The Economics of Liquidated Damage Clauses in Contractual Environments with Private Information

Lars A. Stole
University of Chicago

1. Introduction

Economists have long recognized that agreements freely entered into by all affected parties with full information and cognizance of the terms of trade necessarily improve social welfare in the traditional Pareto sense. It comes as no surprise that economists look at the law with skepticism whenever courts invalidate any mutually agreed-upon terms within a contract absent negative effects upon third parties. Nonetheless, courts routinely invalidate contractually stipulated damages for breach of contract (commonly known as liquidated damages) when such damages are "unreasonably large" relative to actual or expected losses but not when such damages are unreasonably small.¹

The invalidation of excessive stipulated damage clauses is difficult to justify economically. Liquidated damage clauses promote efficiency in contractual relationships by reducing the litigation and judicial costs that accompany breach, by providing the correct incentives for a breaching party, and by

¹ See Uniform Commercial Code, §§ 2-302(1), 2-718(1), and the Restatement of Contracts (Second), §§ 208, 356, where the requirement is that liquidated damage clauses are invalid as penalties if they are unreasonably large relative to both actual and expected loss. Variations of these standards have been embraced by different courts; see Farnsworth (1982:900–01).

© 1992 by Oxford University Press. All rights reserved. ISSN 8756–6222
optimally allocating risk. Most importantly, stipulation of damages by the parties rather than by judicial determination allows parties to utilize efficiently their superior information, which typically courts can access only imperfectly.

The courts have had difficulty motivating the invalidation of excessive stipulated damage clauses as penalties. One theory often presented by legal scholars posits that legal remedies for breach of contract serve only to compensate and never to punish; see Farnsworth (1982:896). Such a principle has economic merit. We ordinarily want parties to breach contracts when it is economically efficient that they do so. By making the promisor more than compensate the loss incurred from his nonperformance, the contract induces a suboptimal level of breach.

Unfortunately, this simple explanation falls short on two points. First, it does not explain why rational individuals would agree to such a contract when there exists another contract that sets damages at the value of performance and that makes both parties better off. Second, it also does not explain why the courts fail to extend this operating principle to situations of undercompensatory stipulated damage agreements, which produce a superoptimal level of breach.

Many courts and legal scholars answer the first point by arguing that excessive liquidated damages are presumptive evidence of a contractual failure such as fraud or mutual mistake. Arguably, courts view excessive damages as evidence that at least one party has wrongly agreed to a contract that is not Pareto improving and respond by striking such clauses. But this does not explain why the court does not also strike extremely low liquidated damage clauses, which presumably are also the product of contractual failures. We are left with an apparent asymmetry in legal principles.

2. Shavell analyzes the use of damage remedies to provide incentives for efficient breach. Although Shavell does not explicitly entertain the idea of stipulated damages, his analysis is closely related. In complementary work, Polinsky has shown that in some instances it is efficient from a risk-allocation viewpoint to contract for stipulated damages in excess of the actual loss from breach. Such conditions require, among other things, that the buyer should bear some of the price risk introduced from third-party, breach-inducing offers. Rea (p. 154), however, has argued that these conditions are rare.

3. Clarkson, Miller, and Muris make an alternative argument: if one party has the ability to induce breach by the other, excessive damage clauses may waste resources, either on breach-inducing activities or on detecting and preventing breach inducement. In cases where these costs are high, liquidated damage clauses should be enforced only if they are reasonable in relation to damages sustained. But such an argument supposes that one party to the contract did not correctly anticipate such a situation, and so there is some contractual inefficiency ex post. Additionally, if neither party has private information, they will renegotiate to a new level of damages so as to eliminate the resource waste.

4. Aghion and Bolton provide an additional story involving negative effects upon third parties who are not part of the contract. Two individuals may desire to sign a contract that assigns excessive liquidated damages for breach so as to foreclose entry by another supplier. Of course, these damages are socially inefficient. In a related paper, Diamond and Maskin consider the joint problems of breach and search for new trading partners. They find that because an individual who breaches can get his new partner to share the burden of the liquidated damage he pays to his old partner, a pair of partners in a contract exerts some monopoly power over potential partners, thereby making liquidated damages supercompensatory.
In this article, I provide a partial explanation for the lack of legal symmetry: Undercompensatory damages are the likely result of the rational decision of two individuals bargaining in an environment where each possesses private information about the exchange. The same cannot be said for supercompensatory damages. Because the low damages do not necessarily represent a contractual failure but can realistically reflect a jointly beneficial contract arrived at under the constraints of asymmetric information, legal institutions are arguably consistent in enforcing such terms. This article’s principal thesis maintains that when each party to a contract possesses private information whose disclosure would adversely affect its position in the contractual bargaining, rationally calculated liquidated damages will be set at undercompensatory levels. Thus, although I do not explain why the court would want to invalidate excessive damages (or why parties would ever want to write such contracts), I do partially answer the riddle of why courts enforce only liquidated damage clauses that are set at or below the level of actual loss from breach.

Economic analysis of liquidated damage clauses generally has been limited to symmetrically informed parties. In many contractual situations, however, the assumption that parties entered into the contract without private information is not palatable. When such asymmetries in information are present, the liquidated damage clause takes on a dual role: (i) providing incentives for efficient breach, and (ii) efficiently screening among different types of buyers and sellers. Specifically, in this article, I demonstrate that when parties have asymmetric information, stipulated damages may be used to communicate valuable information at the pre-contractual stage. As such, the loss from suboptimal or excessive breach may be offset by informational gains. In fact, in the typical buyer–seller contract where each party has private information, stipulated damages will almost always fall short of actual losses from the breach.

I examine the buyer–seller relationship, although its results appear much more general. I assume that the buyer has private information regarding the value of the product to herself, and that the seller has private information regarding alternative markets where the product may be sold absent a sale to the present buyer. The contractual framework is modeled in Section 2. In Section 3, I examine various bargaining situations. In Section 3.1, I analyze

5. Rea provides a related and complementary analysis where he finds that the presence of moral hazard and risk aversion suggest parties may desire to set damages below expected loss.

6. Liquidated damages chosen so as to approximate expected or actual losses should be invariant to changes in bargaining power. Nonetheless, courts have held that liquidated damage clauses that vary with changes in a party’s bargaining power are not necessarily invalid as penalty clauses [In re Bubble Up Delaware, Inc., 684 F.2d 1259, 1263 (9th Cir. 1982)]. This suggests that liquidation terms are currently serving as more than just compensatory measures, and perhaps are innately connected with the bargaining process, as this article argues.

7. Schwartz has established a weaker version of this result for two-type distributions when private information exists only on the buyer’s side and the seller has the bargaining power; this case is similar to the contracts developed in Section 3.2 of this article.
the consequences of placing all of the bargaining power in the hands of the buyer; in Section 3.2, I assume all of the bargaining power resides with the seller. In Section 3.3, I examine what an efficient but uninformed broker would assign as stipulated damages, thereby providing an upper bound on the joint gains from exchange under any mechanism and giving insights into the nature of more general, efficient contracting mechanisms. In each of the three contracting scenarios, I find that there is no role for excessive stipulations, but there is a positive role for undercompensatory terms. Indeed, undercompensatory terms occur with probability 1. These terms provide a valuable method for both parties to reveal their private information, increasing the gains from trade.

As a motivating example, consider the seller of a good (which is costless to produce) who faces a single buyer with a private valuation for the good of 2 with probability 3 and 1 with probability 1. Absent liquidated damage clauses (i.e., the seller pays the buyer’s valuation, which a court perfectly determines in the event of breach), the optimal price is 2 with an expected profit of 3. In this case, the low-valuation buyer does not buy. But now suppose that the seller can offer two contracts with different liquidated damage terms from which the buyer can choose: a high-priced contract with a higher stipulated damage and a low-priced contract with little protection from breach and, consequently, a greater likelihood of nonperformance. Generally, if the prices and levels of protection are correctly chosen, the optimal contracts are designed such that the high-value buyer takes the high-priced contract with complete breach protection (i.e., liquidated damages are 2) and the low-value seller takes the low-priced terms with less than complete protection (i.e., liquidated damages are less than 1). In this example, gains from trade increase when liquidated damages are used to screen among buyers if both types of buyers purchase the item, even though the low-value buyer gets an inefficient level of protection. As will become clear, the precise design of the optimal contract depends upon the relationship between higher damages and the probability of seller breach. The general case of a buyer with a continuum of types, as well as a seller with a continuum of outside opportunities, is examined in this article.

In Section 4, I examine the policy question of whether a perfectly informed court would generally improve matters by requiring that all stipulated damages be exactly compensating. I find that under plausible conditions, even a perfectly informed court can be a menace to the parties’ contract and to social welfare if it naively imposes a requirement that liquidated damages accurately reflect the actual loss from breach. There is a direct benefit and an indirect cost from judicial intervention. Eliminating the agents’ abilities to set liquidated damages below valuation reduces inefficient breach of contract: the seller will breach only if it is efficient to do so. Unfortunately, such a restriction on liquidated damages also restricts the offerer to pooling contracts, which set a single price. As the example above suggests, such a restriction

8. Placing all of the bargaining power in the hands of one party manifests itself as the opportunity of the party to write a contract and make a take-it-or-leave-it offer to the other.
may lead to buyer-designed contracts, which induce trade with only low-opportunity sellers, and seller-designed contracts, which induce trade with only high-value buyers. Consequently, some individuals may be foreclosed from trade, leaving unrealized gains from exchange. These results are analogous to the social planner’s decision of whether to allow second-degree price discrimination in the context of monopoly pricing. I summarize and conclude in Section 5.

2. The Contractual Framework

We examine the contractual relationship between a buyer and a seller, where third-party offers for the seller’s services may induce breach after an agreement has been reached. The buyer and seller recognize this possibility and bargain both over price and over a damage stipulation, which the seller agrees to pay the buyer in event of nonperformance.9

The buyer (she) and a seller (he) contract to trade a single good at date 1. At the time of contracting, the parties are in a bilateral bargaining situation. After contracting, the buyer cannot find other sellers (e.g., the buyer makes relation-specific investments or her outside opportunities disappear), but the seller’s opportunism is constrained only by the nonperformance damage terms of the contract. At date 2 a third-party offer is made to the seller for his wares. The seller can either accept the third party’s offer and pay the buyer the stipulated damages, or deliver the product to the buyer as promised.

The buyer has a positive valuation, v, distributed on some interval [v, V] according to the density function \( f(v) > 0 \), and cumulative distribution, \( F(v) \). Only the buyer knows v, although its distribution is common knowledge. The third party’s offer for the seller’s product is equal to \( \theta + \varepsilon \), where \( \theta \) is privately known by the seller at date 1, and \( \varepsilon \) is an unknown outside valuation shock at date 2. \( \theta \) is distributed on the interval \( [\theta, \dot{\theta}] \) according to the density function, \( g(\theta) > 0 \), and cumulative distribution, \( G(\theta) \). Only the seller knows \( \theta \), although its distribution is also common knowledge. \( \varepsilon \) is distributed on \( [\varepsilon, \bar{\varepsilon}] \) according to the density function, \( h(\varepsilon) > 0 \), and cumulative distribution, \( H(\varepsilon) \). In order to avoid corner solution complications, we also assume that the support of \( \varepsilon \) is sufficiently wide and \( v \) is sufficiently large.10 Neither party observes \( \varepsilon \) at date 1, and only the seller observes \( \varepsilon \) at date 2. The expected value of \( \varepsilon \) is zero, thereby making \( \theta \) an unbiased estimate at date 1 of the alternative market at date 2. Additionally, it is common knowledge that the

9. The character of the results remains unchanged if one considers instead that the buyer may breach after finding an alternative product. In this alternative, the parties negotiate damages that the buyer will pay the seller in case of a lost transaction.

10. It is sufficient, for example, that for every \( \theta \), \( v \) and for any \( \alpha \in [0,1] \),

\[
v - \theta - \left( \alpha \frac{G}{\varepsilon} + (1 - \alpha) \frac{1 - F}{f} \right) \in [\varepsilon, \bar{\varepsilon}],
\]

together with the requirement that \( v \geq \max\{g(\theta)^{-1}, f(\psi)^{-1}\} \).
seller's costs are to be zero, although the assumption of zero costs is only for simplicity. Absent any contract offer, the seller expects to make

$$E[\theta + \varepsilon|\theta + \varepsilon \geq 0] \cdot \text{Prob}[\theta + \varepsilon \geq 0],$$

which will equal $\theta$ if $\theta + \varepsilon \geq 0$. More generally, the option value of the third-party offer is possibly less than $\theta$, as the seller will not sell when the offered price is below cost (i.e., given our normalization, if price is negative). The buyer's outside opportunities have been normalized to zero. Finally, we make the standard distributional assumptions that $[1 - F(\nu)]/f(\nu)$ is nonincreasing in $\nu$ and $G(\theta)/g(\theta)$ is nondecreasing in $\theta$.

A contract consists of a price, $p$, paid at the time of signing, and a stipulated damage payment of $l$ to be paid at date 2 in the event of the supplier's breach. Thus, a contract is a pair given by $\{p,l\}$. Payoffs are not discounted. For now we assume that only the contract and the existence of breach is observable by the court. Later, in Section 4, we relax this assumption to determine if a perfectly informed, but myopic, court could improve contracting among the parties.

Given $l$, the supplier will breach whenever $\theta + \varepsilon > l$, and perform otherwise. Thus, the probability of performance is $H(l - \theta)$ and the probability of breach is $1 - H(l - \theta)$.

It turns out that the presence of a future outside offer can best be thought of as an option on future trade. In general, if an option paying out $\varepsilon$ costs $x$ to exercise, the value of the option is given by

$$\Lambda(x) = E[\varepsilon|\varepsilon \geq x] \cdot \text{Prob}[\varepsilon \geq x] = \int_x^\varepsilon (\varepsilon - x) dH(\varepsilon) \geq 0.$$

Thus, the supplier's expected value from an outside offer given $\theta$ and $l$ is simply $\Lambda(l - \theta)$, and the expected profit of the supplier from a contract, $\{p,l\}$, is

$$\pi^e = p + \Lambda(l - \theta).$$

A supplier who refuses to contract with the buyer can be assured of an expected profit of $\Lambda(-\theta)$, and so this forms a natural reservation value for the supplier's contract-acceptance decision.

The profit of the buyer with value $\nu$ facing a seller with an expected future offer of $\theta$ is simply

$$\pi^b = \nu H(l - \theta) + l[1 - H(l - \theta)] - p.$$

11. The actual requirement is that $\theta + \varepsilon > \max(l,0)$, but given our assumptions on the size of $\nu$, the optimal contract will always have $l \geq 0$, so the more general condition is unnecessary.
We consider three different contracting scenarios to provide a diverse range of environments for analysis. First, the buyer may propose the contract to the seller, and the seller may accept or reject it. Second, the seller may propose the contract, and the buyer may accept or reject it. Finally, an uninformed broker may design a contract that maximizes the joint surplus from trade between the parties. Interestingly, these three different contractual environments share common characteristics in \( l \).

Before considering each case in the following sections, we consider the full-information benchmark solutions for comparisons: In all three cases, the optimal full-information contract involves setting \( l = v \). To see this, note that the gross social surplus for the contract \( \{ p, l \} \) is given by

\[
S(v,l,\theta) = vH(l - \theta) + l[1 - H(l - \theta)] + A(l - \theta).
\]

The first two terms are the expected gains from trade ignoring third parties, while the third term represents the option value of the outside opportunity that is available to the seller whenever \( \theta + \varepsilon > l \). Note that when \( l \geq 0 \), the above expression is necessarily positive.

If the buyer has all of the bargaining power, the buyer’s optimal strategy is to maximize her profits subject to the seller’s acceptance of the conditions [i.e., \( \pi^b \geq A(-\theta) \)]. Substituting for \( p \) and simplifying yield the following program for \( l \):

\[
\max_{l} \; vH(l - \theta) + l[1 - H(l - \theta)] + A(l - \theta) - A(-\theta),
\]

which has \( l = v \) as the unique optimum. The corresponding contract price offered by the buyer is \( p = A(-\theta) - A(l - \theta) \), which just satisfies the supplier’s reservation price. Similarly, if the seller has all of the bargaining power, the seller will maximize profits subject to the buyer’s acceptance of terms (i.e., \( \pi^b \geq 0 \)). Substituting for \( p \) and simplifying yields

\[
\max_{l} \; vH(l - \theta) + l[1 - H(l - \theta)] + A(l - \theta),
\]

which is identical to the buyer’s program above except for a constant, and so we again find \( l = v \). Note, however, that the price-paid by the buyer to the seller under this scheme is \( p = v \), which extracts all of the buyer’s rent. Finally, if a broker proposes a contract to the parties, the broker will maximize the expected gains from trade by choosing \( l \) to maximize the collective surplus \( S(v,l,\theta) \). Again the solution is to set \( l = v \). The broker then chooses a price to allocate the gains from trade with \( p \) lying in the interval \( [A(-\theta) - A(v - \theta), v] \). It is not surprising that the optimal full-information contract specifies \( l = v \) for each contracting environment, since this condition guarantees that breach occurs if and only if it is efficient.

When information is not public, the resulting contract typically has \( l \neq v \). Instead, \( l \) will depend upon \( v \) and \( \theta \) in a manner that will elicit a party’s private information by creating distortions from efficient breach. The precise rela-
tion between l, v, and \( \theta \) will depend on the contractual context: buyer power, seller power, or brokered contracts.

3. Optimal Contracts in Various Environments

3.1 The Buyer's Optimal Contract

Because the buyer does not know the seller's expected outside opportunity \( \theta \), she must take into account the effect of the liquidated damage clause on the seller's gains from trade. If \( l \) is set arbitrarily high, it will effectively lock the seller out of the alternative market; a low \( l \) preserves the option value of breach, which in turn is an increasing function of \( \theta \). A higher liquidated damage term is marginally more costly to a seller with a high outside opportunity than for a seller of a lower type. A high-opportunity seller would be more likely to accept an offer by the buyer with a combination of a low price and limited breach protection than would a low-opportunity seller, because the low damage term has a higher cash value to the high-opportunity seller.

Recognizing this relationship, the buyer can effectively use the damage clause to screen among different types of sellers in much the same way that a price-discriminating monopolist screens among different consumers by offering multiple quality-price packages. The buyer will offer a menu of contracts, from which the seller chooses the one most profitable given his private information, \( \theta \). That is, the buyer may offer a continuum of contracts to the seller represented by a function \( p(l) \). Following the Revelation Principle, we reparameterize according to the seller's outside opportunity, \( \theta \), as \( \{p(v,\theta), l(v,\theta)\} \). The arguments of these functions also include \( v \) in order to capture the dependence of the optimal contract on the buyer's valuation. Accordingly, we solve for a type-v buyer's optimal contract, \( \{p, l\} \), for every \( \theta \), subject to each seller type finding it optimal to choose the contract designed for his type.

The buyer's expected profit from any contract \( \{p(v,\theta), l(v,\theta)\} \) is

\[
\pi^b(v) = E_\theta[vH(l(v,\theta) - \theta) + l(v,\theta)[1 - H(l(v,\theta) - \theta)] - p(v,\theta)].
\]

She maximizes this subject to two sets of constraints: (i) the seller must be willing to sign the contract (i.e., not make a loss from trade), and (ii) the seller must select the contract designed for his type.

Define \( \pi^s(\tilde{\theta}|v,\theta) \) as the profit to a seller with outside opportunity \( \theta \), who selects the contract designed for a seller of type \( \tilde{\theta} \). That is,

\[
\pi^s(\tilde{\theta}|v,\theta) \overset{\text{def}}{=} p(v,\tilde{\theta}) + A(l(v,\tilde{\theta}) - \theta).
\]

The buyer's first constraint requires that every seller's truthful selection must

---

12. The choice of contract by the buyer may possibly reveal information about the buyer's type, \( v \), to the seller. There is no problem with mechanism design by an informed principal in this case, however, as the seller's utility is independent of \( v \) (except insofar as \( v \) affects the contract offered by the buyer).
be profitable relative to refusing to deal and taking the outside option of $A(-\theta)$. Thus,

$$\pi^*(v, \theta) = \pi^*(\theta|v, \theta) \geq A(-\theta),$$

for all $\theta$, which represents the individual rationality (IR) or participation constraint. Second, every seller must select the correct contract from the menu. These incentive compatibility (IC) constraints require

$$\pi^*(\theta|v, \theta) \geq \pi^*(\tilde{\theta}|v, \theta), \forall \theta, \tilde{\theta}.$$  

These IR and IC constraints are intractable in their present form, so we replace them with the significantly simpler representation in Lemma 1. Unfortunately, the standard procedure used to replace the global constraints with a local representation must be augmented to take into account the complication arising from the dependence upon type of the individual rationality constraint. To this end, we have a phenomena induced by an outside option, which is similar to that found by Lewis and Sappington's (1989a, 1989b) work on the effects of “countervailing incentives.” Technically speaking, the presence of the outside option implies that the individual rationality constraint will bind for the highest type seller. All results are proved in the Appendix.

**Lemma 1.** The mechanism $\{p(v, \theta), l(v, \theta)\}$ satisfies the IR and IC constraints only if

$$\pi^*(v, \theta) = \pi^*(v, \tilde{\theta}) - \int_0^{\tilde{\theta}} [1 - H(l(v, t) - t)]dt, \quad (3)$$

$$\pi^*(v, \tilde{\theta}) \geq A(-\tilde{\theta}), \quad (4)$$

$l(v, \theta)$ is nonincreasing in $\theta$. \quad (5)

Additionally, if $l \geq 0$, (3)–(5) are sufficient for satisfying the IR and IC constraints.

Intuitively, Lemma 1 follows from the envelope theorem: Assuming truthful selection is optimal for the seller, differentiating $\pi^*(v, \theta)$ results in $\pi^*_{\theta}(v, \theta) = 1 - H(l(v, \theta) - \theta)$. Integrating this partial derivative produces (3) and (4). The condition in (5) is a second-order condition for truthful selection.

With this simplification, we proceed by substituting (3) into the buyer's expected profits function. Equating (3) with $\pi^*(\theta|v, \theta)$ and solving for $p(v, \theta)$ yield

$$p(v, \theta) = \pi^*(v, \theta) - A(l(v, \theta) - \theta). \quad (6)$$
Taking the expectation of \( p(v, \theta) \) over \( \theta \) and integrating by parts result in

\[
E_\theta[p(v, \theta)] = E_\theta \left[ \pi^*(v, \bar{\theta}) - [1 - H(l(v, \theta) - \theta)] \frac{G(\theta)}{g(\bar{\theta})} - A(l(v, \theta) - \theta) \right].
\]

Finally, substituting this expression into the buyer's objective function yields the unconstrained problem

\[
\max_l E_\theta \left[ S(v, l, \theta) + [1 - H(l(v, \theta) - \theta)] \frac{G(\theta)}{g(\bar{\theta})} - \pi^*(v, \bar{\theta}) \right],
\]

which may be solved by maximizing \( l \) pointwise over \( \theta \) and checking that the solution satisfies (5). This program yields the following proposition.

**Proposition 1.** The optimal menu of contracts, \( \{p(v, \theta), l(v, \theta)\} \), for the buyer consists of a contract with

\[
l(v, \theta) = v - \frac{G(\theta)}{g(\bar{\theta})},
\]

and \( p(v, \theta) \) jointly determined by (3), (6), and \( \pi^*(v, \bar{\theta}) = A(-\bar{\theta}) \).

As Proposition 1 indicates, the actual buyer loss from breach, \( v \), almost always exceeds the amount of stipulated damages in the optimal contract when the buyer has all of the bargaining power. Specifically, the liquidated damage term is set equal to the valuation of the buyer, less an amount that represents a distortion introduced to reduce the seller's rents from his private information. This additional rent term \( G(\theta)/g(\bar{\theta}) \) increases in \( \theta \). Note that the lowest type seller \( \theta \) chooses damages \( l = v \); all sellers with types above \( \theta \) will accept a lower contract price \( p \) in exchange for undercompensatory liquidated damage terms (which they prefer).

### 3.2 The Seller's Optimal Contract

Because the seller does not know the buyer's valuation of the good, he must take into account the effect of the liquidated damage clause on protecting the buyer's value. If \( l \) is set arbitrarily low, it will allow the seller to breach and use the alternative market whenever \( \varepsilon \) is favorable, thereby imposing a loss of \( v \) on the buyer. For buyers with high \( v \)'s, this will produce a lower reserve price because of the lower likelihood that the value of the exchange will materialize. Recognizing this, the seller can effectively use the stipulated damage clause to select among the different types of buyers just as the buyer was previously shown to select among sellers.

In this section, for tractability we also assume that \( \theta \) is observed by the
buyer. The seller may offer a menu of contracts like that in the previous section and allow the buyer to choose the one most profitable given her \( v \). In this case, the menu can be represented by either the function \( p(l) \) or the parametric contract \( \{p(v, \theta), l(v, \theta)\} \).

The seller's expected profit from a contract is

\[
\pi^s(\theta) = E_v[p(v, \theta) + \Lambda(l(v, \theta) - \theta)],
\]

which he maximizes subject to the buyer's IR and IC constraints, just as in the buyer's problem above.

Define \( \pi^b(v|\theta) \) by

\[
\pi^b(v|\theta) \overset{\text{def}}{=} \nu H(l(\bar{v}, \theta) - \theta) + l(\bar{v}, \theta)[1 - H(l(\bar{v}, \theta) - \theta)] - p(\bar{v}, \theta),
\]

which represents the profit to a buyer with value \( v \) who selects the contract designed for a buyer of type \( \bar{v} \). Analogously to the buyer-designed contract case, the IR and IC constraints are, respectively,

\[
\pi^b(v, \theta) \overset{\text{def}}{=} \pi^b(v|\theta) \geq 0,
\]

\[
\pi^b(v|\theta) \geq \pi^b(\bar{v}|\theta),
\]

for all \( v \) and \( \bar{v} \). Again, as in the buyer-designed contract, the above two sets of constraints are difficult to work with but can be greatly simplified, as in Lemma 2.

**Lemma 2.** The mechanism \( \{p(v, \theta), l(v, \theta)\} \) satisfies the buyer's IR and IC constraints if and only if

\[
\pi^b(v, \theta) = \pi^b(v|\theta) + \int_{\bar{v}}^{v} H(l(t, \theta) - \theta) dt,
\]

\[
\pi^b(v, \theta) \geq 0,
\]

\( l(v, \theta) \) nondecreasing in \( v \).

The proof again is an application of the envelope theorem. With this sim-
plification, we proceed by substituting (9) into the seller's objective (8). Equating (9) with \( \pi^b(v|\nu, \theta) \) and solving for \( p(v, \theta) \) yield
\[
p(v, \theta) = vH(l(v, \theta) - \theta) + l(v, \theta)[1 - H(l(v, \theta) - \theta)]
\]
\[\hspace{1cm} - \pi^b(v) - \int_{v}^{\nu} H(l(t, \theta) - \theta) dt.\]
\[\hspace{1cm} \text{ (12)}\]

Taking the expectation of \( p(v, \theta) \) over \( v \), and integrating by parts, produces
\[
E_v[p(v, \theta)] = E_v \left[ vH(l(v, \theta) - \theta) + l(v, \theta)[1 - H(l(v, \theta) - \theta)]
\]
\[\hspace{1cm} - H(l(v, \theta) - \theta) \frac{1 - F(v)}{f(v)} - \pi^b(v) \right].
\]

Substituting this expression into the buyer's objective function yields the unconstrained problem
\[
\max \int E_v \left[ S(v,l(v,\theta),\theta) - H(l(v,\theta) - \theta) \frac{1 - F(v)}{f(v)} - \pi^b(v) \right]. \hspace{1cm} \text{ (13)}
\]

This problem may be solved by maximizing \( l \) pointwise over \( v \), and checking that the resulting solution satisfies (11). The resulting expressions provide Proposition 2.

**Proposition 2.** The optimal menu of contracts, \( \{p(v, \theta), l(v, \theta)\} \), for the seller consists of

\[
l(v, \theta) = v - \frac{1 - F(v)}{f(v)},
\]

and \( p(v, \theta) \) jointly determined by (12) and \( \pi^b(v, \theta) = 0 \).

As Proposition 2 demonstrates, the actual buyer loss from breach, \( v \), almost always exceeds the amount of stipulated damages in the optimal contract when the seller behaves as a monopolist. Specifically, the liquidated damage term is set equal to the valuation of the buyer, less an amount that represents a distortion introduced to reduce the buyer's information rents (or consumer surplus, to use the monopoly analogy) from her private information. This additional rent term \( [1 - F(v)]f(v) \) decreases in \( v \). Note that the highest type buyer \( \tilde{v} \) chooses damages \( l = v \), and all buyers with lower types choose undercompensatory liquidated damage terms in exchange for a lower contract price \( p \).

### 3.3 Broketed Contracts

Rather than place all of the bargaining power in the hands of one agent, we now consider the resulting contract where both agents delegate the contractual
terms to a outside party (e.g., a broker) who knows neither \( \nu \) nor \( \theta \). This broker is concerned with maximizing the total gains from trade when each party knows only its own private information. As a motivation, one might suppose that certain institutions evolve that maximize the joint gains from trade between agents from an ex ante point of view (which is equivalent to an uninformed broker designing a mechanism for exchange). An alternative motivation has the buyer and seller contracting ex ante, before they learn their private information, but subject to a limited-liability constraint where either party can legally walk away from the contract once private information is learned if losses are sufficiently great. This latter explanation appears realistic in the requirements contracting context.

The problem facing the broker is to maximize the joint gains from trade by designing a menu of contracts. The contracts may depend upon both \( \theta \) and \( \nu \). We may parameterize this family of contracts by \((\theta, \nu)\), and envision the contract as a direct revelation mechanism where each party announces his or her private information and the broker selects the appropriate contract accordingly, from the set \( \{p(\nu, \theta), l(\nu, \theta)\} \).

Because there is two-sided asymmetric information, the traditional techniques need to be augmented slightly; we follow Myerson and Satterthwaite in this regard. Lemma 3 is a direct extension of Lemmas 1 and 2 and characterizes the set of all contracts that are incentive compatible and individually rational for both parties.

**Lemma 3.** The mechanism \( \{p(\nu, \theta), l(\nu, \theta)\} \) satisfies IR and IC constraints only if

\[
E_{\nu}[\pi^x(\nu, \theta)] = E_{\nu}[\pi^x(\nu, \theta)] - E_{\nu} \left[ \int_{\theta}^{\hat{\theta}} [1 - H(l(\nu, t) - t)] dt \right],
\]

(14)

\[
E_{\theta}[\pi^b(\nu, \theta)] = E_{\theta}[\pi^b(\nu, \theta)] + E_{\theta} \left[ \int_{\nu}^{\hat{\theta}} H(l(s, \theta) - \theta) ds \right],
\]

(15)

\[
E_{\nu, \theta} \left[ S(\nu, l, \theta) - \frac{1 - F(\nu)}{f(\nu)} H(l - \theta) \right.
\]

\[
+ \frac{G(\theta)}{g(\theta)} [1 - H(l - \theta)] - \Lambda (\hat{\theta}) \right] \geq 0.
\]

(16)

If, in addition,

\( l(\nu, \theta) \) is nonincreasing in \( \theta \) and nondecreasing in \( \nu \),

(17)

and \( l \geq 0 \), then (14)–(16) are sufficient for IR and IC.

---

14. In this section, we return to our assumption that the buyer does not observe \( \theta \).

15. Also see Williams for a fuller treatment and extension of Myerson and Satterthwaite’s model. Williams characterizes the efficient locus of contracts, depending upon the weights attached to the buyer’s and seller’s utilities.
With Lemma 3, we may write the broker’s problem as maximizing the expected profit of each party subject to (14)–(16) above. Let \( \mu \) be the Lagrange multiplier for (16). Bringing the constraint into the integral above and simplifying yield the third party’s objective function:

\[
E_{v,\theta} \left[ S(v,l,\theta) + \frac{\mu}{1+\mu} \left\{ [1 - H(l - \theta)] \frac{G(\theta)}{g(\theta)} 
- H(l - \theta) \frac{1 - F(v)}{f(v)} - A(-\theta) \right\} \right].
\]

(18)

The function \( l(v,\theta) \) that maximizes this integral may be found by maximizing the expression for \( l \) pointwise in \( \theta \) and \( v \), and checking that the solution satisfies (17). The solution results in the following proposition.

**Proposition 3.** The optimal menu of contracts for the brokered buyer–seller relationship consists of

\[
l(v,\theta) = v - \frac{\mu}{1+\mu} \left( \frac{1 - F(v)}{f(v)} + \frac{G(\theta)}{g(\theta)} \right),
\]

where \( \mu \geq 0 \). Additionally, \( \mu > 0 \) whenever

\[
E_{v,\theta} \left[ v + A(v - \theta) - H(v - \theta) \frac{1 - F(v)}{f(v)} 
+ [1 - H(v - \theta)] \frac{G(\theta)}{g(\theta)} - A(-\theta) \right] > 0.
\]

(19)

Proposition 3 provides the solution of the broker’s problem. Before, when one party had all of the bargaining power, that party traded off breach inefficiencies against increased rent extraction. In such a skewed bargaining environment, \( l \) never exceeds \( v \) in the optimal contract. We now find that even when a broker is employed, the optimal contract never involves excessive stipulated damages. The proposition reveals that stipulated damages do not exceed the actual loss from breach of contract and are strictly less than actual loss whenever the expected gains from trade are less than the expected information rents for almost all \( \theta \) and \( v \). As in the monopsony and monopoly contexts above, the liquidated damage term is set equal to the buyer’s valuation less some information rent term. Now, however, this rent is the sum of both \( G(\theta)/g(\theta) \) and \( [1 - F(v)]/f(v) \) rather than just one or the other, but weighted with a coefficient less than unity. It is thus quite analogous to the previous contractual outcomes. Note that when the constraint in (16) binds, \( l < v \) for all cases except \( (v,\theta) = (v,\theta) \). Because the brokered contract yields greater combined gains from trade than either the buyer-designed contract or the seller-designed contract but still utilizes undercompensatory liquidated damage contracts, the previous results appear quite robust.

4. Welfare Implications and Policy Conclusions

We have seen the existence of private information by contracting parties in a wide range of bargaining environments introduces the likelihood that liqui-
dated damages will be below the actual losses caused by breach. Using liquidated damages to select among different types of economic agents is, in a sense, second-degree price discrimination where the monopolist offers price–damage, rather than price–quality, bundles. The question then arises as to whether public policy should require that all damages for breach of contract equal the true losses incurred, providing such information about losses is available to the court after the breach. It is arguable that any intervention based upon ex post information would be precarious at best, especially given our limitations of knowledge about the actual contracting conditions between parties.

To consider the issue of judicial intervention, we posit the strongest possible assumption in favor of activism to determine the most optimistic assessment: assume that courts can perfectly determine actual losses from breach ex post. That is, assume that \( v \) becomes known to the court at date 2 in the event of breach so the court has an informational advantage. For realism, further assume that parties cannot base their contract price on the judicial determination of \( v \); otherwise, the court would become nothing more than an auditing agency for private contracts. That is, we suppose that courts allow the observed \( v \) to be used only in determining \( l \). With these assumptions, we seek to answer the question of whether the court should require \( l = v \) in all breached contracts.\(^{16}\)

There is a benefit and a cost from judicial intervention: Eliminating the agents' abilities to set liquidated damages below valuation reduces inefficient breach of contract, but it may foreclose some buyers and sellers from efficient trade.

If there were only two possible types of buyers and sellers, and assuming the offerer would choose to serve only part of the market if it were not permissible to choose \( l \) different from \( v \), then the posited form of judicial intervention always produces inefficiencies. To see this, note that when the offeree is of a good type (i.e., high-value buyer or low-opportunity seller), liquidated damages are set at actual value. Consequently, for good types there is no inefficiency with or without judicial intervention. When the offerees are of bad type (i.e., low-value buyers or high-opportunity sellers), offerers will set inefficient damage levels when given the option. But while the terms are inefficient, individual rationality implies both parties are better off trading than not trading. Judicial intervention that prevents the use of undercompensatory damages must therefore decrease social welfare.

When a continuum of types of buyers and sellers exists, the analysis is more difficult. For simplicity, assume that when the buyer is able to make a take-it-or-leave-it offer, the seller knows \( v \); hence, in both cases of buyer offers and

\(^{16}\) Alternatively, we could assume that courts require that damages equal the expected value of the loss rather than the actual value. Because either rule restricts parties from using the liquidated damage clause in screening among different types of traders, a similar analysis can be performed on such a rule that involves analogous trade-offs. We examine the actual-loss rule because it seems to make the strongest case for judicial intervention.
seller offers, the asymmetry of information between the parties is over only a single variable. Consider the problem facing the buyer with all of the bargaining power who is constrained to set \( l = v \); ex post, in all offered contracts. If she sets the contract price low, only very low-\( \theta \) sellers will accept the terms, but she will make a larger profit on those contracts where such a sale is made. If she sets the price high, her terms will be accepted by most sellers, but her gains from actual trade will be lower. The problem facing her is much the same as that facing a monopolist setting one price: Higher prices result in fewer sales but greater profits per sale. Her maximization problem is simply

\[
\max_p (v - p)G(\theta_b^*),
\]

subject to the defining condition for \( \theta_b^* \) that \( p + A(v - \theta_b^*) - A(-\theta_b^*) = 0 \); \( \theta_b^* \) is the marginal seller who is indifferent between accepting and rejecting the offered price. Optimization reveals that for an interior solution, the optimal constrained-contract price for a buyer of type \( v \) is implicitly given by

\[
p_b^* = v - \frac{G(\theta_b^*)}{g(\theta_b^*)} [H(v - \theta_b^*) - H(-\theta_b^*)].
\]

We denote the optimal price and the threshold seller type as \( p_b^*(v) \) and \( \theta_b^*(v) \), respectively. Under the policy of requiring \( l = v \), when the buyer has the bargaining power (i.e., behaves as a monopsonist) and is of type \( v \), only seller types lower than \( \theta_b^*(v) \) will sell the good to the buyer, even though the socially efficient cutoff level is in fact higher than \( \theta_b^*(v) \).

With this formulation it is straightforward to measure the benefits of a policy of requiring nonperformance damages to equal actual losses. Specifically, the benefits in the buyer-designed contract regime are

\[
\Delta W^bd = E_v \left\{ \int_{\theta_b^*(v)}^{\theta_b^*(v)} [S(v,v,\theta) - S(v,l,\theta)] dG(\theta) \\
- \int_{\theta_b^*(v)}^{\theta_b^*(v)} [S(v,l,\theta) - A(-\theta)] dG(\theta) \right\}. \tag{20}
\]

The first term in brackets represents the gain from more efficient breach; the second term indicates the expected losses from reduced trade.

Consider now the seller’s problem when the seller has all of the bargaining power. Since the buyer will buy whenever \( v \geq p \) if \( l = v \), the seller’s problem is to choose \( p \) to solve

\[
\max_p \int_p^\phi [p + A(v - \theta)] dF(v) + F(p)A(-\theta).
\]
Maximization reveals that the seller's optimal constrained-contract price $p_s^\ast(\theta)$ satisfies the implicit relationship

$$p_s^\ast + A(p_s^\ast - \theta) - A(-\theta) = \frac{1 - F(p_s^\ast)}{f(p_s^\ast)}.$$  

Under a policy of $l = \nu$, when the seller has the bargaining power (i.e., behaves as a monopolist) and is of type $\theta$, only buyers with valuations above $p_s^\ast(\theta)$ will buy the good, even though the socially efficient cutoff level is in fact lower.

Under policies of judicially mandated actual-damage policy, the social welfare changes from seller-designed contracts are given by

$$\Delta W^{\text{sd}} = E_\theta \left\{ \int_{p_s^\ast(\theta)}^\delta [S(\nu, l, \theta) - S(\nu, l, 0)]dF(\nu) - \int_{\nu}^{p_s^\ast(\theta)} [S(\nu, l, \theta) - A(-\theta)]dF(\nu) \right\}. \quad (21)$$

As before with buyer-designed contracts, there are a cost and a benefit from judicial intervention. The first term represents the expected efficiency gain from more efficient breach of contract; the judicial restriction has the effect of reducing the occurrence of breach to more efficient levels. The second term represents the expected loss from inefficient trade due to the inability of sellers to effectively screen buyers by type.

We can transform our expressions for both $\Delta W^{\text{bd}}$ and $\Delta W^{\text{sd}}$ into an alternative form by substituting for the corresponding optimal $l$ contracts that will be offered by the party with bargaining power. We state the results in the following proposition.

**Proposition 4.** The benefits under a policy of judicial intervention that requires perfect compensation in the event of breach for the buyer-power and seller-power contracting scenarios are given by

$$\Delta W^{\text{bd}} = E_{\nu, \theta} \left[ \frac{G(\theta)}{g(\theta)} [1 - H(l - \theta)] + A(\nu - \theta) - A(l - \theta) \right]$$

$$- E_{\nu} \left[ \int_{\theta(\nu)}^{\theta} [\nu + A(\nu - \theta) - A(-\theta)]dG(\theta) \right], \quad (22)$$

$$\Delta W^{\text{sd}} = E_{\nu, \theta} \left[ \frac{1 - F(\nu)}{f(\nu)} [1 - H(l - \theta)] + A(\nu - \theta) - A(l - \theta) \right]$$

$$- E_{\theta} \left[ \int_{\nu}^{p_s^\ast(\theta)} [\nu + A(\nu - \theta) - A(-\theta)]dF(\nu) \right]. \quad (23)$$
The central question is under what general conditions are these expressions either positive (i.e., judicial intervention is good) or negative (i.e., judicial intervention is bad). Unfortunately, there are no clear general conditions. Rather, the sign of each equation depends fundamentally on the distributions of \( \nu \), \( \theta \), and \( \varepsilon \). To illustrate the results of this and the preceding section, consider the following numerical example. Suppose that \( \nu \) and \( \theta \) are uniformly distributed on the interval \([100,200]\) and \( \varepsilon \) is distributed uniformly on \([-200,200]\). By Propositions 1 and 2, the optimal damage terms for both the buyer-designed and seller-designed contracts are \( l_b = \nu - \theta + 100 \) and \( l_s = 2\nu - 200 \), respectively. Numerically, \( \Delta W^{bd} = -89.9 \) and \( \Delta W^{sd} = -14.0 \) in this simple example, and so judicial intervention will decrease welfare under either bargaining regime.

More generally, as in the two-type example mentioned above, when the distributions are such that there is a sufficiently large mass of high-valuation buyers (low opportunity-cost sellers) in the distribution function, the net benefits of judicial intervention in the seller-designed (buyer-designed) contracting scenario are negative. Furthermore, it should be noted that these results depend upon the optimistic assumption that the courts know perfectly the value of loss. If courts make errors, the above loss in welfare may be even greater.17

5. Extensions and Conclusions

The modeling approach taken in this article was to assume that the seller may breach with some probability and that the buyer's valuation needed protection from such behavior. Alternatively, I could have chosen a framework where the buyer breaches with some probability and the seller's sunk production costs need to be protected. In this case, I would obtain similar results: Liquidated damages would never exceed the seller's production costs and would frequently fail to protect the seller's investment fully. In this sense my explanation regarding the asymmetric treatment of liquidated damage clauses by the courts is robust.

I have also assumed that at the initial bargaining date, only one buyer and one seller exist. In an alternative case where, for example, many buyers exist and the unique seller of the good has all of the bargaining power, the seller will find it profitable to conduct an auction at the contract-signing stage. As in Laffont and Tirole's related analysis of a government auction of natural monopoly incentive contracts, the contract that is auctioned off will still display the same types of distortions as in the case of one buyer; the difference is that the buyers' rents are reduced by the presence of competition. In the limit as the market converges to perfect competition, the liquidated damage clause will converge to the actual value. A similar analysis applies to the case of many sellers and a single buyer making a take-it-or-leave-it offer.

Liquidated damage clauses serve as a screening device for contracting parties that can be used in the construction of optimal contracts for monopoly

17. The court's determination of value does not enter the expressions linearly, and so even an unbiased estimate by the court introduces additional nonlinear effects.
sellers, monopsony buyers, or a surplus-maximizing broker. It is also plausible that other screening devices exist, such as quality, which may interact with the marginal valuation of the buyer or the marginal cost (outside opportunity cost, etc.) of the seller, and the analysis would appear similar. Even so, it is doubtful that quality screening dominates damage-based screening to the extent that liquidated damages would not be used in this capacity; most likely, some mixture of screening instruments would be employed in any optimal contract. Additionally, in situations where breach commonly occurs, the use of liquidated damages terms to screen seems compelling and economically significant.

In this article, I have demonstrated that when information of contracting parties is private, liquidated damage clauses serve a dual role of promoting efficient breach and increasing the likelihood of trade. Furthermore, even if the judicial system had perfect information, intervention in the form of prohibiting undercompensatory damages does not necessarily improve social welfare. This may explain why courts have not found it necessary to invalidate undercompensatory damage clauses but have continued to strike overcompensatory clauses. The former may be the result of a belief that bargaining parties made rational choices, while the latter may be best explained as a belief that excessive damage clauses are symptomatic of contractual failure.

Appendix

Proof of Lemma 1. Necessity of (3)–(5): Incentive compatibility and the definition of \( \pi^*(\theta|\theta) \) imply

\[
\pi^*(\theta|v,\theta) \geq \pi^*(\theta|v,\theta) = \pi^*(\theta|v,\theta) + A(l(v,\theta) - \theta) - A(l(v,\theta) - \bar{\theta}).
\]

Integrating by parts and simplifying yield

\[
\pi^*(\theta|v,\theta) - \pi^*(\theta|v,\bar{\theta}) \geq (\theta - \bar{\theta}) - \int_{l(v,\bar{\theta})}^{l(v,\theta) - \theta} H(e)de.
\]

Interchanging \( \theta \) and \( \bar{\theta} \) similarly provides

\[
\pi^*(\bar{\theta}|v,\bar{\theta}) - \pi^*(\theta|v,\bar{\theta}) \geq (\bar{\theta} - \theta) - \int_{l(v,\theta)}^{l(v,\bar{\theta}) - \bar{\theta}} H(e)de.
\]

And so combining the inequalities results in

\[
(\theta - \bar{\theta}) - \int_{l(v,\theta)}^{l(v,\theta) - \theta} H(e)de \geq \pi^*(v,\theta) - \pi^*(v,\bar{\theta})
\]

\[
\geq (\theta - \bar{\theta}) - \int_{l(v,\bar{\theta}) - \theta}^{l(v,\theta) - \theta} H(e)de,
\]
which implies (5). Without loss of generality, take \( \theta > \bar{\theta} \), divide through this inequality, taking limits (using l'Hôpital's rule) as \( \theta \to \bar{\theta} \) to obtain

\[
\frac{\partial \pi^v(v, \theta)}{\partial \theta} = 1 - H(l(v, \theta) - \theta).
\]

\( \pi^v(v, \theta) \) is monotonic in \( \theta \), and therefore Riemann integrable; integrating, we obtain (3). Finally, IR implies (4) directly.

Sufficiency of (3)–(5) when \( l \geq 0 \): (5) implies that \( l(v, \theta) \) is Riemann integrable and almost everywhere differentiable, and hence there exist \( p(v, \theta) \) and \( l(v, \theta) \) that satisfy (3), which requires that

\[
p_\theta(v, \theta) + A'(l(v, \theta) - \theta)l_\theta(v, \theta) = 0
\]

for all \( \theta \). Suppose that (3)–(5) were true but the contract was not IC. Then there exist some \( \theta \) and \( \tilde{\theta} \) such that

\[
p(v, \tilde{\theta}) + A(l(v, \tilde{\theta}) - \theta) > p(v, \theta) + A(l(v, \theta) - \theta).
\]

Integrating,

\[
\int_\theta^{\tilde{\theta}} [p_\theta(v, s) + A'(l(v, s) - \theta)]l_\theta(v, s)] ds > 0.
\]

Using (3) again,

\[
\int_\theta^{\tilde{\theta}} \{[p_\theta(v, s) + A'(l(v, s) - \theta)]l_\theta(v, s)]
\]

\[
- [p_\theta(v, s) + A'(l(v, s) - s)]l_\theta(v, s)] ds > 0.
\]

Integrating again,

\[
\int_\theta^{\tilde{\theta}} \int_s^\theta - A''(l(v, s) - t)]l_\theta(v, s)dt ds > 0.
\]

Since \( A''(x) = h(x) > 0 \), (5) implies the integrand is nonnegative, which yields a contradiction.

Now consider the IR constraint:

\[
\Delta(v, \theta) = \pi^v(v, \theta) - A(-\theta) \geq 0, \quad \forall \theta.
\]

Differentiating, we find the net utility function \( \Delta \) is decreasing since

\[
\Delta_\theta(v, \theta) = \frac{\partial \pi^v(v, \theta)}{\partial \theta} + A'(-\theta) = H(-\theta) - H(l - \theta) \leq 0,
\]

given our hypothesis that \( l \geq 0 \). Hence, if the IR constraint binds, it binds at \( \theta \), and thus (4) guarantees that the IR constraint is satisfied globally. \( \square \)
Proof of Proposition 1. Suppose that the buyer wishes to sell to an interval \([\theta^-, \theta^+]\). Pointwise maximization of

\[
\max_l \int_{\theta^-}^{\theta^+} \left\{ S(v, l(v, \theta), \theta) + [1 - H(l(v, \theta) - \theta)] \frac{g(\theta)}{G(\theta)} \pi^\varepsilon(\theta^+) \right\} dG(\theta),
\]

ignoring monotonicity and nonnegativity constraints on \(l\), yields the given expression for \(l(\theta)\), which in turn satisfies the monotonicity and nonnegativity restrictions given our assumptions on the distribution of \(\theta\). Lemma 1 implies that (3), (4), and (6) uniquely determine \(\rho(\theta)\) such that IC and IR are satisfied.

We now need to verify that it is optimal for the buyer to choose \([\theta^-, \theta^+] = [\bar{\theta}, \bar{\theta}]\). It is sufficient to show that for any \(\theta\) the derivatives of (A1) with respect to \(\theta^-\) and \(\theta^+\) are negative and positive, respectively. The \(\theta^-\) derivative is

\[
- g(\theta) \left\{ S(v, l, \theta) + [1 - H(l(v, \theta) - \theta)] \frac{G(\theta)}{g(\theta)} \pi^\varepsilon(\theta^+, v) \right\} < 0,
\]

and the \(\theta^+\) derivative condition is

\[
g(\theta) \left\{ S(v, l, \theta) + [1 - H(l(v, \theta) - \theta)] \frac{G(\theta)}{g(\theta)} - \pi^\varepsilon(\theta^+, v) \right\} - G(\theta)[1 - H(l(v, \theta) - \theta)] > 0.
\]

Both inequalities hold if \(S(v, l, \theta) \geq \pi^\varepsilon(\theta^+, v)\). Because \(\pi^\varepsilon\) is increasing in \(\theta\) and \(S\) is decreasing in \(\theta\), it is sufficient to show that \(S(v, l, \bar{\theta}) \geq \pi^\varepsilon(\bar{\theta}, v) = A(\bar{\theta})\). Using the fact that \(l = v - G(\theta)/g(\theta)\) and simplifying yield

\[
S(v, l, \bar{\theta}) - A(\bar{\theta}) = \nu H(l - \bar{\theta}) - \int_0^l \varepsilon dH(\varepsilon) = 0.
\]

Thus all types will be served. \(\square\)

Proof of Lemma 2. Necessity of (9)–(11): Incentive compatibility and the definition of \(\pi^b(v|v, \theta)\) imply

\[
\pi^b(v|v, \theta) \geq \pi^b(\bar{v}|v, \theta) = \pi^b(\bar{v}|\bar{v}, \theta) + (v - \bar{v})H(l(\bar{v}, \theta) - \theta).
\]

Interchanging \(v\) and \(\bar{v}\) similarly provides

\[
\pi^b(\bar{v} | \bar{v}, \theta) \geq \pi^b(v | \bar{v}, \theta) = \pi^b(v | v, \theta) + (\bar{v} - v)H(l(v, \theta) - \theta).
\]

Combining the inequalities results in

\[
(v - \bar{v})H(l(v, \theta) - \theta) \geq \pi^b(v, \theta) - \pi^b(\bar{v}, \theta) \geq (v - \bar{v})H(l(\bar{v}, \theta) - \theta),
\]
which implies (11). Without loss of generality, take \( v > \bar{v} \), divide through the above inequality, taking limits as \( v \to \bar{v} \) to obtain

\[
\frac{\partial \pi^b(v, \theta)}{\partial v} = H(l(v, \theta) - \theta).
\]

\( \pi^b(v, \theta) \) is monotonic, and therefore Riemann integrable. Integrating, we obtain (9). Finally, IR implies (10) directly.

Sufficiency of (9)-(11): (11) implies that \( l(v, \theta) \) is Riemann integrable and almost everywhere differentiable on \( v \), and hence \( p(v, \theta) \) and \( l(v, \theta) \) exist that satisfy (9), which requires that

\[
\{(v - l)h(l - \theta) + [1 - H(l - \theta)]l_\iota(v, \theta) - p_\iota(v, \theta) = 0,
\]

for all \( v \). Suppose that \( (9)-(11) \) were true but the contrast was not IC. Then there exist some \( v \) and \( \bar{v} \) such that

\[
vH(l(\bar{v}, \theta) - \theta) + l(\bar{v}, \theta)[1 - H(l(\bar{v}, \theta) - \theta)] - p(\bar{v}, \theta)
\]

\[
> vH(l(v, \theta) - \theta) + l(v, \theta)[1 - H(l(v, \theta) - \theta)] - p(v, \theta).
\]

Integrating,

\[
\int_\nu^\bar{v} \{(v - l(s, \theta))h(l(s, \theta) - \theta) + [1 - H(l(s, \theta) - \theta)]l_\iota(s, \theta)
\]

\[
- p_\iota(s, \theta)\}ds > 0.
\]

Using (9) again,

\[
\int_\nu^\bar{v} (v - s)h(l(s, \theta) - \theta)l_\iota(s, \theta)ds > 0.
\]

Integrating again,

\[
\int_\nu^\bar{v} \int_s^v h(l(s, \theta) - \theta)l_\iota(s, \theta)dt \, ds > 0.
\]

Because \( l \) is nondecreasing in \( v \), the integrand is nonnegative, which yields a contradiction.

Finally, since \( \pi^b \) is increasing in \( v \), it is sufficient for the IR constraint that (10) holds.

\[ \square \]

Proof of Proposition 2. Suppose that the seller decides to serve the interval of buyers in \( [v^-, v^+] \). Pointwise maximization of
\[ \max_{l} \int_{v^{-}}^{v^{+}} \left\{ S(v,l(v,\theta),\theta) - H(l(v,\theta) - \theta) \frac{1 - F(v)}{f(v)} - \pi b(v^{-}) - A(-\theta) \right\} dF(v), \]  

(A2)

ignoring the monotonicity constraint on \( l \), yields the given expression for \( l(v,\theta) \), which in turn satisfies the monotonicity restriction given our assumptions on the distribution of \( v \). Lemma 2 implies that (9), (10), and (12) uniquely determine \( p(v,\theta) \) such that IC and IR are satisfied.

Last, we check that the optimal interval is \([v,v]\). It is sufficient that the derivatives of (A2) with respect to \( v^{-} \) and \( v^{+} \) are negative and positive, respectively, evaluated at any \( v \). Noting that \( \pi b(v^{-}) = 0 \), the sufficient conditions on the derivatives are

\[-f(v) \left\{ S(v,l(v,\theta),\theta) - H(l(v,\theta) - \theta) \frac{1 - F(v)}{f(v)} - A(-\theta) \right\} < 0,\]

\[ f(v) \left\{ S(v,l(v,\theta),\theta) - H(l(v,\theta) - \theta) \frac{1 - F(v)}{f(v)} - A(-\theta) \right\} > 0,\]

and so it is sufficient that

\[ S(v,l(v,\theta),\theta) - H(l(v,\theta) - \theta) \frac{1 - F(v)}{f(v)} > A(-\theta).\]

Simplifying the relationship, given that \( l = v - [1 - F(v)]/f(v) \), yields

\[ l + A(l - \theta) - A(-\theta) = lH(l - \theta) - \int_{0}^{l} \epsilon \ dH(\epsilon) > 0,\]

which is satisfied. Thus, all buyers are served. \( \square \)

\textit{Proof of Lemma 3 (sketch).} The necessity of (14) and (15) follows along similar arguments as in the proofs to Lemmas 1 and 2. By construction,

\[ E_{v,\theta}[S(v,l(v,\theta),\theta)] = E_{v,\theta} \left[ \pi s(v,\theta) + \pi b(v,\theta) - \int_{0}^{\delta} [1 - H(l(v,t) - t)] dt \right. \]

\[ + \left. \int_{v}^{\delta} H(l(s,\theta) - \theta) ds \right], \]

or, rearranging,
where the inequality is implied by the IR constraint. Integrating by parts yields (16).

To see that (14)–(17) and $l \geq 0$ are sufficient for incentive compatibility and individual rationality, we construct $p(v, \theta)$ such that $l(v, \theta)$ is IC and IR. From (14), $p$ must satisfy the partial differential equations

$$E_v[p_\theta(v, \theta)] = E_v[[1 - H(l(v, \theta) - \theta)]l_\theta(v, \theta)], \quad \text{(A3)}$$

$$E_\theta[p_v(v, \theta)] = E_\theta[((v - l(v, \theta))h(l(v, \theta) - \theta) + [1 - H(l(v, \theta) - \theta))]l_v(v, \theta)]. \quad \text{(A4)}$$

If the constructed price function satisfies these two partial differential equations, we know from similar arguments to those made in Lemmas 1 and 2 that the monotonicity conditions expressed in (17) together with $l \geq 0$ are sufficient for incentive compatibility. One possible construction of $p$ has $p(v, \theta)$ such that $E_\theta[\pi^b(v, \theta)] = 0$. That is,

$$E_\theta[\nu H(l(\theta, \nu) - \theta) + l(\theta, \nu)[1 - H(l(\theta, \nu) - \theta)]] = E_\theta[p(\theta, \nu)].$$

Define

$$p(v, \theta) \overset{\text{def}}{=} E_\theta \left[ \int_0^\nu \left\{ [s - l(s, \theta)]h(l(s, \theta) - \theta) + [1 - H(l(s, \theta) - \theta)]l(s, \theta)ds \right\} ight. $$

$$+ E_\theta[H(l(\nu, \theta) - \theta) + [1 - H(l(\nu, \theta) - \theta)]l(\nu, \theta)]$$

$$- E_v \left[ \int_0^\delta [1 - H(l(v, t) - t)]l_\theta(v, t)dt \right]$$

$$+ E_v, \theta \left[ [1 - H(l(v, \theta) - \theta)]l_\theta(\nu, \theta) \frac{G(\theta)}{\theta} \right].$$

The first two expressions represent the expectation over $\theta$ of the integral $p_v(v, \theta)$ and the endpoint $p(\theta, \nu)$. The second pair of expressions are zero in expectation. It is straightforward to check that (14) and (15) are satisfied by this price function. By construction, $E_\theta[\pi^b(\nu, \theta)] = 0$, and so $E_\theta[\pi^b(\nu, \theta)] \geq 0$. Also via this construction, (16) implies that $E_v[\pi^b(v, \hat{\theta})] \geq \Lambda(-\hat{\theta})$. Fol-
lowing the arguments presented in Lemma 1, this is sufficient for participation by the seller.

Proof of Proposition 3. Using Lemma 3 and substituting yield the following expression for expected surplus from a mechanism, where $\mu \geq 0$ is the Lagrange multiplier associated with the constraint in (16):

$$E_{\nu, \theta} \left[ S(\nu, l, \theta) + \frac{\mu}{1 + \mu} \left\{ [1 - H(l - \theta)] \frac{G(\theta)}{g(\theta)} - H(l - \theta) \frac{1 - F(v)}{f(v)} - A(-\theta) \right\} \right].$$

Pointwise maximization with respect to $l$ yields the unique expression for $l(\nu, \theta)$. Since $l$ is nonnegative and satisfies the monotonicity properties in (17), and since $p(v, \theta)$ can be constructed so as to satisfy (14)–(16), the contract is IC and IR. Finally, to prove that $\mu > 0$ under the integral condition (19) above, suppose to the contrary that $\mu = 0$. Then $l(\nu, \theta) = \nu$, and, by our hypothesis, (19) must fail, indicating that $\mu > 0$.

References


