Game Theory I
# A Strategic Situation
*(due to Ben Polak)*

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<thead>
<tr>
<th></th>
<th>(\alpha)</th>
<th>(\beta)</th>
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<tbody>
<tr>
<td>(\alpha)</td>
<td>B-, B-</td>
<td>A, C</td>
</tr>
<tr>
<td>(\beta)</td>
<td>C, A</td>
<td>A-, A-</td>
</tr>
</tbody>
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Player 1

Player 2
**Selfish Students**

No matter what Selfish 2 does, Selfish 1 wants to choose $\alpha$ (and vice versa).

$(\alpha, \alpha)$ is a sensible prediction for what will happen.

<table>
<thead>
<tr>
<th></th>
<th>Selfish 1</th>
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<tbody>
<tr>
<td>$\alpha$</td>
<td>1, 1</td>
<td>3, 0</td>
</tr>
<tr>
<td>$\beta$</td>
<td>0, 3</td>
<td>2, 2</td>
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</tbody>
</table>
No matter what Selfish 2 does, Selfish 1 wants to choose $\alpha$ (and vice versa)
**SELFISH STUDENTS**

<table>
<thead>
<tr>
<th></th>
<th>Selfish 2</th>
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<tbody>
<tr>
<td><strong>α</strong></td>
<td><strong>β</strong></td>
</tr>
<tr>
<td><strong>α</strong></td>
<td>1, 1</td>
</tr>
<tr>
<td><strong>β</strong></td>
<td>0, 3</td>
</tr>
</tbody>
</table>

- No matter what Selfish 2 does, Selfish 1 wants to choose $\alpha$ (and vice versa).
- $(\alpha, \alpha)$ is a sensible prediction for what will happen.
Nice Students

Each nice student wants to match the behavior of the other nice student: $(\alpha, \alpha)$ or $(\beta, \beta)$ seem sensible.

We need to know what people think about each other’s behavior to have a prediction.
Each nice student wants to match the behavior of the other nice student
Each nice student wants to match the behavior of the other nice student

$(\alpha, \alpha)$ or $(\beta, \beta)$ seem sensible.
Nice Students

Each nice student wants to match the behavior of the other nice student

- \((\alpha, \alpha)\) or \((\beta, \beta)\) seem sensible.

- We need to know what people think about each other’s behavior to have a prediction
Selfish vs. Nice

Nice

\[
\begin{array}{c|cc}
\text{Selfish} & \alpha & \beta \\
\hline
\alpha & 1, 2 & 3, 0 \\
\beta & 0, 1 & 2, 3 \\
\end{array}
\]
**Selfish vs. Nice**

<table>
<thead>
<tr>
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<tbody>
<tr>
<td>Selfish</td>
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<td>1, 2</td>
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- Nice wants to match what Selfish does
### Selfish vs. Nice

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- Nice wants to match what Selfish does
- No matter what Nice does, Selfish wants to play \(\alpha\)
Selfish vs. Nice

Nice

\[
\begin{array}{ccc}
\text{Selfish} & \alpha & \beta \\
\alpha & 1, 2 & 3, 0 \\
\beta & 0, 1 & 2, 3 \\
\end{array}
\]

- Nice wants to match what Selfish does
- No matter what Nice does, Selfish wants to play \( \alpha \)
- If Nice can think one step about Selfish, she should realize she should play \( \alpha \)
## Selfish vs. Nice

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- Nice wants to match what Selfish does
- No matter what Nice does, Selfish wants to play \(\alpha\)
- If Nice can think one step about Selfish, she should realize she should play \(\alpha\)
- \((\alpha, \alpha)\) seems the sensible prediction
OUTLINE

STRATEGIC FORM GAMES

SOLVING A GAME: NASH EQUILIBRIUM
COMPONENTS OF A GAME

Players: Who is involved?

Strategies: What can they do?

Payoffs: What do they want?
### Chicken

<table>
<thead>
<tr>
<th></th>
<th>Player 2</th>
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<tbody>
<tr>
<td><strong>Player 1</strong></td>
<td>Straight</td>
</tr>
<tr>
<td>Straight</td>
<td>0, 0</td>
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<tr>
<td>Swerve</td>
<td>1, 3</td>
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</table>
Choosing a Restaurant

<table>
<thead>
<tr>
<th></th>
<th>Rebecca</th>
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<tbody>
<tr>
<td><strong>P</strong></td>
<td>4, 3</td>
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<tr>
<td><strong>V</strong></td>
<td>0, 0</td>
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</table>
Working in a Team

2 players

Player $i$ chooses effort $s_i \geq 0$

Jointly produce a product. Each enjoys an amount

$$\pi(s_1, s_2) = s_1 + s_2 + \frac{s_1 \times s_2}{2}$$

Cost of effort is $s_i^2$

$$u_i(s_1, s_2) = \pi(s_1, s_2) - s_i^2$$
Player 1’s payoffs as a function of each player’s strategy

\[ u_1(s_1, 6) \]
\[ u_1(s_1, 3) \]
\[ u_1(s_1, 0.5) \]
Demand Bargaining

$N$ players

Each player “demands” a real number in $[0, 10]$

If the demands sum to 10 or less, each player’s payoff is her bid

Otherwise players’ payoffs are 0
Outline

Strategic Form Games

Solving a Game: Nash Equilibrium
Nash Equilibrium

A strategy profile where no individual has a unilateral incentive to change her behavior

Before we talk about why this is our central solution concept, let’s formalize it
**Notation**

Player $i$’s strategy

- $s_i$

Set of all possible strategies for Player $i$

- $S_i$

Strategy profile (one strategy for each player)

- $s = (s_1, s_2, \ldots, s_N)$

Strategy profile for all players except $i$

- $s_{-i} = (s_1, s_2, \ldots, s_{i-1}, s_{i+1}, \ldots, s_N)$

Different notation for strategy profile

- $s = (s_i, s_{-i})$
**Selfish Students**

\[ S_i = \{\alpha, \beta\} \]

4 strategy profiles: \((\alpha, \alpha), (\alpha, \beta), (\beta, \alpha), (\beta, \beta)\)
**CHICKEN**

\[
\begin{array}{cc|cc}
\text{Player 1} & \text{Player 2} & \text{Straight} & \text{Swerve} \\
\hline
\text{Straight} & 0, 0 & 3, 1 \\
\text{Swerve} & 1, 3 & 2, 2 \\
\end{array}
\]

\[S_i = \{\text{Straight, Swerve}\}\]

4 strategy profiles: (Straight, Straight), (Straight, Swerve), (Swerve, Straight), (Swerve, Swerve)
## Choosing a Restaurant

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<td>1,1</td>
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<td><strong>V</strong></td>
<td>0,0</td>
<td>3,4</td>
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$S_E = ?$  \quad S_R = ?$

Strategy profiles: ?
Demand Bargaining with 3 Players

$S_i = [0, 10]$
- Player $i$ can choose any real number between 0 and 10

$s = (s_1 = 1, s_2 = 4, s_3 = 7) = (1, 4, 7)$
- An example of a strategy profile

$s_{-2} = (1, 7)$
- Same strategy profile, with player 2’s strategy omitted

$s = (s_{-2}, s_2) = ((1, 7), 4)$
- Reconstructing the strategy profile
Notating Payoffs

Players’ payoffs are defined over strategy profiles

- A strategy profile implies an outcome of the game

Player $i$’s payoff from the strategy profile $s$ is

$$u_i(s)$$

Player $i$’s payoff if she chooses $s_i$ and others play as in $s_{-i}$

$$u_i(s_i, s_{-i})$$
Nash Equilibrium

Consider a game with $N$ players. A strategy profile $s^* = (s_1^*, s_2^*, \ldots, s_N^*)$ is a Nash equilibrium of the game if, for every player $i$

$$u_i(s_i^*, s_{-i}^*) \geq u_i(s_i', s_{-i}^*)$$

for all $s_i' \in S_i$
A strategy, $s_i$, is a best response by Player $i$ to a profile of strategies for all other players, $s_{-i}$, if

$$u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i})$$

for all $s'_i \in S_i$
Best Response Correspondence

Player $i$’s **best response correspondence**, $\text{BR}_i$, is a mapping from strategies for all players other than $i$ into subsets of $S_i$ satisfying the following condition:

- For each $s_{-i}$, the mapping yields a set of strategies for Player $i$, $\text{BR}_i(s_{-i})$, such that $s_i$ is in $\text{BR}_i(s_{-i})$ if and only if $s_i$ is a best response to $s_{-i}$.
An Equivalent Definition of NE

Consider a game with $N$ players. A strategy profile $s^* = (s_1^*, s_2^*, \ldots, s_N^*)$ is a **Nash equilibrium** of the game if $s_i^*$ is a best response to $s_{-i}^*$ for each $i = 1, 2, \ldots, N$.  


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SELFISH vs. NICE

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<tr>
<td>( \alpha )</td>
<td>( \alpha )</td>
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<tr>
<td>1, 2 ( \checkmark )</td>
<td>3, 0</td>
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<td><strong>Selfish</strong></td>
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<td><img src="https://via.placeholder.com/150" alt="" /></td>
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<tr>
<td>α</td>
<td>1✓, 2✓</td>
<td>3✓, 0</td>
</tr>
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Chicken

Player 1

Straight

Swerve

Player 2

Straight

0, 0

3, 1

Swerve

1, 3

2, 2
Chicken

Player 1

<table>
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<tr>
<th>Straight</th>
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<tbody>
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<td>Swerve</td>
<td>3, 1</td>
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Player 2

Swerve
### Chicken

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<td>Swerve</td>
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<td>Player 1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Straight</td>
<td>0, 0</td>
<td>3(\checkmark) , 1(\checkmark)</td>
</tr>
<tr>
<td>Swerve</td>
<td>1(\checkmark) , 3</td>
<td>2, 2</td>
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## Chicken

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<tr>
<td>0, 0</td>
<td>3✓, 1✓</td>
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Player 1

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<td>3✓, 1✓</td>
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<td>2, 2</td>
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<tr>
<td></td>
<td>Straight</td>
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<tr>
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<td>1✓, 3✓</td>
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**Chicken**
You Solve Choosing a Restaurant

<table>
<thead>
<tr>
<th></th>
<th>Rebecca</th>
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<tbody>
<tr>
<td><strong>Ethan</strong></td>
<td></td>
</tr>
<tr>
<td><strong>P</strong></td>
<td>P</td>
</tr>
<tr>
<td><strong>P</strong></td>
<td>4, 3</td>
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<tr>
<td><strong>V</strong></td>
<td>0, 0</td>
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</table>
### Another Practice Game

<table>
<thead>
<tr>
<th></th>
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<tbody>
<tr>
<td><strong>L</strong></td>
<td>10, 2</td>
</tr>
<tr>
<td><strong>R</strong></td>
<td>3, 4</td>
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</tbody>
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<table>
<thead>
<tr>
<th>Player 1</th>
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<tbody>
<tr>
<td><strong>U</strong></td>
<td>10, 2</td>
</tr>
<tr>
<td><strong>D</strong></td>
<td>-1, 0</td>
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</tbody>
</table>

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<tr>
<th>Player 1</th>
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<tbody>
<tr>
<td><strong>D</strong></td>
<td>5, 7</td>
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**WORKING IN A TEAM**

\[ u_1(s_1, s_2) = \pi(s_1, s_2) - s_1^2 = s_1 + s_2 + \frac{s_1 s_2}{2} - s_1^2 \]

Find Player i’s best response by maximizing for each \( s_2 \)

\[ \frac{\partial u_1(s_1, s_2)}{\partial s_1} = 1 + \frac{s_2}{2} - 2s_1 \]
Working in a Team

\[ u_1(s_1, s_2) = \pi(s_1, s_2) - s_1^2 = s_1 + s_2 + \frac{s_1s_2}{2} - s_1^2 \]

Find Player \( i \)’s best response by maximizing for each \( s_2 \)

\[ \frac{\partial u_1(s_1, s_2)}{\partial s_1} = 1 + \frac{s_2}{2} - 2s_1 \]

First-order condition sets this equal to 0 to get \( BR_1(s_2) \)

\[ 1 + \frac{s_2}{2} - 2 BR_1(s_2) = 0 \]
Working in a Team

\[ u_1(s_1, s_2) = \pi(s_1, s_2) - s_1^2 = s_1 + s_2 + \frac{s_1s_2}{2} - s_1^2 \]

Find Player \( i \)'s best response by maximizing for each \( s_2 \)

\[ \frac{\partial u_1(s_1, s_2)}{\partial s_1} = 1 + \frac{s_2}{2} - 2s_1 \]

First-order condition sets this equal to 0 to get \( BR_1(s_2) \)

\[ 1 + \frac{s_2}{2} - 2 BR_1(s_2) = 0 \]

\[ BR_1(s_2) = \frac{1}{2} + \frac{s_2}{4} \]

\[ BR_2(s_1) = \frac{1}{2} + \frac{s_1}{4} \]
Player 1’s Best Response

utility

$u_1(s_1, 6)$

$u_1(s_1, 3)$

$u_1(s_1, 0.5)$

$BR_1(0.5)$  $BR_1(3)$  $BR_1(6)$  $s_1$
Nash Equilibrium

\[ BR_1(s_2) = \frac{1}{2} + \frac{s_2}{4} \]

\[ BR_2(s_1) = \frac{1}{2} + \frac{s_1}{4} \]
SOLVING FOR NE

Since best responses are unique, a NE is a profile, \((s_1^*, s_2^*)\) satisfying

\[
s_1^* = BR_1(s_2^*) = \frac{1}{2} + \frac{s_2^*}{4} \quad s_2^* = BR_2(s_1^*) = \frac{1}{2} + \frac{s_1^*}{4}
\]

Substituting

\[
s_1^* = \frac{1}{2} + \frac{1}{2} + \frac{s_1^*}{4}
\]

\[
s_1^* = \frac{3}{2} + \frac{s_1^*}{4}
\]

\[
s_1^* = \frac{3}{2} + \frac{3}{2} = \frac{6}{2} = 3
\]

\[
s_1^* = \frac{3}{3} \quad s_2^* = \frac{3}{3}
\]
2 players

Each player, $i$, chooses a real number $s_i$

There is a benefit of value 1 to be divided between the players

At a strategy profile $(s_i, s_{-i})$, Player $i$ wins a share

$$\frac{s_i}{s_i + s_{-i}}$$

The cost of $s_i$ is $s_i$
Solving

Write down Player 1’s payoff from \((s_1, s_2)\)

\[ u_1(s_1, s_2) = s_1 s_1 + s_2 \times 1 - s_1 = s_2 (s_1 + s_2) - 1 = 0 \]

Set equal to zero to maximize

\[ \text{BR}_1(s_2) = \sqrt{s_2 - s_2^{3/8}} \]
Solving

Write down Player 1’s payoff from \((s_1, s_2)\)

\[
u_1(s_1, s_2) = \frac{s_1}{s_1 + s_2} \times 1 - s_1
\]
SOLVING

Write down Player 1’s payoff from \((s_1, s_2)\)

\[
    u_1(s_1, s_2) = \frac{s_1}{s_1 + s_2} \times 1 - s_1
\]

Calculate Player 1’s best response correspondence
**Solving**

Write down Player 1’s payoff from \((s_1, s_2)\)

\[
    u_1(s_1, s_2) = \frac{s_1}{s_1 + s_2} \times 1 - s_1
\]

Calculate Player 1’s best response correspondence

\[
    \frac{\partial u_1(s_1, s_2)}{\partial s_1} = \frac{s_1 + s_2 - s_1}{(s_1 + s_2)^2} \times 1 - 1 = \frac{s_2}{(s_1 + s_2)^2} - 1
\]

Set equal to zero to maximize

\[
    \frac{s_2}{(BR_1(s_2) + s_2)^2} - 1 = 0 \Rightarrow BR_1(s_2) = \sqrt{s_2} - s_2
\]
Solving

Player 2 is symmetric to Player 1, so write down both players’ best response correspondences
Player 2 is symmetric to Player 1, so write down both players’ best response correspondences

\[ \text{BR}_1(s_2) = \sqrt{s_2} - s_2 \quad \text{BR}_2(s_1) = \sqrt{s_1} - s_1 \]
Player 2 is symmetric to Player 1, so write down both players’ best response correspondences

\[ \text{BR}_1(s_2) = \sqrt{s_2} - s_2 \quad \text{BR}_2(s_1) = \sqrt{s_1} - s_1 \]

At a NE each player is playing a best response to the other. Write down two equations that characterize equilibrium.
Player 2 is symmetric to Player 1, so write down both players’ best response correspondences

\[ \text{BR}_1(s_2) = \sqrt{s_2^* - s_2} \quad \text{BR}_2(s_1) = \sqrt{s_1^* - s_1} \]

At a NE each player is playing a best response to the other. Write down two equations that characterize equilibrium.

\[ s_1^* = \sqrt{s_2^* - s_2^*} \quad s_2^* = \sqrt{s_1^* - s_1^*} \]
Solving

\[ s_1^* = \sqrt{s_2^* - s_2^*} \quad s_2^* = \sqrt{s_1^* - s_1^*} \]

Use substitution to find Player 1’s equilibrium action
Solving

\[ s_1^* = \sqrt{s_2^* - s_2^*} \quad s_2^* = \sqrt{s_1^* - s_1^*} \]

Use substitution to find Player 1’s equilibrium action

\[ s_1^* = \sqrt{\sqrt{s_1^* - s_1^*} - \left(\sqrt{s_1^* - s_1^*}\right)} \Rightarrow s_1^* = \frac{1}{4} \]

Now substitute this in to find Player 2’s equilibrium action
Solving

\[ s_1^* = \sqrt{s_2^* - s_2^*} \quad s_2^* = \sqrt{s_1^* - s_1^*} \]

Use substitution to find Player 1’s equilibrium action

\[ s_1^* = \sqrt{\sqrt{s_1^* - s_1^*} - \left( \sqrt{s_1^* - s_1^*} \right)} \Rightarrow s_1^* = \frac{1}{4} \]

Now substitute this in to find Player 2’s equilibrium action

\[ s_2^* = \sqrt{\frac{1}{4} - \frac{1}{4}} = \frac{1}{2} - \frac{1}{4} = \frac{1}{4} \]
Why Nash Equilibrium?

No regrets

Social learning

Self-enforcing agreements

Analyst humility
A Nash Equilibrium is a strategy profile where each player is best responding to what all other players are doing.

You find a NE by calculating each player’s best response correspondence and seeing where they intersect.

NE is our main solution concept for strategic situations.