true (in effect, by using the modal principle $-p \supset -\Box p$). But, I argued, this only shows that the statements of Euclidean geometry are synthetic; I suggested that to identify "empirical" and "synthetic" is to lose a useful distinction. The way in which I proposed to draw that distinction is as follows: call a statement *empirical relative to a body of knowledge B* if possible observations (including observations of the results of experiments people with that body of knowledge could perform) would be *known* to disconfirm the statement (without drawing on anything outside of that body of knowledge). It seemed to me that this captures pretty well the traditional notion of an empirical statement. Statements which belong to a body of knowledge but which are not empirical relative to that body of knowledge I called "necessary relative to the body of knowledge." The putative truths of Euclidean geometry were, prior to their overthrow, simultaneously synthetic and necessary (in this relativized sense). The point of this new distinction was, as I explained, to emphasize that there are at any given time some accepted statements which cannot be overthrown merely by *observations,* but can only be overthrown by thinking of a whole body of alternative theory as well. And I insisted (and still insist) that this is a distinction of methodological significance.

If I were writing "It Ain't Necessarily So" today, I would alter the terminology somewhat. Since it seems odd to call statements which are false "necessary" (even if one adds "relative to the body of knowledge B"), I would say "*quasi*-necessary relative to body of knowledge B." Since a "body of knowledge," in the sense in which I used the term, can contain (what turn out later to be) false statements, I would replace "body of knowledge" with "conceptual scheme." And I would further emphasize the nonpsychological character of the distinction by pointing out that the question is not a *mere* question of what some people can imagine or not imagine; it is a question of what, given a conceptual scheme, one *knows* how to falsify or at least disconfirm. Prior to Lobachevski, Riemann, and others, no one knew how to disconfirm Euclidean geometry, or even knew if anything *could* disconfirm it. Similarly, I would argue, we do not today know how to falsify or disconfirm (3), and we do not know if anything could (or would) disconfirm (3). But we do know, at least in a rough way, what would disconfirm (1), and probably we know what would disconfirm (2). In this sense, there is a qualitative difference between (1), and probably (2), on the one hand, and (3) on the other. I do not urge that this difference be identified with analyticity; Quine is surely right that the old notion of analyticity has collapsed, and I see no point in reviving it. But I do believe that this distinction, the distinction between what is necessary and what is empirical relative to a conceptual scheme, is worth studying even if (or especially if) it is not a species of analytic-synthetic distinction. Here I shall confine myself to its possible significance for the philosophy of mathematics. First, however, I shall use it to try to clarify, and possibly to supplement,[15] some insights in Wittgenstein's *On Certainty*.

Some Thought Experiments in *On Certainty*

There are a number of places in *On Certainty* at which Wittgenstein challenges the very conceivability of what look at first blush like empirical possibilities. I will consider just two of these. The statement that water has boiled in the past, that is, that it has on many occasions boiled, or even more weakly that it has boiled on at least *one* occasion, looks like a paradigmatic "empirical statement." The conventional wisdom is that its degree of confirmation should, therefore, be less than one; its falsity should be conceivable, even given our experience so far. But is it? Can we, that is to say, so much as make sense of the possibility that we are deceived about *this;* conceive that our entire recollection of the past is somehow mistaken, or (alternatively) that we have all along been subject to a collective hallucination?[16]

A different kind of case: Can I be mistaken in thinking that my name is Hilary Putnam,[17] in thinking, that is, that that is the name by which I am called and have been called for years?

To take the second case first: Certainly I can imagine experiences as of waking up and discovering that what I call my life (as Hilary Putnam, as husband, father, friend, teacher, philosopher) was "all a dream." One might "make a movie" in which just that happened. And Wittgenstein (who, of course, wrote "Ludwig Wittgenstein" and not "Hilary Putnam") admits[18] that such experiences might *convince* one. (Of course, they might not convince one; one might break down mentally, or one might commit suicide—there are many ways of telling such a story.) But, Wittgenstein points out, saying that such experiences might convince one is one thing; saying that they *justify* the conclusion that it was all a hallucination, that I am not Hilary Putnam, is something else. Why should I not say that *those* experiences are the hallucination (if I come to have them)? If experiences call into question *everything* that I take for granted—including the evidence

for every single scientific theory I accept, by the way—then what is left of notions like "justification" and "confirmation"?

The question Wittgenstein raises here has a significance that reaches far beyond these examples. There is a sense in which they challenge not just the truth but the very intelligibility of the famous Quinian slogan that no statement is immune from revision. *Can* "Water has sometimes boiled" be revised? Can I (rationally) revise my belief that my name is Hilary Putnam?

In one sense, of course, we can revise the *sentences.* We could change the very meaning of the words. In that sense it is trivial that any *sentence* can be revised. And, since Quine rejects talk of "meaning" and "synonymy" at least when fundamental metaphysical issues are at stake, it might seem that the very question Wittgenstein is raising cannot even make sense for Quine (when Quine is doing metaphysics), depending, as that question does, on speaking of "beliefs" rather than "sentences." But things are not so simple.

Quine's Philosophy of Logic

The reason they are not so simple is that Quine himself has at times suggested[19] that it is difficult to make sense of the notion of revising the laws of classical logic. The problem is that—at least in the case of truth-functions—the fact that a translation manual requires us to impute violations of these laws to speakers calls into question the very adequacy of the translation manual. By so much as raising this question, Quine has opened the door to the sort of question I saw Wittgenstein as raising two paragraphs back. Can we now conceive of a community of speakers (1) whose language we could make sense of, "translate," in Quine's sense, who (2) assent to a sentence *which we would translate as* "Water has never boiled"? Can we now conceive of a community of speakers whom (1) we could interpret and understand, and who (2) assent to a sentence *which we would translate as* "7 + 5 = 13"?

Well, suppose we cannot. What significance does it have if we admit that we cannot do this? Here I would like to recall again what I wrote in "It Ain't Necessarily So." In my view, if we cannot *describe* circumstances under which a belief would be falsified, circumstances under which we would be prepared to say that -B had been confirmed, then we are not presently able to attach a clear *sense* to "B can be revised."[20] In such a case we cannot, I grant, say that B is "unrevisable," but neither can we intelligibly say "B can be revised." Since this point is essential to my argument, I shall spend a little more time on it.

Consider a riddle. A court lady once fell into disfavor with the king. (One easily imagines how.) The king, intending to give her a command impossible of fulfillment, told her to come to the Royal Ball "neither naked nor dressed." What did she do? (Solution: she came wearing a fishnet.)

Concerning such riddles, Wittgenstein says[21] that we are able to give the words a sense only after we know the solution; the solution bestows a sense on the riddle-question. This seems right. It is true that I could translate the sentence "She came to the Royal Ball neither naked nor dressed" into languages which are related to English, languages in which the key English words "naked" and "dressed" have long-established equivalents. But if I didn't know the solution, could I *paraphrase* the question "How could she come to the ball neither naked nor dressed?" *even in English itself?* I would be afraid to make *any* change in the key words, for fear of losing exactly what the riddle might turn on. Similarly, I would be afraid to translate the riddle into a foreign language which was not "similar" to English in the sense of having obvious "equivalents" to "naked" and "dressed." And if someone asked me, "In what sense, exactly, was she neither naked nor dressed?" I could not answer if I did not know the solution.

But are we not in the same position with respect to a sentence like "In the year 2010 scientists discovered that 7 electrons and 5 electrons sometimes make 13 electrons"? Or with respect to "In the year 2010 scientists discovered that there are exceptions to $5 + 7 = 12$ in quantum mechanics"? If this is right, and I think it is, then perhaps we can see how to save something that is right in the Kant-Frege-early Wittgenstein line that I described earlier.

Kant-Frege-Wittgenstein (Again)

Before trying to say what might be saved of the position I attributed to Kant, possibly Frege, and the early Wittgenstein at the beginning of this essay, it is important to specify what is metaphysical excess baggage that should be jettisoned. According to these thinkers, logical (and mathematical) truths are true by virtue of the nature of thought (or judgment) as such. This is a highly metaphysical idea, and it receives a somewhat different inflection in the writings of each of them.

In Kant's case, the metaphysics is complicated by the need to distinguish the truths of logic not only from empirical truths, but also from synthetic *a priori* truths. In the case of a synthetic *a priori* judgment, say, "Every event has a cause," Kant tells us that what makes the judgment true is not the way the world is—that is, not the way the world is "in itself"—but the way our reason functions; but this talk of the function and constitution of human reason has to be distinguished (by Kant) from talk of the nature of thought, and of the (normative) laws of thought, alluded to above. There is, according to Kant, such a thought as the thought that there is an event with no cause; but I can know *a priori* that that thought is false, because the very constitution of my reason ensures that the data of the senses, as those data are represented to my mind, will fit into a certain structure of objects in space and time related by causality. There is a sense in which the negations of synthetic *a priori* truths are no more descriptions of a way the world could be than are the negations of logical truths. Yet there is an enormous difference (for Kant) between the negation of a synthetic *a priori* truth and a logical contradiction. The negation of a synthetic *a priori* truth is thinkable; and the reason such a statement could never turn out to be a truth is explainable—to provide the explanation is precisely the task of the *Critique of Pure Reason.* The negation of a logical truth is, in a sense, unthinkable; and it is unthinkable precisely *because* it is the negation of a logical truth. Explanation goes no further. "Logical truth" is, as it were, itself an ultimate metaphysical category.[22]

Frege's views are less clear, although he too seems to have retained the notion of synthetic *a priori* truth.[23] At the same time, Frege prepares the way for Wittgenstein by identifying the Kantian idea of the nature of thought with the structure of an ideal language. The early Wittgenstein, however, tried (if my reading is correct) to marry a basically Kantian conception of logic with an empiricist rejection of the synthetic *a priori.* For the Wittgenstein of the *Tractatus,* the opposition is between logical truths and empirical truths, not between logical truths and synthetic truths in the Kantian sense. The problem of distinguishing the way in which the structure of thought (which, as just remarked, becomes the structure of the ideal language) guarantees the unrevisability of logic from the way in which the structure of reason guarantees the unrevisability of the synthetic *a priori* no longer arises, because either a judgment is about the world, in which case its negation is not only thinkable but, in certain possible circumstances, confirmable, or it is not about the world, in which case it is *sinnlos.*

My suggestion is not, of course, that we retain this idea of a nature of thought (or judgment, or the ideal language) which metaphysically guarantees the unrevisability of logic. But what I *am* inclined to keep from this story is the idea that logical truths do not have negations that we (presently) understand. It is not, on this less metaphysically inflated story, that we can say that the theorems of classical logic are "unrevisable"; it is that the question "Are they revisable?" is one which we have not yet succeeded in giving a sense. I suggest that the "cans" in the following sentences are not intelligible "cans": "Any statement can be held true come what may, if we make drastic enough adjustments elsewhere in the system. Even a statement very close to the periphery can be held true in the face of recalcitrant experience by pleading hallucination or by amending certain statements of the kind called logical laws. Conversely, by the same token, no statement is immune from revision."

A Few Clarifications

Let me spend a few moments in explaining how I am using some key terms, to prevent misunderstandings. I have already illustrated the idea that a question may not have a sense (or, at any rate, a sense we can grasp), until an "answer" gives it a sense, with the example of the riddle. And I want to suggest that, in the same way, saying that logic or arithmetic may be "revised" does not have a sense, and will never have a sense, unless some concrete piece of theory building and applying *gives* it a sense. But saying this leaves me open to a misunderstanding; it is easy to confuse talk of "senses" with talk of meanings in the sense in which translation manuals are supposed to be recursive specifications of meanings. But the word "sense" (in "In what sense do you mean *p?*") is much broader and much less specific than the term "meaning." When I learned the sense of "She came to the ball neither naked nor dressed" I did not learn anything that would require me to revise my dictionary entries for either the word *naked* or the word *dressed*.[24] Knowing the "sense" of a statement (or a question) is knowing how the words are used in a particular context; this may turn out to be knowing that the words had a "different meaning," but this is relatively rare. (Yet knowing the sense of the question or statement is connected with our ability to paraphrase discourses intelligently.) I may know the meaning of words, in the sense of knowing their "literal meaning," and not understand what is said on a particular occasion of the use of those words.

It follows that "giving words a sense" is not always a matter of giving them a new literal meaning (although it can be). "Momentum is not the product of mass and velocity" once had no sense; but it is part of Einstein's achievement that the sense he gave those words seems now *inevitable.* We "translate" (or read) old physics texts homophonically, for the most part; certainly we "translate" *momentum* homophonically.[25] We do not say that the word "momentum" used not to refer, or used to refer to a quantity that was not conserved; rather we say that the old theory was wrong in thinking that momentum was exactly *mv*. And we believe that wise proponents of the old theory *would* have accepted our correction had they known what we know. So this is not a case of giving a word a new meaning, but, as Cavell put it (using a phrase of Wittgenstein's), "knowing how to go on."[26] But that does not alter the fact that the sense we have given those words (or the use we have put them to) was not available before Einstein.

A different point of clarification. There is an old (and, I think futile) debate about whether contradictions are "meaningless." When I suggested that Frege was attracted to (and the Wittgenstein of the *Tractatus* held) the position that the negation of a theorem of logic violates the conditions for being a thinkable thought or judgment, I do not mean to exclude contradictions from "meaning" in the sense of well-formedness in the language, or in an ideal language. (For Kant, of course, it would be anachronistic to raise this issue.) The point is rather that a contradiction cannot be used to express a judgment by itself. Frege would perhaps say that it has a degenerate *Sinn,* that a contradiction functions as a mode of presentation of the truth value $\perp$ (falsity). (This would explain how it can contribute to the meaning of a complex judgment, say, $q \supset (p \cdot -p)$, which is just a way of saying $-q$.) But for Frege, as for Kant, the notion of *thinking that* $(p \cdot -p)$ makes no sense (except as "a hitherto unknown form of madness").

In any case, it is well to remember that part of the price we pay for talking as if science were done in a formalized language is that we make it harder to see that in every language that human beings *actually* use, however "scientific" its vocabulary and its construction, there is the possibility of forming questions and declarative sentences to which we are not presently able to attach the slightest sense. If we formalize English, then in the resulting formal idiom, "John discovered last Tuesday that $7 + 5 = 13$," or the formula that corresponds to that sentence in regimented notation, may be "well formed," but it does not follow that one could *understand* that sentence if one encountered it.

Arithmetic

One last point of clarification: it may seem that, given any p of the kind I have been discussing, any p that has the status of being "quasi-necessary relative to our present conceptual scheme," there must be a fact of the matter as to whether p is *merely* quasi-necessary or truly "necessary." But a review of the considerations I have employed should dispel this impression. Whether a given statement, say, the parallels postulate of Euclidean geometry, or some formulation of determinism, or an ordinary arithmetic truth "could be revised" depends on whether an "alternative theory" could really be constructed and confirmed (and on whether or not we would translate p homophonically into our present language, were such an alternative theory described to us); and all of the crucial terms—"theory," "confirmation," "acceptable translation manual"—have far too much indeterminacy to make application of the principle of bivalence convincing. The illusion that there is in all cases a fact of the matter as to whether a statement is "necessary or only quasi-necessary" is the illusion that there is a God's-Eye View from which all possible epistemic situations can be surveyed and judged; and that is indeed an illusion.

We can now, perhaps, see how the position I have been suggesting differs from the position of Rudolf Carnap. On Carnap's position, an arithmetic truth, say $5 + 7 = 12$, or a set-theoretic truth for that matter, is *guaranteed* to be unrevisable—guaranteed by a recursive linguistic stipulation.[27] We *know* that the truths of mathematics are unrevisable (that any revision would be a change of meaning), and we know this because we have *stipulated* that they are unrevisable (and changing the stipulations just *is* changing the meaning of the terms). As Quine has pointed out,[28] Carnap wanted a notion of analyticity that would have epistemological clout! By contrast, what I have suggested is simply that, as a matter of descriptive fact about our present cognitive situation, we *do not know* of any possible situation in which the truths of mathematics (as we take them to be) would be disconfirmed, save for situations in which the meanings of terms are (by our present lights) altered. To insist that these statements *must* be falsifiable, or that *all* statements must be falsifiable—is to make falsifiability a third (or is it a fourth by now?) dogma of empiricism.

It might be argued,[29] however, that there must be more to the truth of the theorems of mathematics than my story allows, for the following reason: even if the theorems of mathematics are consequences of principles whose negations could not, as far as we now know, intelligibly be true—principles such that the idea of *finding out that they are false* has not been given a sense—still there are statements of mathematics whose truth value no human being may ever be able to decide, even if more axioms become accepted by us on grounds of "intuitive evidence" or whatever. Certainly there are (by Gödel's theorem) sentences whose truth value cannot be decided on the basis of the axioms we presently accept. Yet, given that we accept the principle of bivalence, many of these sentences are true (as many as are false, in fact). And nothing *epistemic* can explain the truth of such undecidable statements, precisely because they *are* undecidable. To this objection, I can only answer that I am not able to attach metaphysical weight to the principle of bivalence; but a discussion of *that* issue would take another essay at least as long as this one.

The Existence of Mathematical Objects

It is time to consider the effect of the position I am considering (and tentatively advocating) on the issues with which this essay began. Some of Quine's doctrines are obviously unaffected if this line of thinking is right. What I have called the metaphysical analytic-synthetic distinction, that is, the idea of a notion of "analyticity" which will do foundational work in epistemology, is still jettisoned. Indeed, I have made no use of the idea of "truth by virtue of meaning," and the only use made of the notion of sense is the claim that there are some "statements" to which we are presently unable to attach any sense—something which I take to be a description of our lives with our language, rather than a piece of metaphysics. The principal effect of this line of thinking is on the idea, described at the beginning of this essay, that the existence of mathematical entities needs to be justified.

To begin with, let me say that, even apart from the issues I have been discussing here, talk of the "existence" of mathematical entities makes me uncomfortable. It is true that when we formalize mathematics, we at once get (as well-formed formulas, and as theorems) such sentences as "Numbers exist," in addition to sentences which might really occur in a mathematics text or class, sentences like "There exist prime numbers greater than a million." For Quine, this shows that arithmetic commits us to the existence of numbers; I am inclined to think that the notion of "ontological commitment" is an unfortunate one. But I will not discuss that issue here. What is clear, even if we accept "Numbers exist" as a reasonable mathematical assertion, is that if it makes no sense to say or think that we have discovered that arithmetic is *wrong,* then it also makes no sense to offer a reason for thinking it is not wrong. A reason for thinking mathematics is not wrong is a reason which excludes nothing. Trying to justify mathematics is like trying to say that whereof one cannot speak one must be silent; in both cases, it only looks as if something is being ruled out or avoided.

If this is right, then the role of applied mathematics, its utility in prediction and explanation, is not at all like the role of a physical theory. I can imagine a "possible world" in which mathematics—even number theory, beyond the elementary counting that a nominalist can account for without difficulty—serves no useful purpose. (Think of a world with only a few thousand objects, and no discernible regularities that require higher mathematics to formulate.) Yet imagining such a world is not imagining a world in which number theory, or set theory, or calculus, or whatever is *false;* it is only imagining a world in which number theory, or set theory, or calculus, or whatever, is not *useful.* It is true that if we had not ever found any use for applied mathematics, then we might not have developed pure mathematics either. The addition of mathematical concepts to our language enlarges the expressive power of that language; whether that enlarged expressive power will prove useful in empirical science is an empirical question. But that does not show that the truth of mathematics is an empirical question.

The philosophy of logic and mathematics is the area in which the notion of "naturalizing epistemology" seems most obscure. The suggestion of this essay is that the problem may lie both with "naturalize" and with "epistemology." The trouble with talk of "naturalizing" epistemology is that many of our key notions—the notion of understanding something, the notion of something's making sense, the notion of something's being capable of being confirmed, or infirmed, or discovered to be true, or discovered to be false, or even the notion of something's being capable of being stated—are normative notions, and it has never been clear what it means to naturalize a normative or partly normative notion. And the trouble with talk of epistemology in the case of mathematics is that this talk depends on the idea that there is a problem of *justification* in this area. But perhaps mathematics does not require justification; it only requires theorems.

Notes

I am indebted to Warren Goldfarb and Charles Parsons for valuable discussions of previous drafts of this paper.

1. W. V. Quine, "Carnap on Logical Truth" *Synthese* (1960), p 364.
2. "The Scope and Language of Science," *British Journal of the Philosophy of Science,* 8 (1957), 16.
3. I am aware that many people (for instance, Michael Dummett) read Frege as a Platonist rather than as a Kantian. However, two of my colleagues at Harvard, Burton Dreben and Warren Goldfarb, convinced me long ago that this is a mistake. For a reading of Frege which is close to the views of Dreben and Goldfarb, see Joan Weiner's *Frege in Perspective* (Ithaca: Cornell University Press, 1991).
4. Immanuel Kant, *Logic,* trans. R. Hartman and W. Schwarz (Mineola, N.Y.: Dover, 1974).
5. A term F is defined to "occur essentially" in a sentence (after that sentence has been "regimented" in the style of quantification theory) if there is a term G of the same syntactic class (e.g., a monadic predicate if F is a monadic predicate, a dyadic predicate if F is a dyadic predicate . . .) such that replacement of every occurrence of F in the sentence by an occurrence of G results in a sentence with a different truth-value. Notice that this Quinian definition of "occurs essentially" uses no *modal* notions.
6. Quine himself draws this conclusion, for example on p. 40 of "Two Dogmas of Empiricism" in *From a Logical Point of View* (Cambridge: Harvard University Press, 1953): "Any statement can be held true come what may, if we make drastic enough adjustments elsewhere in the system. Even a statement very close to the periphery can be held true in the face of recalcitrant experience by pleading hallucination or by amending certain statements of the kind called logical laws. Conversely, by the same token, no statement is immune to revision. Revision even of the logical law of the excluded middle has been proposed as a means of simplifying quantum mechanics; and what difference is there between such a shift and a shift whereby Kepler superseded Ptolemy, or Einstein Newton, or Darwin Aristotle?"
7. "Truth by Convention," pp. 350–351, in Benacerraf and Putnam, eds., *Philosophy of Mathematics: Selected Readings,* 2nd ed., ed. Hilary Benacerraf and Hilary Putnam (Cambridge: Cambridge University Press, 1984).
8. For example, in "Carnap on Logical Truth" Quine writes (p. 354), "Deductively irresoluble disagreement as to logical truth is evidence of deviation in usage (or meanings) of words" (emphasis in the original). In *Word and Object* (Cambridge: MIT Press, 1960), it is part of translation practice to translate others so that the verdict-tables for their truth functions come out the same as ours (otherwise, one simply does not attribute truth func-

tions to them) and so that "stimulus analytic" sentences have stimulus analytic translations. Indeed, in *Philosophy of Logic,* 2nd ed. (Cambridge: Harvard University Press, 1986), Quine seems to hold that one *cannot* revise propositional calculus without losing simplicity; but he has later rejected this view.

9. On this, see my "Two Dogmas Revisited," in *Realism and Reason,* vol. 3 of my *Philosophical Papers* (New York: Cambridge University Press, 1983).
10. I recall that Herbert Feigl asked me once, shortly after the appearance of "Two Dogmas," "Is the difference between a difference in kind and a difference in degree a difference in kind or a difference in degree?" This essay is, in a way, a return to this issue.
11. I do not claim, by the way, that *no* revisions in classical logic are conceivable. I myself have expressed sympathy for both quantum logic (which gives up the distributive law) and intuitionist logic. But the principle of contradiction seems to me to have quite a different status. On this see "There Is at Least One A Priori Truth" in my *Realism and Reason.*
12. I am aware that some people think such a logic—paraconsistent logic—has already *been* put in the field. But the lack of any convincing application of that logic makes it, at least at present, a *mere* formal system, in my view.
13. Reprinted in *Mathematics, Matter, and Method,* vol. 1 of my *Philosophical Papers* (New York: Cambridge University Press, 1975).
14. One way to see that they were false, regarded as descriptions of the space in which bodies are situated, is that they imply that space is infinite, whereas, according to our present physics, space is finite. See the discussion in "It Ain't Necessarily So."
15. In my view, the propositions Wittgenstein calls "grammatical" are often better conceived of as necessary relative to our present conceptual scheme. The use of the term "grammatical," with its strong linguistic connotations, inevitably suggests to many readers something like the notion of analyticity.
16. *On Certainty,* ed. G. E. M. Anscombe and G. H. von Wright (Oxford: Basil Blackwell, 1969), §§338–341.
17. Ibid., §515.
18. See ibid., §517.
19. See n. 8 above for references.
20. The way I expressed this in "It Ain't Necessarily So" (in *Mathematics, Matter, and Method*), using the example of Euclidean geometry, was to say that prior to the construction of an alternative geometry plus physics, the statement "There are only finitely many places to get to, travel as you will" did not express anything we could conceive.
21. In an unpublished manuscript cited by Cora Diamond in *The Realistic Spirit* (Cambridge: MIT Press, 1991), p. 267. Readers of that book will see how much my thinking has been influenced by Diamond's brilliant essays on Frege and Wittgenstein.

22. I do not, of course, use the word "category" here in the sense of the Kantian table of categories.
23. For example, Frege never doubted that the truths of *geometry* are synthetic *a priori.*
24. But see Donald Davidson on "passing theories" in "A Nice Derangement of Epitaphs," in *Truth and Interpretation: Perspectives on the Philosophy of Donald Davidson,* ed. Ernest Lepore (Oxford: Basil Blackwell, 1984), p. 446.
25. Likewise, we treat "place" (in "There are only finitely many places to get to, travel as you will" homophonically, when we are relating the Euclidean view of the world—infinite space—with the Einsteinian. Cf. "It Ain't Necessarily So," p. 242. That is why I wrote, "Something that was literally inconceivable has turned out to be true."
26. See Cavell, *The Claim of Reason* (Oxford: Oxford University Press, 1979), pp. 121–122.
27. See Quine's "Truth by Convention."
28. See Quine's "Reply to Geoffrey Hellman," in *The Philosophy of W. V. Quine,* ed. L. E. Hahn and P. A. Schilpp (La Salle, Ill.: Open Court, 1986).
29. I argued this way myself in "What Is Mathematical Truth?" in *Mathematics, Matter, and Method.*