Game Theory I
A Strategic Situation
(due to Ben Polak)

<table>
<thead>
<tr>
<th>Player 1</th>
<th>Player 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha$</td>
<td>$\alpha$</td>
</tr>
<tr>
<td>B-, B-</td>
<td>A, C</td>
</tr>
<tr>
<td>$\beta$</td>
<td>$\beta$</td>
</tr>
<tr>
<td>C, A</td>
<td>A-, A-</td>
</tr>
</tbody>
</table>
**SELFISH STUDENTS**

No matter what Selfish 2 does, Selfish 1 wants to choose $\alpha$ (and vice versa)

$(\alpha, \alpha)$ is a sensible prediction for what will happen
Nice Students

Each nice student wants to match the behavior of the other nice student

$(\alpha, \alpha)$ or $(\beta, \beta)$ seem sensible.

We need to know what people think about each other’s behavior to have a prediction.
Selfish vs. Nice

Nice

<table>
<thead>
<tr>
<th></th>
<th>α</th>
<th>β</th>
</tr>
</thead>
<tbody>
<tr>
<td>α</td>
<td>1,2</td>
<td>3,0</td>
</tr>
<tr>
<td>β</td>
<td>0,1</td>
<td>2,3</td>
</tr>
</tbody>
</table>

Nice wants to match what Selfish does

No matter what Nice does, Selfish wants to player α

If Nice can think one step about Selfish, she should realize she should play α

(α, α) seems the sensible prediction
Components of a Game

Players: Who is involved?

Strategies: What can they do?

Payoffs: What do they want?
# Chicken

<table>
<thead>
<tr>
<th>Player 1</th>
<th>Straight</th>
<th>Swerve</th>
</tr>
</thead>
<tbody>
<tr>
<td>Straight</td>
<td>0, 0</td>
<td>3, 1</td>
</tr>
<tr>
<td>Swerve</td>
<td>1, 3</td>
<td>2, 2</td>
</tr>
</tbody>
</table>

Player 2
# Choosing a Restaurant

<table>
<thead>
<tr>
<th></th>
<th>P</th>
<th>V</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rebecca</td>
<td>4,3</td>
<td>1,1</td>
</tr>
<tr>
<td>Ethan</td>
<td>0,0</td>
<td>3,4</td>
</tr>
</tbody>
</table>
Demand Bargaining

$N$ players

Each player “demands” a real number in $[0, 10]$.

If the demands sum to 10 or less, each player’s payoff is her bid.

Otherwise players’ payoffs are 0.
Nash Equilibrium

A strategy profile where no individual has a unilateral incentive to change her behavior

Before we talk about why this is our central solution concept, let’s formalize it
**Notation**

Player $i$’s strategy

- $s_i$

Set of all possible strategies for Player $i$

- $S_i$

Strategy profile (one strategy for each player)

- $s = (s_1, s_2, \ldots, s_N)$

Strategy profile for all players except $i$

- $s_{-i} = (s_1, s_2, \ldots, s_{i-1}, s_{i+1}, \ldots, s_N)$

Different notation for strategy profile

- $s = (s_i, s_{-i})$
**Selfish Students**

\[
\begin{array}{ccc}
\text{Player 1} & \alpha & \beta \\
\hline
\alpha & 1,1 & 3,0 \\
\beta & 0,3 & 2,2 \\
\end{array}
\]

\[
S_i = \{\alpha, \beta\}
\]

4 strategy profiles: \((\alpha, \alpha), (\alpha, \beta), (\beta, \alpha), (\beta, \beta)\)
## Chicken

<table>
<thead>
<tr>
<th>Player 1</th>
<th>Player 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Straight</td>
<td>0, 0</td>
</tr>
<tr>
<td></td>
<td>3, 1</td>
</tr>
<tr>
<td>Swerve</td>
<td>1, 3</td>
</tr>
<tr>
<td></td>
<td>2, 2</td>
</tr>
</tbody>
</table>

\[ S_i = \{ \text{Straight, Swerve} \} \]

4 strategy profiles: (Straight, Straight), (Straight, Swerve), (Swerve, Straight), (Swerve, Swerve)
Choosing a Restaurant

<table>
<thead>
<tr>
<th></th>
<th>P</th>
<th>V</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ethan</td>
<td>4, 3</td>
<td>1, 1</td>
</tr>
<tr>
<td>Rebecca</td>
<td></td>
<td></td>
</tr>
<tr>
<td>P</td>
<td></td>
<td></td>
</tr>
<tr>
<td>V</td>
<td>0, 0</td>
<td>3, 4</td>
</tr>
</tbody>
</table>

$S_E = ?$  $S_R = ?$

Strategy profiles: ?
Demand bargaining with 3 players

\[ S_i = [0, 10] \]

- Player \( i \) can choose any real number between 0 and 10

\[ s = (s_1 = 1, s_2 = 4, s_3 = 7) = (1, 4, 7) \]

- An example of a strategy profile

\[ s_{-2} = (1, 7) \]

- Same strategy profile, with player 2’s strategy omitted

\[ s = (s_{-2}, s_2) = ((1, 7), 4) \]

- Reconstructing the strategy profile
Notating Payoffs

Players’ payoffs are defined over strategy profiles.

- A strategy profile implies an outcome of the game.

Player $i$’s payoff from the strategy profile $s$ is

$$u_i(s)$$

Player $i$’s payoff if she chooses $s_i$ and others play as in $s_{-i}$

$$u_i(s_i, s_{-i})$$
Consider a game with \( N \) players. A strategy profile \( s^* = (s_1^*, s_2^*, \ldots, s_N^*) \) is a **Nash equilibrium** of the game if, for every player \( i \)

\[
u_i(s_i^*, s_{-i}^*) \geq u_i(s_i', s_{-i}^*)\]

for all \( s_i' \in S_i \)
A strategy, $s_i$, is a best response by Player $i$ to a profile of strategies for all other players, $s_{-i}$, if

$$u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i})$$

for all $s'_i \in S_i$
Best Response Correspondence

Player $i$’s **best response correspondence**, $\text{BR}_i$, is a mapping from strategies for all players other than $i$ into subsets of $S_i$ satisfying the following condition:

- For each $s_{-i}$, the mapping yields a set of strategies for Player $i$, $\text{BR}_i(s_{-i})$, such that $s_i$ is in $\text{BR}_i(s_{-i})$ if and only if $s_i$ is a best response to $s_{-i}$
Consider a game with $N$ players. A strategy profile $s^* = (s_1^*, s_2^*, \ldots, s_N^*)$ is a Nash equilibrium of the game if $s_i^*$ is a best response to $s_{-i}^*$ for each $i = 1, 2, \ldots, N$.
**Selfish vs. Nice**

<table>
<thead>
<tr>
<th></th>
<th>Nice</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Selfish</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>α 1^, 2^</td>
<td>β 3^, 0</td>
</tr>
<tr>
<td></td>
<td>β 0, 1</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>β 2, 3^</td>
</tr>
</tbody>
</table>

\[\alpha\]
# Chicken

<table>
<thead>
<tr>
<th></th>
<th>Straight</th>
<th>Swerve</th>
</tr>
</thead>
<tbody>
<tr>
<td>Player 1</td>
<td>0, 0</td>
<td>1(^\vee), 3(^\vee)</td>
</tr>
<tr>
<td>Player 2</td>
<td>3(^\vee), 1(^\vee)</td>
<td>2, 2</td>
</tr>
</tbody>
</table>
### You Solve Choosing a Restaurant

<table>
<thead>
<tr>
<th></th>
<th>Rebecca</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>P</strong></td>
<td>4, 3</td>
</tr>
<tr>
<td></td>
<td>1, 1</td>
</tr>
</tbody>
</table>

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>P</strong></td>
<td><strong>V</strong></td>
<td></td>
</tr>
<tr>
<td>P</td>
<td>0, 0</td>
<td>3, 4</td>
</tr>
<tr>
<td>V</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
### Another Practice Game

<table>
<thead>
<tr>
<th></th>
<th>L</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>U</strong></td>
<td>10,2</td>
<td>3,4</td>
</tr>
<tr>
<td><strong>D</strong></td>
<td>−1,0</td>
<td>5,7</td>
</tr>
</tbody>
</table>
The War of Attrition

2 countries (1 and 2) are fighting over a territory

Each country $i$ decides how long it is willing to hold out, $t_i \geq 0$

The winner is the country that is willing to hold out for the longest time

- If both hold out the same amount of time, they split the territory

The war ends as soon as one country gives in
Country $i$’s Payoffs

Value of winning whole territory is $v_i > 0$

Value of winning half the territory is $\frac{v_i}{2}$

Cost of holding out for length of time $t_i$ is $t_i$
\[ u_1(t_1, t_2) = \begin{cases} 
-t_1 & \text{if } t_1 < t_2 \\
\frac{v_1}{2} - t_1 & \text{if } t_1 = t_2 \\
v_1 - t_2 & \text{if } t_1 > t_2 
\end{cases} \]
Country 1’s Best Response if $t_2 < v_1$

If Country 1 chooses $t_1 < t_2$, its payoff is $-t_1$

- Maximized at 0

If Country 1 chooses $t_1 = t_2$, its payoff is $\frac{v_1}{2} - t_1$

If Country 1 chooses $t_1 > t_2$, its payoff is $v_1 - t_2$

Any $t_1 > t_2$ is a best response
Country 1’s Best Response if \( t_2 = v_1 \)

If Country 1 chooses \( t_1 < t_2 \), its payoff is \(-t_1\)

- Maximized at 0

If Country 1 chooses \( t_1 = t_2 = v_1 \), its payoff is \( \frac{v_1}{2} - t_1 < 0 \)

If Country 1 chooses \( t_1 > t_2 \), its payoff is \( v_1 - t_2 = 0 \)

\( t_1 = 0 \) or any \( t_1 > t_2 \) are best responses
**Country 1’s Best Response if**

\( t_2 > v_1 \)

If Country 1 chooses \( t_1 < t_2 \), its payoff is \(-t_1\)

- Maximized at 0

If Country 1 chooses \( t_1 = t_2 \), its payoff is \( \frac{v_1}{2} - t_1 < 0 \)

If Country 1 chooses \( t_1 > t_2 \), its payoff is \( v_1 - t_2 < 0 \)

\( t_1 = 0 \) is the best response
Nash Equilibria

\[ t_1 = 0 \text{ and } t_2 > v_1 \]

\[ t_1 > v_2 \text{ and } t_2 = 0 \]
Why Nash Equilibrium?

No regrets

Social learning

Self-enforcing agreements

Analyst humility
Take Aways

A Nash Equilibrium is a strategy profile where each player is best responding to what all other players are doing.

You find a NE by calculating each player’s best response correspondence and seeing where they intersect.

NE is our main solution concept for strategic situations.