Optimal stationary contract with two-sided imperfect enforcement and persistent adverse selection

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HIGHLIGHTS

• The optimal stationary contract in an infinitely repeated relationship is proposed.
• The contract is made of two distinct pieces.
• For the most efficient types of the agent, the contract entails bunching.
• For less efficient types, the contract exhibits downward output distortions.
• Distortions are set below the Baron–Myerson level.

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ABSTRACT

We consider an infinitely-repeated principal–agent relationship run with stationary contracts. The agent has private information on his persistent cost parameter and, under limited enforcement, both parties can breach the contract. The optimal stationary contract with limited enforcement is made of two distinct pieces. For the most efficient types of the agent, the contract entails bunching with a fixed payment and a fixed output. For less efficient types, the contract exhibits downward output distortions below the Baron–Myerson level that would have been achieved had enforcement been costless.

1. Introduction

We consider an infinitely repeated principal–agent relationship. The agent has private information on his cost parameter. This cost is persistent over time and drawn once for all from a continuous distribution. The contract can be breached at any point in time by either party. The principal may choose not to pay once delivered the good. The agent may choose not to deliver the requested quantity/quality of the good to be traded. Enforcement of the contract is restricted to the use of limited penalties for breach on the party who does not fulfill his obligations. On top, future trades are disrupted following a breach by either party.

Similar settings have already been studied in the literature. In a framework where no formal contract can ever be enforced, Levin (2003) considered a symmetric information setting (with either complete information or types which are i.i.d. over time). He showed that the optimal relational contract is stationary and may entail output and wage compression. Baron and Besanko (1984) studied a repeated contracting environment with full commitment and perfect enforcement. They showed that there is no loss of generality in considering stationary contracts and that the optimal long-term contract is indeed the replica of the well-known static optimum found in Baron and Myerson (1982). Allowing for limited enforcement but formal contracting, Martimort et al. (2017) considered the case of a two-type discrete distribution. We showed there that the optimal contract is actually non-stationary when enforcement constraints are binding and that long-run distortions encapsulate the shadow cost of limited enforcement. Kwon (2016) derives the optimal relational contract with persistent adverse selection in Levin (2003)’s environment and shows that it is no longer stationary. The non-stationarity of relational contracts may
also come from learning persistent types as in a model of the labor market proposed by Yang (2013).

Unfortunately, in some structured environments, such non-stationary profiles of payments and outputs may be hard to implement. To illustrate, consider the ongoing relationships between an upstream manufacturer (the principal) and a collection of downstream retailers (her agents). If the retailers initiate contracts with the principal at different points in time, then even if each of those relationships might a priori be ruled under different time-varying contracts, a prohibition against discriminatory contracts at each point in time would force the use of stationary contracts throughout the whole vertical structure. Similarly, when there are possibilities for the downstream retailers to further trade and subcontract with each other, arbitrage opportunities would make it difficult to credibly enforce trades at different prices with different retailers providing the same quality good. As an empirical matter, stationarity in vertical relationships is not uncommon. LaFontaine and Shaw (1999) found in their study of vertical franchise contracts, for example, that royalty rates and franchise fees do not change over time.

In this note, we characterize the optimal enforceable stationary contract and analyze its main properties. In a model with continuous types, we show that, under a mild assumption on the distribution of types that generalizes the well-known monotonicity of the hazard rate property, that the optimal contract, when constrained by enforcement, has two main features. First, bunching arises for the most efficient types who all produce the same quantity and receive the same payment. Second, less efficient types are separated, though at outputs below the Baron-Myerson allocation that would be achieved had enforcement been costless. Although apparently similar to Levin (2003)'s result on wage/output compression, our results are significantly different and hinge on a more subtle trade-off. Indeed, in Levin (2003), the fact that there is symmetric information between the principal and the agent means that the enforcement constraint can only be satisfied provided that current payments are small compared with the continuation value of the relationship. Since types are i.i.d. over time, this continuation value is "fixed" and the enforcement constraint is akin to an exogenous upper bound on payments (as in Thomas (2002)). Instead, in our screening scenario, payments play a second role as a screening instrument. Higher payments help to induce information revelation from the most efficient types of the agent. With imperfect enforcement, the principal mediates the desire to raise payments for screening reasons against the incentives to renego on such large promises. Wages and output compressions follow from this trade-off.

2. Model

We consider an infinitely-repeated trading environment between a principal (she, the buyer) and an agent (he, the seller). Time is indexed by \( \tau = 0, 1, \ldots, \infty \), and \( \delta < 1 \) is the common discount factor. In each period, a quantity \( q_\tau \) can be traded. Before (resp. after) trade takes place, the principal makes some payment \( t_{1,\tau} \) (resp. \( t_{2,\tau} \)). The agent has private information on his cost parameter \( \theta \) which is drawn once for all before the relationship starts. A contract is thus an array \( C = \{ (t_{1,\tau}, t_{2,\tau}, q_\tau), \theta \}_{\theta \in \Theta} \}_{\tau = 0, \ldots, \infty} \) that stipulates for each trading period payments (respectively before and after current trade) \( t_{1,\tau}(\theta), t_{2,\tau}(\theta) \) and a quantity \( q_\tau(\theta) \) that are contingent on the agent’s report \( \theta \) on his cost parameter.

The principal’s and the agent’s utility functions are respectively given by:

\[
(1 - \delta) \sum_{\tau = 0}^{\infty} \delta^\tau (S(q_\tau(\theta)) - t_{1,\tau}(\theta) - t_{2,\tau}(\theta))
\]

\[
(1 - \delta) \sum_{\tau = 0}^{\infty} \delta^\tau (t_{1,\tau}(\theta) + t_{2,\tau}(\theta) - \theta q_\tau(\theta)).
\]

We assume that \( S(\cdot) > 0 > S'(\cdot) \) with \( S(0) = 0 \) and where \( S(\cdot) \) is also large enough to avoid corner solutions and avoid shut-down of the least-efficient types. The principal only knows the cumulative distribution \( F(\cdot) \) whose support is \( \theta = \{ \underline{\theta}, \overline{\theta} \} \) and whose positive and atomless density is denoted by \( f(\cdot) = F'(\cdot) \).

We consider stationary contracts of the form: \( q_\tau(\theta) = q(\theta), t_{1,\tau}(\theta) = t_{1}(\theta) \) and \( t_{2,\tau}(\theta) = t_{2}(\theta) \) for all \( \tau \). The timing of the contracting game unfolds as follows.

1. At date \( \tau = 0 \) the agent learns his cost parameter \( \theta \). The principal offers a contract \( C = (t_{1}(\theta), t_{2}(\theta), q(\theta))_{\theta \in \Theta} \). The agent accepts or rejects \( C \). If the agent rejects, then both parties get their reservation values that are normalized at zero. If the agent accepts, he reports having a type \( \theta \).
2. At any date \( \tau \geq 0 \), trade takes place. First, the principal pays an advance payment \( t_{1}(\theta) \). Second, the agent produces \( q(\theta) \). If he does not deliver this requested quantity, the contract is breached and the agent must pay the penalty \( L \). If \( q(\theta) \) is delivered, the principal pays the after-sale payment \( t_{2}(\theta) \). If she does not, the contract is again breached and the principal pays the penalty \( K \). Following breach by either party, the contract is terminated.

3. Enforcement and incentive compatibility

Denote by \( U(\theta) \) the agent’s average per-period payoff with a stationary contract, i.e., \( U(\theta) = t_{1}(\theta) + t_{2}(\theta) - \theta q(\theta) \) where \( t(\theta) = t_{1}(\theta) + t_{2}(\theta) \). By the Revelation Principle it is without the loss of generality to focus on contracts that induce incentive compatible, stationary allocations \((q(\theta), U(\theta))\). A standard argument characterizes incentive compatible allocations in this environment:

Lemma 1. An allocation \((q(\theta), U(\theta))_{\theta \in \Theta}\) is incentive compatible if and only if \( U(\theta) \) is absolutely continuous, convex and satisfies at any point of differentiability (i.e., almost everywhere)

\[
U'(\theta) = -q(\theta),
\]

\[
q(\theta) \text{ is non-negative and non-increasing.}
\]

Eq. (3.1) implies that \( U \) is a non-increasing function. Hence, a contract induces participation for all types if it does so for the least-efficient one, namely:

\[
U(\overline{\theta}) \geq 0.
\]

1 Antitrust authorities have repeatedly addressed price discrimination in intermediate-good markets with suspicion. In the U.S., the Robinson–Patman was enacted to put on equal foot small businesses and large buyers in intermediate-good markets. Although this more complex motivation is not part of our model, it justifies our focus on non-discriminatory contracts.

2 The no-renegotiation assumption can be justified on several grounds. First, it allows us to compute an upper bound on the possible gains from trade that can be achieved under asymmetric information and limited enforcement. Second, as an empirical matter, it is not uncommon for contractual breaches to lead to the termination of a productive relationship without any attempt at renegotiation. For example, such termination is commonly observed in construction contracts. This behavior may represent an equilibrium of a richer reputation game which we choose to leave unmodeled in this note. Finally, as a practical matter, the assumption of commitment to no-renegotiation is a standard assumption in the law and economics literature (see Edlin, 1998 and Shavell, 2004, p. 315), and an obvious starting point for richer studies.

3 The proof is standard and is thus omitted. See for example Laffont and Martimort (2002).
With limited enforcement, the possibility of contract breach by either party adds new constraints to the characterization of incentive-feasible allocations so as to prevent such behavior. Turning first on the agent’s side, the contract should be designed so that an agent with type \( \theta \) prefers to take the corresponding contract rather than mimicking a type \( \hat{\theta} \) for \( r-1 \) periods and then not delivering at date \( \tau \) the requested quantity. Such “take-the-money-and-run” strategy is prevented when the following agent-enforceability constraint is satisfied:

\[
U(\theta) \geq (1 - \delta)\left(t(\hat{\theta}) - \theta q(\hat{\theta})\right) + \cdots + \delta^{r-1}\left(t(\hat{\theta}) - \theta q(\hat{\theta})\right) + \delta^r(t(\hat{\theta}) - L) = (1 - \delta^r)\left(U(\hat{\theta}) + (\hat{\theta} - \theta)q(\hat{\theta})\right) + (1 - \delta^r)q(\hat{\theta}) - \delta^r(t(\hat{\theta}) - L) \quad \forall (\theta, \hat{\theta}) \in \Theta^2. 
\]

(3.4)

From (3.4) at \( r = 0 \), we get:

\[
U(\theta) \geq (1 - \delta)\left(t(\theta) - L\right) \quad \forall (\theta, \theta) \in \Theta^2. 
\]

(3.5)

Taking instead \( \tau = +\infty \) into (3.4) yields the familiar incentive constraints:

\[
U(\theta) \geq U(\hat{\theta}) + (\hat{\theta} - \theta)q(\hat{\theta}) \quad \forall (\theta, \theta) \in \Theta^2. 
\]

(3.6)

Reciprocally, it is straightforward to check that (3.4) is a linear combination of (3.5) and (3.6). Thus, we can replace the \textit{a priori} more complex constraints in (3.4) by (3.5) and (3.6).

We now turn to the principal’s incentives to breach the contract and how contracts can be designed so as to prevent such behavior. The principal abides and pays \( t_{\tau-r}(\theta) \) after delivery whenever the continuation value of the relationship he can secure thereby exceeds the penalty for the current period. The principal’s enforceability can thus be written as:

\[
t_\tau(\theta) + \frac{\delta}{1 - \delta}(S(q(\theta)) - t(\theta)) \geq -K. 
\]

Expressing this constraint in terms of the agent’s per-period rent, we obtain:

\[
t_\tau(\theta) - \theta q(\theta) - U(\theta) + \frac{\delta}{1 - \delta}(S(q(\theta)) - t(\theta)) \geq -K. 
\]

(3.7)

Several simple observations allow us to further simplify the set of incentive-feasible allocations. First, starting from a contract \( \hat{C} = (t_1(\theta), t_2(\theta), q(\theta))_{\theta \in \Theta} \), we may consider the new contract \( \tilde{C} = (t_1(\theta), t_2(\theta), q(\theta))_{\theta \in \Theta} \) whose transfers are defined as follows:

\[
t_1(\theta) = t_1 \equiv \max_{\hat{\theta} \in \Theta} t_1(\hat{\theta}), \quad t_2(\theta) \equiv t_1(\theta) - t_2(\theta) - \max_{\hat{\theta} \in \Theta} t_1(\hat{\theta}).
\]

Because \( \tilde{C} \) and \( \hat{C} \) stipulate the same overall payment and output, incentive compatibility is preserved and the principal’s expected payoff remains unchanged. Constraint (3.5) also remains the same for both contracts while (3.7) is relaxed for \( \tilde{C} \). Thus, there is no loss of generality in assuming that the pre-delivery payment is fixed, i.e., \( t_1(\theta) = t_1 \).

Second, suppose that \( t_1 > L \). Then, we could decrease \( t_1 \) by \( \epsilon > 0 \) and \( U(\theta) \) by \( (1 - \delta)\epsilon \) for all \( \theta \). Constraint (3.5) would not change. The left-hand side of (3.7) is not modified either. Since the value of the objective function increases with such change and the constraints are satisfied, we obtain a contradiction. If \( t_1 < L \) then we can increase \( t_1 \) by \( \epsilon > 0 \). It does not affect constraint (3.5) and (3.7) is relaxed. Because the principal’s objective function does not depend on \( t_1 \), this modification cannot reduce the principal’s payoff. We summarize those findings in the next lemma.

**Lemma 2.** There is no loss of generality in restricting the analysis to enforceable incentive-compatible allocations such that:

1. \( t_1(\theta) = L \) \( \forall \theta \in \Theta \).
2. The principal’s and the agent’s enforceability constraints can be pooled into a single constraint

\[
h(\theta, q(\theta), U(\theta)) = \delta S(q) - \theta q(\theta) - U(\theta) + (1 - \delta)M \geq 0 \quad \forall \theta \in \Theta
\]

(3.8)

where \( M = K + L \).

By making the pre-delivery price just equal to the breach penalty that the agent could incur, the principal ensures that the agent is just indifferent between breaching or not. Everything thus happens as if the principal ends up paying the agent’s penalty for breach.

**4. The optimal contract**

The principal’s problem \( \mathcal{P} \) can be expressed as:

\[
\langle \mathcal{P} \rangle : \max_{\langle U(\theta), q(\theta) \rangle} \int_\theta (S(q(\theta)) - \theta q(\theta) - U(\theta))dF(\theta)
\]

subject to (3.1)–(3.3) and (3.8).

Without the aggregate enforcement constraint (3.8), the solution is the familiar Baron–Myerson (1982) optimal second-best output defined as:

\[
S^*(q^{BM}(\theta)) = \theta + F(\theta)\int_{\theta}^\infty (4.1)
\]

\[
S'(q^{BM}(\theta)) = \theta + F(\theta)\int_{\theta}^\infty \frac{U'(q) + \Psi}{F'(\theta)}d\theta, \quad U^{BM}(\theta) = t^{BM}(\theta) - \delta q^{BM}(\theta)
\]

Monotonicity of \( q^{BM}(\theta) \) is guaranteed by the well-known monotonicity of the optimal contract. With limited enforcement, as in our context, this property has to be generalized. Indeed, the characterization of the solution to \( \langle \mathcal{P} \rangle \) is simplified when the following condition holds.

**Assumption 1.** For any \( \Psi \geq 0 \) the generalized hazard rate \( \frac{F(\theta) + \Psi}{F'(\theta)} \) satisfies

\[
\frac{d}{d\theta} \left( \frac{F(\theta) + \Psi}{F'(\theta)} \right) \geq 1 - \frac{\delta}{\delta} \quad \forall \theta \in \Theta.
\]

(4.2)

This assumption always holds when the density \( f(\theta) \) is non-increasing and the discount factor \( \delta \geq \frac{1}{2} \). For future references, we also generalize the Baron–Myerson’s output \( q^{BM}(\theta) \) defined as:

\[
S'(q^{BM}(\theta)) = \theta + F(\theta)\int_{\theta}^\infty \frac{U'(q) + \Psi}{F'(\theta)}d\theta, \quad \forall \theta \in \Theta.
\]

**Definition 1.** Let \( \Psi \geq 0 \). The generalized Baron–Myerson output \( q^{BM}(\theta) \) is defined as:

\[
S'(q^{BM}(\theta)) = \theta + F(\theta)\int_{\theta}^\infty \frac{U'(q)}{F'(\theta)}d\theta, \quad \forall \theta \in \Theta.
\]

We are now ready to characterize the optimal contract with limited enforcement.

**Theorem 1.** Suppose that Assumption 1 holds. Two regimes for the optimal contract solution of \( \langle \mathcal{P} \rangle \) are possible.

1. **Strong enforcement.** If

\[
h(\theta, q^{BM}(\theta), 0) \geq 0 \text{ and } h(\theta, q^{BM}(\theta), U^{BM}(\theta)) \geq 0
\]

then the optimal contract is Baron–Myerson.

2. **Weak enforcement.** If

\[
h(\theta, q^{BM}(\theta), 0) < 0 \text{ or } h(\theta, q^{BM}(\theta), U^{BM}(\theta)) < 0
\]

then there exist a threshold type \( \theta^* > \theta \) and an output level \( q^* = q_{(1-\delta)M}(\theta^*) < q^{BM}(\theta) \) such that the optimal contract is
continuous and satisfies

\[
\overline{q}(\theta) = \begin{cases} 
q^* & \text{if } \theta \leq \theta^* \\
q_{1-\delta M}(\theta) & \text{if } \theta > \theta^*
\end{cases}
\quad \text{and } U(\theta) = \int_\theta^{\theta^*} \overline{q}(\xi)d\xi.
\tag{4.3}
\]

When the enforcement constraint (3.8) is binding on a non-empty interval (the case of weak enforcement), the principal makes this constraint cheaper by reducing the agent’s information rent. It means that both the agent’s requested payment and his output have to be reduced below the Barro–Myerson outcome. The principal is torn between his desire to reward the most efficient types and facilitate screening and the fact that such large payments exacerbate his incentives to reneg on his contractual obligations. The enforcement constraint is akin to an upper bound on possible payments. As such, the characterization of the optimal contract has much in common with Thomas (2002) who analyzed the case of an exogenous upper bound on payments in a nonlinear pricing environment. In our context, instead this upper bound is endogenously derived from an enforcement problem. Technically, this means that the enforcement constraint becomes a somewhat complex mixed-constraint.\(^4\) The upper bound on payments is binding for a whole set of types on the lower tail of the distribution of the agent’s efficiency parameter. Those types are all bunched together and produce the same quantity \(q^*\). By incentive compatibility, compressing payments for those most efficient types also requires further downward output distortions for the least efficient ones. The output requested from those types is distorted downward below the Barro–Myerson level even though the enforcement constraint is not binding for those types. Our results thus suggest that optimal contracts with limited enforcement are tilted towards low-powered incentives. These results also point out how simple pooling contracts, an extreme form of incompleteness, may perform well in such context.

Appendix

Consider the problem \((\widetilde{P})\) obtained from \((P)\) by omitting (3.2). The Lagrangian is

\[
L(\theta, U, q, \lambda, \psi) = (S(q) - \theta q - Uf(\theta) - \lambda q + \psi h(\theta, q, U)
\]

where \(\lambda\) is the co-state variable associated with (3.1) and \(\psi\) is the multiplier of constraint (3.8). Observe that the necessary conditions (Seierstad and Sydsaeter, 1987, p. 276) are also sufficient conditions. The solution \((U(\theta), q(\theta), \lambda(\theta), \psi(\theta))\) is such that the pair \((U(\theta), q(\theta))\) is admissible, the function \(\lambda(\theta)\) is continuous and piecewise continuously differentiable and the function \(\psi(\theta)\) is piecewise continuous. The optimality conditions are

\[
f(\theta)[S'(q(\theta)) - \theta] + \psi(\theta)[S'(q(\theta)) - \theta] = \lambda(\theta),
\tag{A.1}
\]

\[
\dot{\lambda}(\theta) = f(\theta) + \psi(\theta), \quad \text{a.e.},
\tag{A.2}
\]

\[
\lambda(\theta) = 0, \quad U(\theta) = 0,
\tag{A.3}
\]

\[
\psi(\theta) \geq 0, \quad h(\theta, q(\theta), U(\theta)) \geq 0, \quad \psi(\theta)h(\theta, q(\theta), U(\theta)) = 0.
\tag{A.4}
\]

Using (A.2) and (A.3), we obtain

\[
\lambda(\theta) = F(\theta) + \psi(\theta) \quad \forall \theta \in \Theta
\]

where \(\psi(\theta) = \int_\theta^{\theta^*} \overline{q}(\xi)d\xi\), \(\theta^*\) is a constant over \(\Theta\).

We consider a partition of \(\Theta\) as minimal set of intervals \([\theta_1, \theta_2, \ldots, \theta_n]\) such that \(\Theta = \Theta_1 \cup \Theta_2 \cup \ldots \cup \Theta_n\) with \(\psi(\theta) = 0 \quad \forall \theta \in \Theta_2\) and \(\psi(\theta) > 0 \quad \forall \theta \in \Theta_2 \cup \Theta_2 \cup \ldots \cup \Theta_n\). Denote the boundaries of an interval \(\Theta_i\) as \(\theta_{i-1}\) and \(\theta_i\), \(\theta_0 = 0, \theta_n = \theta^*\) and \(\theta_{i-1} \leq \theta_i \leq \ldots \leq \theta_n = \theta^*\).

Because \(\psi(\theta) = 0\) for all \(\theta \in \Theta_2\), (A.6) implies

\[
S'(q(\theta)) = \theta + \frac{\theta \psi(\theta) + F(\theta) + \psi(\theta)}{f(\theta)}.
\tag{A.6}
\]

Using (A.2) and (A.3), we obtain

\[
S'(q(\theta)) = \theta + \frac{\theta \psi(\theta) + F(\theta) + \psi(\theta)}{f(\theta)}.
\tag{A.6}
\]

We prove first that \(n \leq 3\). Suppose that \(\theta_4\) is non-empty. Then we have

\[
S'(q_3) = \theta_2 + \frac{F(\theta_2) + \psi_2}{f(\theta_2)} \quad \text{and } S'(q_4) = \theta_3 + \frac{F(\theta_3) + \psi_4}{f(\theta_3)}
\]

where \(\psi_4 > \psi_2 > \psi_2 > \theta_2\). This implies

\[
0 = \theta_3 - \theta_2 + \frac{F(\theta_3) + \psi_4}{f(\theta_3)} - \frac{F(\theta_2) + \psi_2}{f(\theta_2)}.
\tag{A.8}
\]

A contradiction. Hence, \(\theta_4 = \emptyset\).

Suppose that \(\Theta_2 = \{\theta_2, \theta_3\}\) is non-empty and \(\psi(\theta) = \psi_3 \quad \forall \theta \in \Theta_2\). Consider a strictly concave function \(h(\theta, q, U(\theta)) = M(q) = (1 - \delta)M, h(\theta) = (1 - \delta)M > 0, q^* = \text{arg max } h(\theta)\). We have \(q^* > 0\) and \(\delta S'(q^*) = \theta^*\). There exist a unique value \(q^* > q^*\) such that \(h(q^*) = 0\). By concavity of \(h(\cdot)\) we have \(S'(q^*) > S'(q^*)\), or, equivalently.

\[
\delta S'(q^*) < \theta^*.
\tag{A.9}
\]

Because \(h(\theta, q, U(\theta)) = 0 \quad \forall \theta \in \Theta_2\), we have \(S'(q_3) = \theta_3 - \theta_{q_3 + (1 - \delta)M = 0}. \quad \text{Thus } q_3 = \hat{q}\). By continuity of \(q(\theta)\) at

\[\text{Seierstad and Sydsæter (1987).}\]
Lemma A.3. Suppose \( \Theta_1 \neq \emptyset \). Then \( \psi(\theta) \) is a continuous non-negative function given by

\[
\psi(\theta) = \begin{cases} 
\frac{(S'(q_1) - \theta) f(\theta) - F(\theta)(\theta - S'(q_1)) - (S'(q_1) - \theta) f'(\theta)}{(\theta - S'(q_1))^2} & \text{if } \theta \in \Theta_1, \\
0 & \text{if } \theta \in \Theta_2. 
\end{cases}
\] (A.10)

Proof. Because \( \Theta_1 \) is non-empty we have \( \psi(\theta) > 0 \). Equations (A.1) and (A.3) yield

\[
(S'(q_1) - \theta) f(\theta) + \psi(\theta) (\delta S'(q_1) - \theta) = \lambda(\theta) = 0.
\]

Thus \( S'(q_1) - \theta > 0 \) and \( \delta S'(q_1) - \theta < 0 \).

By continuity of \( q(\theta) \) at \( \theta_1 \) we have \( f(\theta_1) S'(q_1) - \theta_1 = F(\theta_1) + \psi(\theta_1) \) and

\[
f(\theta_1) S'(q_1) - \theta_1 + \psi(\theta_1) (\delta S'(q_1) - \theta) = F(\theta_1) + \psi(\theta_1).
\]

Therefore, \( \delta S'(q_1) - \theta_1 = 0 \). However, by (A.11) \( \delta S'(q_1) < 0 \). Hence \( \psi(\theta_1) = \psi(\theta_1) = 0 \) and \( \psi(\theta) \) is a continuous function.

By (A.1) we have \( \theta > \delta S'(q_1) \). Hence \( \theta - \delta S'(q_1) > 0 \). From (A.1) we obtain

\[
\psi(\theta) + \frac{1}{\theta - \delta S'(q_1)} \psi(\theta) = \frac{F(\theta) + f(\theta)(\theta - S'(q_1))}{\delta S'(q_1) - \theta}.
\] (A.12)

Taking into account that \( \psi(\theta) = 0 \), the solution to this differential equation is

\[
\psi(\theta) = \frac{(S'(q_1) - \theta) f(\theta)}{\theta - \delta S'(q_1)}.
\] (A.13)

Differentiating this function we obtain

\[
\psi'(\theta) = \frac{(S'(q_1) - \theta) f'(\theta) - F(\theta)(\theta - S'(q_1)) - (S'(q_1) - \theta) f'(\theta)}{(\theta - S'(q_1))^2}.
\] (A.14)

We show that \( \psi(\theta) \geq 0 \) for \( \theta \in \Theta_1 \). Because \( \psi(\theta_1) = 0 \) it sufficient to show that the nominator of the right-hand-side of (A.14) is a decreasing function. We have

\[
(\theta - \delta S'(q_1))((S'(q_1) - \theta) f'(\theta) - 2f(\theta)) < 0. \quad \Box
\]

To ensure that the constructed output is optimal, we need that the inequality (3.8) holds \( \forall \theta \in \Theta_2 \).

Lemma A.4. \( h(\theta, q(\theta), U(\theta)) > 0 \) for \( \theta \in \text{int}\Theta_2 \).

Proof. On \( \Theta_2 \) the output is given by (A.7). Thus

\[
\delta S'(q(\theta)) - \theta = \frac{1}{\delta} \left( \frac{F(\theta) + \psi(\theta) - 1 - \delta}{\psi(\theta) - \theta} \right).
\]

By Assumption 1 the function \( \delta S'(q(\theta)) - \theta \) is increasing on \( \Theta_2 \). Note that \( h(\theta, q(\theta), U(\theta)) = 0 \) and \( \delta S'(q(\theta)) - \theta \geq 0 \). The derivative of \( h(\theta, q(\theta), U(\theta)) \) is equal to \( \xi(q(\theta), U(\theta)) - \psi(\theta) \). Because \( \psi(\theta) < 0 \) and \( \delta S'(q(\theta)) - \theta \) is increasing on \( \Theta_2 \) the function \( h(\theta, q(\theta), U(\theta)) \) cannot have more than one extrema on \( \Theta_2 \). Thus \( h(\theta, q(\theta), U(\theta)) > 0 \) for all \( \theta \in \text{int}\Theta_2 \). \( \Box \)

We are now ready to prove Theorem 1.

Strong Enforcement. Consider the Baron–Myerson allocation with \( \lambda(\theta) = f(\theta) + \psi(\theta) \). In this case \( \Theta = \Theta_2 \) and, taking together the assumptions in that case that determine the sign of \( h \) on boundaries \( \theta \) and \( \Theta \) and Lemma A.4 which determines its sign on \( (\theta, \Theta) \) we get

\[
h(\theta, q^{BM}(\theta), U^{BM}(\theta)) \geq 0 \quad \forall \theta \in \Theta.
\]

Hence, the Baron–Myerson allocation is optimal.

Weak Enforcement. By assumptions \( h(\theta, q^{BM}(\theta), U^{BM}(\theta)) < 0 \) either for \( \theta = \theta \) or for \( \theta = \Theta \), and, therefore, the Baron–Myerson allocation no longer satisfies constraint (3.8). Thus \( \Theta_1 \neq \emptyset \) and there must be bunching for the most efficient types. The optimal solution has a form (A.3) with thresholds \( \theta_1 = \theta^* > \Theta \) and \( \bar{q} = q^* < q^{BM}(\theta) \). By Lemma A.3 the values of \( \theta^* \) and \( q^* \) are solutions to

\[
\psi(\theta^*, \psi) = 0 \quad \text{and} \quad h(\theta^*, q^*) = 0
\] (A.15)

where \( \psi(\theta^*, \psi) \) is given by (A.14) with \( \theta^* = \theta_1 \) and \( q^* \). If Filipov–Cesari Existence Theorem (Seierstad and Sydsæter, 1987 p. 285) the solution to (P) exists. Thus, the optimal \( (\theta^*, \psi^*) \) must satisfy the system (A.15). Hence the four-tuple \( (q(\theta), U(\theta), \lambda(\theta), \psi(\theta)) \), where \( U(\theta) = \int_0^\theta q(x) dx, \lambda(\theta) = F(\theta) + \psi(\theta) \), \( \psi(\theta) = \psi(\theta) \) and \( \psi(\theta) \) is given by (A.13) if \( \theta \geq \theta^* \) and \( \psi(\theta) = 0 \) if \( \theta \leq \theta^* \), gives the solution to the program (P).

Finally, note that the optimal output in non-decreasing and by the assumption that \( S(0) \) is large enough, it is non-negative.

References