Intra-firm Bargaining under Non-binding Contracts

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We present a new methodology for studying the problem of intra-firm bargaining, based on the notion that contracts cannot commit the firm and its agents to wages and employment. We develop and analyze a general non-cooperative multilateral bargaining framework between the firm and its employees and consider outcomes which are immune to renegotiations by any party. Equilibrium firm profits are characterizable as both a weighted average of a neo-classical (non-bargaining) firm's profits and a generalization of Shapley value for a corresponding cooperative game. Furthermore, the resulting payoffs induce economically significant distortions in the firm's input and organizational-design decisions.

1. INTRODUCTION

The importance of bargaining and hold-up costs in bilateral and multilateral relationships has long been recognized, both in practice and in the economic literature. In particular, the Theory of the Firm literature has identified such issues to be of central importance in determining the scope and nature of the firm.1 Similarly, labour economics has examined the importance of the nature of bargaining between unions and a firm as a determinant of employee compensation and the input choice of firms.2 Despite this attention, however, formalizations of the intra-firm bargaining process between the firm and its individual employees, and the consequent organizational implications, appear to be in a very preliminary stage. Whereas the Theory of the Firm literature has explored employee hold-up problems given employees' inability to commit to future employment (see Jensen and Meckling (1976) and Hart and Moore (1990) in particular), the focus has been primarily on how such issues affect the boundaries and ownership of the firm. Little attention has been given to questions of hiring, technology choice, and organizational design—variables which this literature typically takes as given. And while the labour literature has been interested in collective bargaining issues involving the negotiations between firms and unions, little has been said about negotiations between the firm and its individual employees (e.g., between managers, who typically are excluded from collective bargaining arrangements, and the owner), or between different bargaining units. Nevertheless, it seems likely that such firm and employee interactions may have profound effects on many important aspects of organizational decision making.

In this paper we present a new theory which is derived from a detailed examination of intra-firm bargaining given employee hold-up power, and explore the subsequent implications on the firm’s hiring and technological choice. This paper carefully develops a theoretical framework with which to examine the bargaining process between employees and the firm in the context of non-binding contracts. In particular, we develop a non-cooperative bargaining game and demonstrate that its unique equilibrium has a number of appealing properties. Among these, we show that in the simplest specification to our bargaining game the firm’s payoffs can be characterized either as the simple average of the neoclassical (i.e., non-bargaining) firm’s profits or as the Shapley value of the corresponding cooperative game. Furthermore, we demonstrate that these results elegantly generalize to a continuum of workers, arbitrary distribution of bargaining power and worker heterogeneity, yielding a weighted average of neoclassical profits as the firm’s payoff. The results we obtain characterizing the outcome of the bargaining game, the consequent labour market implications for the firm, and the relationship between the production process and the distortions that this bargaining implies, are rather striking. While our implications differ sharply from those for a neoclassical firm, they have straightforward interpretations and explain a number of instances in which firm behaviour seemingly departs from neoclassical predictions. Although presently we only provide a few simple applications of our theory to issues of labour demand, in a companion paper, Stole and Zwiebel (1996), we explore many further economic applications.

As a starting point for our analysis, we presume that labour contracts are incomplete, with essentially no capability to bind either party to the relationship. Our working definition for the firm is simply a singularly-owned set of productive assets which produces an output that depends upon the allocation of employees to those assets. Dissatisfied workers are generally free to quit at will, and firms are typically able to dismiss part or all of their labour force. In this spirit, we take labour contracts as non-binding in nature. The inability of a firm to bind its employees through long-term contracts implies that employees may be able to bargain ex post over their rents (wages). This, of course, will only be possible insofar as the firm cannot costlessly replace such employees with new equivalent workers. In practice, one might suspect that some employees can be replaced at little or no disruption to production, while other employees—high-level managers, key product engineers, or perhaps blue-collar workers with significant firm-specific training—would be costly to replace, either due to their possession of some unique irreplaceable skills, or more likely, due to firm-specific training or human capital. As such, our analysis is most applicable in settings where employees possess a high degree of firm specialization. The prospect of such an employee or group of employees threatening to leave and using their (at least temporary) irreplaceability to their benefit in negotiations appears quite plausible.

We take a labour contract to be an agreement for a wage which the firm will pay the employee conditional on the employee providing the contracted productive services. At any time before production, an employee may approach the firm and enter into wage negotiations. Likewise, the firm may choose at any moment before production to call the employee in for wage negotiations. The nature of our contractual incompleteness is the inability of either party to commit to a future wage and employment decision. Because

3. This general approach that we take bears some similarity to the research programme laid out in Jackson and Wolinsky (1994), where they analyse how strategic considerations of “network formation” can affect organizational design.

4. As should become clear below, under the weak conditions that we impose, our results are identical if it is only the employees who cannot commit to future production.
the pre-production firm-worker relationship is characterized by such “employment-at-will”, we refer to our stylized firm as an “at-will firm”; this distinguishes our wage-negotiating environment from the standard neoclassical setting with complete labour contracts.

Our employment-at-will restriction on any agreement between the firm and an employee sets the paper apart from a number of related incomplete contracting papers. In particular, Hart and Moore (1988), examine the ex ante inefficiencies for optimal investment associated with incomplete contracts given renegotiation, where a contract can only specify two payments: one in the event of trade and another if instead trade breaks down. Following this result, a number of papers demonstrate that ex ante efficiency can be restored under additional contractual or information assumptions. MacLeod and Malcomson (1993) show that in three different investment settings the first-best efficient investment emerges when a contract can consist of a payment in the event of trade, a second payment if there is no trade, and a third payment if outside options are exercised. Similarly, Chung (1991), Aghion et al. (1994), and Edlin and Reichelstein (1996) assume contracts can prescribe specific performance remedies which are enforceable by the courts, thereby obtaining efficient investment. Also relevant, Nöldeke and Schmidt (1995) assume that courts can verify delivery of the good by the seller, thereby allowing the seller the use of delivery-option contracts, once again yielding the first-best.

In marked contrast to all of these papers, we do not allow for any contract to be enforceable between the employee and the firm. Most critically, this rules out payments from one party to the other in the event of a breakdown in trade, which plays a crucial role in all the papers considered above. While extreme, such an assumption seems well worth considering insofar as the ability of an employee to terminate a labour agreement without punishment is the typically observed arrangement.5 Indeed, liquidity constraints together with involuntary servitude restrictions that rule out specific performance remedies make any significant punishment provision problematic.

Note that our extreme employment-at-will assumption ruling out all potentially binding wage contracts, together with employee hold-up power, yields inefficiencies in hiring decisions even in a setting without specific investments. The firm must take into account that, after locking in a specified number of employees, these workers will be able to extract rents from the firm given their irreplaceability, thereby leading the firm to distort its hiring decisions. Such considerations form the foundation of our analysis. As such, the nature of our inefficiencies are quite distinct from those considered in the above-mentioned papers on incomplete contracts, even if their ultimate source—hold-up power in specific relationships—is the same. Such inefficiencies could be avoided and the first-best could be obtained if the firm could extract a fixed payment up front from workers—to compensate for their future bargaining power—and commit to hiring the first-best number of employees and level of non-labour inputs. Nonetheless, such a scheme is unworkable if either employees are liquidity-constrained or if a commitment to a specified level of employment and non-labour inputs is infeasible (say, because different “types” of workers are indistinguishable to a court). Without such a commitment, a fixed entry fee by workers would only serve to magnify the hiring distortions which we examine.

Formally, we study equilibrium wage contracts immune to intra-firm pairwise renegotiations, where the outcome of such a bargaining process depends on the outside options

5. Posner (1986, p. 306) states, “employment at will is the usual form of [the] labour contract”. While exceptions exist to allowing employees unlimited freedom to terminate a labour agreement, they appear quite specialized and are often unenforced by the courts. See Stole and Zwiebel (1996) for further discussion on this point.
of the firm and the worker in question in the event of a breakdown in negotiation. For the worker, this outside option is the reservation wage; for the firm, the outside option is the equilibrium outcome of the process with one less worker. In this manner, our equilibrium outcome is recursive in structure.

We allow for the outside options to affect negotiations in a general manner as threat points. In particular, we characterize the equilibrium outcome of this bargaining process for any given split of the bilateral surplus between an employee and the firm, including surplus divisions which vary across different employees or with the number of employees. All such specifications yield results with qualitatively similar economic implications. Our companion paper (Stole and Zwiebel (1996)) demonstrates the broad potential scope of such implications through a wide range of economic applications.

The analysis of this paper is organized as follows. Section 2 presents our bargaining game and equilibrium notion for the simple specification whereby bilateral surplus is divided evenly between bargaining parties. More specifically, we introduce the bargaining game and the notion of stable outcomes. While for interpretive reasons we generally speak of our equilibrium in terms of a stable wage and profit profile immune to renegotiations, we provide a rigorous extensive-form game for which the stable outcome is the unique subgame-perfect equilibrium. In addition to characterizing the outcome of the bargaining process and noting its intuitive features, we extend the bargaining game to the setting where labour is taken to be a continuum, obtaining similar results. Moreover, we demonstrate that our non-cooperative equilibrium outcome is equivalent to the Shapley values of a corresponding cooperative game, thereby yielding an elegant economic interpretation to the bargaining process. This non-cooperative foundation for the Shapley value differs from previous results in the literature in that, for our employment setting, we do not need to introduce any form of randomization over the bargaining order, and all bargaining is conducted through a central agent (i.e., the firm). We are able to do so because the firm is essential to the production process in our game.

In Section 3, we further generalize results of Section 2 in a few significant directions. In Section 3.1., we generalize the results to allow for arbitrary specification of bargaining power. We demonstrate that the characterization of the stable outcome profiles takes on an elegant form: the firm's profit outcome is the weighted average of neoclassical profits over the feasible interval of labour. Furthermore, we demonstrate any weighting of the neoclassical firm's profits can be generated by our bargaining game (with the right choice of corresponding bargaining power) if and only if the weights are a probability measure satisfying a simple elasticity property: the elasticity of the measure with respect to labour.

6. We also provide a further generalization of our model in an appendix for the case where the outside options enter only as constraints on the negotiated wages.

7. Several related strands in the labour literature are worth mentioning. First, a large number of papers have been written regarding the impact of bargaining over wages with employees who have already been hired. Among the first was Becker (1975), who explicitly considers the incentives to invest in specific capital in light of future wage determinations, but who does not theoretically model the bargaining process, focusing in large part on competitive wage pressures to determine the equilibrium wage. Another approach, that of Mortensen (1985) and Pissarides (1987) for example, is search-theoretic in nature and models the costly matching process between firms and workers and the resulting equilibrium wage in light of these search costs. Here, bargaining power emerges because both parties wish to avoid additional costly search. Although bargaining power in our approach similarly derives from the ability of either party to force a separation, the multilateral nature of our bargaining process is crucial to our analysis. A third related strand in the labour economics literature considers the influence of "insiders" on the wage and employment outcomes of the firm, especially the effect of such insiders on the ability of "outsiders" to obtain employment. Lindbeck and Snower (1988) make the argument that the incumbent labour force (e.g., a union) has a favoured bargaining position over outside labour, and therefore may negotiate a wage decision with the firm that favours themselves but is not globally efficient. In our setting, at the time of bargaining we essentially only consider insiders (they are irreplaceable).
is constant over its support. In addition, it is seen that this elasticity property generalizes weighted Shapley values to the case where bargaining power varies with the number of workers. In Section 3.2, we extend our description of the production function to allow any (finite) number of labour arguments, yielding the interpretation that firms either employ heterogeneous employees or assign employees to distinct assets or groups. Here as well, we show that the firm’s profit outcome can again be characterized both as the Shapley value of a corresponding cooperative game and a weighted average of the neo-classical profit function over all feasible labour configurations.

With the general results for the bargaining process in hand, we turn to optimal labour and factor choices by the firm in Section 4. First we demonstrate that for any specification of bargaining power, the firm’s optimal labour choice is distorted in a distinct and significant economic manner. For example, under some very simple conditions on the production function, we demonstrate that excess employment attributable to bargaining always occurs. We then introduce a simple statistic, the frontload factor, which characterizes the extent of the distortion that bargaining induces. This statistic, which summarizes the productivity of initial inputs of labour relative to later units, also characterizes firms’ preferences for “inefficient technologies” which better their bargaining position. We then turn to characterizing the nature of the distortions in the multi-asset (or heterogeneous labour) setting and the resulting cross-asset mis-allocations of inputs. Finally, while we defer many economic applications to a companion paper, we present a few applications of labour choice which demonstrate the nature distortions for both the single and multi-asset settings.

2. BARGAINING IN THE NON-BINDING CONTRACTUAL ENVIRONMENT

In this section, we first consider an underlying game form which is designed to capture the notion that any worker or the firm can start wage renegotiations at any time before production takes place. We begin in Section 2.1 with a simple notion of stability to capture intuitively this notion of bargaining. After defining the bargaining environment and the nature of stable outcome profiles, we fully characterise the wage and profit outcome for any initial workforce. Having developed an understanding of the results, we proceed in Section 2.2 to a more formal non-cooperative underpinning of our bargaining solution. Specifically, we present an extensive-form game whose unique subgame-perfect equilibrium corresponds exactly to our stable wage profile. In doing so, we can examine precisely what specific renegotiation sessions are important for our result. Strikingly, we find that allowing for renegotiations with all remaining workers only when any one worker leaves the firm (due to failed negotiations), serves as a sufficient amount of renegotiations to yield our stable outcome. In Section 2.3, we extend our bargaining outcome to the case of continuous labour. Having thus characterized the equilibrium to our non-cooperative bargaining game, in Section 2.4 we demonstrate that this outcome can also be characterized by the Shapley values of a corresponding cooperative game. This, consequently, implies that the extensive-form game developed in Section 2.2 provides a non-cooperative foundation for Shapley values over the restricted class of games considered here that, unlike previous results in the literature, does not require randomizations and expectations.

Throughout the present section, consider a firm with \( n \) potential employees; we endog-

enize the choice of \( n \) later. We first consider a production function with a single argument for labour input, which we refer to as the firm operating a single asset. Later, when we consider heterogeneous labour (i.e., productively asymmetric labour), we use subscripts to distinguish the labour categories (i.e., \( n_1, \ldots, n_m \)). The productive nature of any asset can be described by a production (or revenue) function, \( F(n) : \mathbb{N} \rightarrow \mathbb{R}_+ \). We let \( \Delta \) represent
the first difference operator: $\Delta F(n) \equiv F(n) - F(n-1)$. In later sections, we also consider an extension of the bargaining model to the case of continuous labour, in which case $F$ represents a continuously differentiable function with bounded first derivative, $F: \mathbb{R}_+ \rightarrow \mathbb{R}_+$. We make comparisons throughout to the neoclassical firm's profit function, defined as the payoff obtained by a firm which can write binding contracts with its $n$ workers at the outside wage of $\bar{w}$: $\pi(n) \equiv F(n) - wn$. We assume that there exists a sufficiently high $n$ such that the marginal productivity of labour is less than $\bar{w}$ at higher levels of labour. For the employment-at-will firm, we make use of $\hat{\bar{w}}(n)$ to denote each employee's wage outcome in a firm with $n$ remaining workers. With this notation, the payoffs received by our at-will firm are simply $\hat{\bar{w}}(n) \equiv F(n) - \hat{\bar{w}}(n)n$.

2.1. Stable outcome profiles of bargaining

The firm selects $n$ employees and offers each a wage for employment. These wage offers are unenforceable in the following strong sense: any time before production takes place, the firm may fire an employee, in which case the employee is forced to return to the external labour market where the outside option $\bar{w}$ is obtained; alternatively, an employee may quit the firm and return to the external labour market. Once negotiations begin the firm cannot hire additional employees from the external market. A plausible motivation for such a setting is that it takes a period of time to train an employee, during which time, the productivity of the workers is zero.

Intuitively, one can think of a setting where employees and the firm can engage in an arbitrary number of pairwise negotiations prior to production in which any employee can re-open negotiations over his or her individual wage contract with the firm and vice versa. Given this notion, formally, we look for an outcome profile (division of production into wages and profits) for any given number of employees that neither any employee nor the firm will wish to alter; we say such a profile is "stable." We will show in an extensive-form game developed in Section 2.2 below that this effectively corresponds with allowing for the opportunity for all employees to engage in renegotiations after the separation of any other employee.

We first assume that in any pairwise negotiation the firm and employee split the difference relative to their outside options and thereby obtain the payoffs associated with the Nash bargaining solution. This equal split is not necessary for the economic implications that follow. Indeed, in Section 3, we generate several interesting results by generalizing this division of surplus to allow for any arbitrary division (even allowing for the split to depend upon the number of remaining employees). After such wages are set, the firm and its retained employees enter the production phase, at which point the wage agreements become binding. The following definitions are useful in characterizing the outcome to this bargaining process.

Definition 1. An outcome profile is a collection of wages and profit levels, $\{\hat{\bar{w}}(i), \hat{\pi}(i)\}_{i=1}^n$, one pair for each employee configuration, such that $i\hat{\bar{w}}(i) + \hat{\pi}(i) = F(i)$, for all $i \leq n$.

8. Although a priori we must allow for the possibility that otherwise symmetric workers might receive asymmetric wages, it follows immediately from induction and the uniqueness of stable outcomes that all productively-symmetric workers must necessarily receive the same wage for a given firm size.

9. We generally interpret employees (e.g., mid-level managers) as the minimal bargaining units for labour. One could, however, instead interpret the minimal bargaining units to be binding collections of employees (e.g., unions).
Definition 2. A stable outcome profile for \( n \) employees is an outcome profile such that for every workforce size, \( i \leq n \), given split-the-difference renegotiations, no individual worker can improve upon his wage in a pairwise renegotiation with the firm, and the firm cannot improve profits renegotiating with any worker.

Below we obtain the stable-outcome profile inductively by considering increasingly larger numbers of workers in the firm and apply the requirement of stability at each step. For a firm with \( n \) workers, we denote the resulting wage of this process by \( \hat{w}(n) \).

Some care must be taken to ensure that the outcome to this bargaining process does not violate the employees' outside option constraint. Intuitively, if the joint surplus between the firm and the employee engaged in a pairwise negotiation is negative, stability prescribes a wage less than \( w \) for the employee, violating this constraint. This can happen in our setting for two reasons. First, even if \( \hat{w}(n) \geq w \), it may nonetheless be the case that \( \hat{w}(i) < w \) for some \( i < n \). That is, over some intermediate range less than \( n \), the firm may prefer to fire employees rather than pay them all \( w > \hat{w}(i) \). We generally ignore this problem for as shown in Section 3, it does not arise when the firm optimally chooses its input levels and the underlying neoclassical profit function (i.e., \( \pi(n) \equiv \Pi(n) - \eta w \)) is quasi-concave. 10

Somewhat more problematic is the possibility that if the firm has "too many" employees on hand, the outside option will bind. In such an event, if the firm wishes to retain all such employees, it will have to pay the outside option of \( w \) instead of \( \hat{w}(n) < w \). Our analysis below characterizes at precisely what level of employment this occurs, and further indicates that when the firm optimally chooses its employment, this does not occur: it is never optimal for the firm to hire workers beyond the point where \( \hat{w}(n) = w \). However, it is important to keep in mind that beyond this employment level, wages in our bargaining game would be given by \( w \) and not by \( \hat{w}(n) \). The following definition is useful in dealing with this issue.

Definition 3 (Feasibility). An outcome profile \( \{\hat{w}(i), \hat{x}(i)\}_{i=0}^{n} \) is feasible at \( n \) with respect to \( w \) if \( \hat{w}(i) \geq w \) for all \( i \leq n \).

With this definition, if the unique outcome profile for \( n \) employees that satisfies stability is feasible, then it gives the outcome to our bargaining process with \( n \) workers. If instead, feasibility is violated at the endpoint, workers earn \( w \) and the firm's profits are instead identical with that of a neoclassical firm with \( n \) workers (which always pays wages of \( w \)).

In order to characterize stable outcomes, we begin by considering the single employee and proceed inductively over the number of employees. Stability requires that for any worker in an \( n \)-worker firm the net surplus is split equally between the firm and that worker, thereby satisfying

\[
\hat{x}(n) - \hat{x}(n-1) = \hat{w}(n) - w.
\]

(1)

Consider the outcome if only one employee is present, and either the employee or firm decides to open wage negotiations. The net surplus the employee receives from staying with the firm and taking a wage \( w \) is \( w - \hat{w} \); the net surplus the firm receives from retaining the employee is \( \Delta F(1) - w \); the total net surplus from exchange is \( \Delta F(1) - w \). In such a

10. Departing from quasi-concavity does not alter our fundamental conclusions qualitatively, though quantitatively the optimization programme becomes more cumbersome. This issue is briefly explored in Section 4.
one-employee setting, the resulting wage is
\[\hat{w}(1) = \frac{1}{2}(\Delta F(1) + w),\]
providing that \(\Delta F(1) \geq w\); i.e., providing \(\hat{w}(1)\) is feasible with respect to \(w\).

Now consider the outcome with two employees present. Suppose an employee wishes to alter his or her wage with the firm when the wage is currently at \(\hat{w}(2)\). Although the net surplus of the employee at a wage of \(w\) is still given by \(w - w\), the firm's net surplus differs from the one-employee setting. If the employee leaves, the firm immediately loses \(\Delta F(2) - w\) on the margin, but in addition a wage renegotiation ensues with the remaining employee, leading to a wage of \(\hat{w}(1)\) instead of \(\hat{w}(2)\). This secondary wage effect is crucial to our analysis. Rather than reaching a wage \(w\) such that \(\Delta F(2) - w = w - w\), the firm and employee negotiate for a wage \(\hat{w}(2)\) satisfying
\[\Delta F(2) - \hat{w}(2) + [\hat{w}(1) - \hat{w}(2)] = \hat{w}(2) - w,\]
providing again that \(\hat{w}(1) \geq w\) and \(\hat{w}(2) \geq w\); i.e., providing feasibility is maintained. Substituting in the result for \(\hat{w}(1)\) we obtain the wage:
\[\hat{w}(2) = \frac{1}{2}[\Delta F(2) + \hat{w}(1) + w] = \frac{1}{2}\Delta F(2) + \frac{1}{2}\Delta F(1) + \frac{1}{2}w.\]
Note that while the coefficients on the marginal products still sum to \(\frac{1}{2}\), now \(\hat{w}(2)\) depends on both the marginal output \(\Delta F(2)\), as well as the inframarginal output \(\Delta F(1)\).

Generalizing the above argument for any \(n\), by induction this outcome can be characterized by the difference equation,
\[\hat{w}(n) = \frac{1}{n + 1} [\Delta F(n) + (n - 1)\hat{w}(n - 1) + w], \tag{2}\]
providing that for all \(i \leq n\), \(\hat{w}(i) \geq w\).

It is straightforward to obtain as the solution to the first-order difference equation in (2) the following expressions for \(\hat{w}(n)\) and \(\hat{w}(n)\):
\[\hat{w}(n) = \frac{1}{n(n + 1)} \sum_{i=0}^{n} i\Delta F(i) + \frac{1}{2}w, \tag{3}\]
\[\hat{w}(n) = \sum_{i=0}^{n} \left(1 - \frac{i}{n + 1}\right)\Delta F(i) - \frac{n}{2}w. \tag{4}\]
This solution represents the outcome of the bargaining process whenever \(F\) is such that \(\hat{w}\) is feasible with respect to \(w\) at \(n\). Of particular interest is equation (3) which indicates that employee wages are given by a weighted average of marginal products, with decreasing weight the further infra-marginal the product. Intuitively, given that the outside option for the firm if negotiations with an employee break down is production with one less employee, it is not surprising that employees capture a fraction of the infra-margins of production as well as the marginal product. Equation (3) indicates that the fraction of each infra-margin that an employee captures falls as the infra-margin becomes more remote from the actual labour configuration. With an equal pairwise division of surplus, these weights in fact take on a particularly simple form—linear in \(i\). And while this linearity is a direct consequence of equal bargaining power, the monotonicity is robust to arbitrary divisions of surplus as demonstrated in Section 3. This monotonicity, in turn, provides the intuition behind the particular economic distortions which arise in our applications.

Whereas equations (3) and (4) give resulting wages and profits in terms of a weighted average of marginal production, we find it useful to refer to an alternative characterization
directly using production functions rather than marginal products. In particular, summing by parts equations (3) and (4) yields

\[
\tilde{w}(n) = \frac{1}{n} \left( F(n) - \frac{1}{n+1} \sum_{i=0}^{n} F(i) \right) + \frac{1}{2} w, \tag{5}
\]

\[
\tilde{\pi}(n) = \frac{1}{n+1} \sum_{i=0}^{n} F(i) - \frac{n}{2} w. \tag{6}
\]

By further rearranging (6) and using the definition of the neoclassical firm’s profit function, \(\pi(n) \equiv F(n) - wn\), we have a strikingly simple characterization of the solution.

**Theorem 1.** Suppose that there are \(n\) employees in the firm and the solution to difference equation (2) is feasible with respect to \(w\). Then the stable wage and profit profiles are

\[
\tilde{w}(n) \equiv w + \left( \frac{\pi(n) - \tilde{\pi}(n)}{n} \right). \tag{7}
\]

\[
\tilde{\pi}(n) = \frac{1}{n+1} \sum_{i=0}^{n} \pi(i). \tag{8}
\]

The first equation follows by definition. It simply states that for any given number of employees \(n\), the firm’s profits plus the employees’ wages must be identical in the employment-at-will and the neoclassical firm; that is, since the production function is fixed, total surplus is unchanged. The second equation, however, is a very powerful result: The profit to a firm with non-binding contracts is the uniform average of the neoclassical firm’s profits as labor varies over \(i = 0, 1, \ldots, n\). Thus, unlike the neoclassical firm which cares only about the “marginal” \(\pi(n)\), the at-will firm cares about the inframarginal profits (of the neoclassical firm) as well. We return to these characterizations (and generalizations of them) below. They are particularly helpful in further characterizations of the at-will firm’s decisions and give rise to straightforward and intuitive economic predictions.

It is notable that the above theorem applies to a variety of economic environments in addition to the canonical neoclassical firm. For example, rather than considering a firm facing a fixed outside wage, \(w\), it may be that \(w\) depends explicitly upon \(n\) because the firm operates in an environment where it has monopsony power in the labour market. A neoclassical firm in this setting would behave as a labour monopsonist, maximizing \(\pi(n) = F(n) - nw(n)\). Nonetheless, straightforward application of the above argument demonstrates that the characterization of \(\tilde{\pi}(n) = (1/(n+1)) \sum_{i=1}^{n} \pi(i)\) remains true, where now, \(\pi(n)\) takes into account the effect of the firm’s scale on the outside wage.

2.2. The extensive-form game

The notion of stability allows us to consider bargaining environments in a rather heuristic manner. Here we are more rigorous and formally define an extensive-form bargaining game whose unique subgame perfect equilibrium corresponds with the stable outcome described above over the feasible range. This extensive-form game will serve to highlight what renegotiations are important to obtain our outcome. In particular, we find that it is enough to allow all remaining employees to renegotiate when any employee leaves the firm following failed negotiations. Although for intuition we focus on our notion of
stability after this section, it should be understood that the following formally defined game underlies the heuristic story we tell.

In particular, consider a firm with \( n \) employees. Choose any fixed ordering for the \( n \) employees to bargain with management over wages.\(^{11}\) Bargaining proceeds as a finite sequence of pairwise bargaining "sessions" between an employee and the firm. In all sessions, either the employee and the firm will reach agreement, or negotiations "break down" and the employee exits the game forever. We describe what occurs in such a negotiating "session" shortly—first we specify the sequence of the bargaining sessions.

Initially, the first employee in the ordering meets with the firm for the first bargaining session. Any time an agreement between an employee and the firm is reached in any bargaining session, negotiations between the firm and the following worker in the ordering ensues. If instead a negotiating session ends in a breakdown, the employee in question leaves the firm forever, bargaining starts over from the beginning with a session between the firm and the first employee who has not left the firm, following the same order over all remaining employees. The game ends when the firm reaches an agreement with the last remaining employee in the ordering (or when all employees have dropped out following failed bargaining sessions).\(^{12}\)

This sequence of bargaining sessions is depicted in Figure 1. Each box represents a bargaining session, numbered by the worker who is negotiating with the firm. A represents Agreement and B represents Breakdown. Here, \( \Gamma(\cdot) \) denotes the subgame which begins with the indicated sequence of workers. For example, \( \Gamma(1, 2, 4, \ldots, n) \) is the subgame that begins after player 2 has exited the game and players 1 and 3-through-\( n \) line up in order to negotiate with the firm. Except in the simple subgames between the firm and a single remaining worker, a breakdown initiates a fresh sequence of bargaining beginning with the first remaining worker and continuing through the sequence of remaining workers in their original order. One such subgame is shown in Figure 1 above for \( \Gamma(2, \ldots, n) \).

11. Our result follows for any ordering chosen, without resorting to taking expectations over a set of possible orderings.
12. Under this specification, at most \( n(n+1)/2 \) bargaining sessions can occur before the game terminates.
We now turn to describing play within each bargaining session. Within each bargaining session the firm and worker play the alternating-offer bargaining game of Binmore, Rubinstein and Wolinsky (1986) (henceforth BRW) in which there is an exogenous probability of breakdown following a rejected offer. In particular, starting with the firm, the firm and worker alternate proposals for the worker’s wage, providing there are net gains from employment. If a proposal is accepted, negotiations terminate; if it is rejected, with probability $q$ negotiations break down, the bargaining session ends without an agreement and the employee leaves the firm forever. If there are no positive net gains from employment, negotiations break down immediately. When breakdown does not occur, the rejecting party makes a counterproposal. Proposals are made until either one is accepted or a breakdown occurs; there is no discounting. We look for the limiting outcome as $q \to 0$. BRW shows that for such a bargaining session, as $q$ approaches zero, the Nash bargaining solution emerges: each party receives half of the joint surplus net of the payoff they would obtain in the event of breakdown (i.e., their outside option). For robustness, in Appendix B we consider the alternating offers game of Rubinstein (1982) as an alternative to BRW. As is well known, in such a game outside options do not affect equilibrium outcomes unless they are binding, in contrast to BRW. Nonetheless, under several additional technological assumptions, most of the qualitative analysis remains unchanged with this change in the bargaining game.

This completes our description of the game. This specification captures the intuition behind our stability criterion, in which we seek an outcome immune to any renegotiations which split the surplus relative to the outside option. We can now state the following result; all omitted proofs are presented in Appendix A.

**Theorem 2.** For any ordering of workers, the unique subgame-perfect equilibrium of the n-worker extensive-form game described above induces an outcome in which all workers receive wages given by the stable wage profile $\tilde{w}(n)$ whenever $\tilde{w}(n)$ is feasible at $n$ with respect to $w$.

It is worth emphasizing that our extensive-form bargaining game involves only pairwise bargaining between the firm and individual workers, and we obtain this unique outcome for any ordering of the workers. Loosely speaking, order is not strategically relevant because if a worker quits, renegotiations ensue with all remaining workers, and this in turn allows a worker to obtain the same share of surplus associated with workers prior to him in the order as those after him. This structure allows workers, through their ability to renegotiate if breakdown occurs later, to effectively achieve the same outcome as a wage agreement up front that is contingent on which workers are ultimately present.

While our notion of stability in the previous section was devised to correspond with arbitrary renegotiations, Theorem 2 indicates that allowing for renegotiations with all remaining employees upon breakdown in negotiations with any given employee is all that is needed to obtain the stable outcome. It is worth noting that we could add further

13. Alternatively, one could consider a BRW bargaining session here as well. However, with negative net surplus, agreement would never be reached and breakdown would ensue. Such sessions with negative net surplus arise precisely when the stable solution is infeasible. We deal with several subtleties which arise from this issue below. For here, it is sufficient to note that insofar as a unique subgame-perfect equilibrium exists in all subgames, there is no ambiguity in specifying the firm’s net surplus conditional on agreement with an employee—such surplus is given by the profit in the equilibrium of such a subgame net the profits in an equilibrium of the subgame where an agreement is not reached.

14. One can think of alternating offers being made at times $t, t+\frac{1}{k}, t+\frac{2}{k}, \ldots, t+(k-1)/k, \ldots$, to ensure that each bargaining session ends with probability 1 in one unit of time.
negotiating sessions to our extensive-form game, in many manners, without changing our result. Thus, for example, additionally allowing all employees who have previously agreed on a wage to renegotiate each time a new employee reaches an agreement would not alter our result. What is crucial to the outcome are the following: (i) all remaining employees get the opportunity to renegotiate any time another employee leaves; (ii) the number of potential renegotiation sessions is finite; and (iii) when breakdown occurs between an employee and the firm, this employee cannot return later to the firm and participate in further negotiations.15,16

Note that in our game a previously-agreed upon wage between the firm and an employee does not affect the outcome of any future bargaining sessions between this same pair if breakdown occurs. This differs from several papers in the incomplete-contracting and renegotiation literature—among them, Hart and Moore (1988) and MacLeod and Malcomson (1993). In these papers, even without binding contracts, a prior wage agreement can affect the final outcome in the renegotiation games. The reason is that these papers consider games in which there is a "last period" of renegotiation and in the absence of a new agreement, a previous agreement can be enforced. In the penultimate period, providing that the current contract satisfies the worker's outside option, the firm realizes that absent renegotiation the worker will show up at production time to supply labour at the previously agreed wage. Consequently, it may be in the firm's interest to refuse to renegotiate. Applying backwards induction, an initial non-binding contract can have direct effects on the bargaining outcomes throughout the game.

In contrast, we presume a potentially infinite number of offers in every negotiation session with no fixed last renegotiation date. As such, there is no last period for an offer at which the firm can credibly refuse to renegotiate and, therefore, no backwards induction argument. Rather, if an employee makes a renegotiation offer in a bargaining session, the firm cannot credibly ignore it, insofar as each period without agreement leads to some fixed chance of breakdown under which each party receives its outside outcome rather than that specified in any previous agreement.17 Consequently, a prior unenforceable agreement cannot affect the outcome in any renegotiation round. The outcome in our setting is as if

---

15. We hypothesize that even with an infinite number of potential renegotiation sessions, provided that conditions (i) and (iii) are satisfied, the stable outcome will be the only possible equilibrium where production occurs. However, in a setting with a potential for an infinite number of renegotiation sessions (and no discounting), it is possible that bargaining sessions would cycle, and therefore production would never take place. The third condition, while central to a number of bargaining models and seemingly frequent in practice, is somewhat harder to motivate economically. Specifically, we do not provide an answer to why breakdown precludes future negotiations. To answer this satisfactorily, we would need a rigorous underpinning of what the "breakdown" state entails, which no paper to our knowledge has addressed.

16. Theorem 2 states that the outcome to our extensive-form game corresponds with stability whenever the stable solution \( \bar{w}(n) \) is feasible. Some minor subtleties arise when this is not the case. Suppose that \( n \) is large enough such that feasibility is not satisfied; i.e., stability prescribes a wage \( \bar{w}(n) < \bar{w} \). By the definition of stability, this only occurs when the net surplus between the worker and the firm is negative. In such a setting no agreement will be reached in a bargaining session, and hence, breakdown occurs in the extensive-form game. This highlights one out-of-equilibrium difference between our notion of stability and our extensive-form game. If the firm happens to have more employees than is optimal for an employment-at-will firm at the start of the bargaining game, those excess workers will separate from the firm via breakdown in the bargaining game. On the other hand, stability, together with the constraining influence of the outside option, defines a wage outcome for each configuration of workers engaged in bargaining, including the case of too many workers. Nonetheless, since these two outcomes only fail to coincide over the infeasible range where hiring is inefficient for the at-will firm, maximizing the stable profit outcome \( \pi(n) \) over \( n \) will be identical with the firm choosing \( n \) to maximize profits in the extensive-form game.

17. Similarly, in Appendix B where we consider the standard Rubinstein bargaining framework instead of the BRW variation, the firm does not move to the production stage until either an agreement is reached, or the employee and the firm permanently separate. In such a setting as well, it is not credible for the firm to refuse to bargain.
either party can credibly disavow the prior agreement even if no further agreement is reached. In contrast, in the papers mentioned above, when a party initiates a renegotiation it cannot commit to forgo the option of accepting the terms of the previous agreement in the absence of any new agreement by the other party.

Having fully characterized the equilibrium outcome profile in the discrete setting, we turn briefly to the continuous extension.

2.3. Continuous labour allocations

Although the results of this paper can be stated while rigorously taking into account integer concerns, we find it useful at times to assume a continuous version of (8) above. The most natural extension is to think of units of labour as subdivisible, with each smaller unit capable of bargaining with the firm in pairwise meetings. Suppose that \( F(n) \) is defined over \( \mathbb{R}_+ \), rather than only \( \mathbb{N} \). Then our solution is defined for any finite subdivision, as also is the neoclassical firm’s profit, \( \pi(n) \). Formally, let labour \( n \) be subdivided into equal subdivisions of size \( h \). Then rewriting (1),

\[
\tilde{\pi}(n) - \tilde{\pi}(n-h) = (\tilde{w}(n) - \tilde{w})h,
\]

making use of the equivalence \( \tilde{\pi}(n) - \pi(n) = n(\tilde{w}(n) - \tilde{w}(n)) \) and taking the limit as \( h \to 0 \), we obtain

\[
\tilde{\pi}'(n)n + \tilde{\pi}(n) = \pi(n).
\] (9)

The unique solution to this first-order differential equation (satisfying the initial condition \( \tilde{\pi}(0) = \pi(0) \)) characterizes the outcome of the continuous-labour game as stated in the following theorem.\(^\text{18}\)

**Theorem 3.** Suppose that there are \( n \) units of infinitely divisible labour and the solution to (9) is feasible (i.e., \( \tilde{w}(s) \geq w \) on \( s \in [0, n] \)), then

\[
\tilde{\pi}(n) = \frac{1}{n} \int_0^n \pi(s) ds.
\] (10)

2.4. Shapley values

Returning to our discrete setting, we now demonstrate the equivalence between the non-cooperative equilibrium payoffs and the Shapley value in a corresponding cooperative game.

Consider the cooperative game, \((\mathbb{N}_n, v)\), where \( \mathbb{N}_n = \{0, 1, 2, \ldots, n\} \); we let 0 index the firm and the positive integers index individual employees. \( v \) is the characteristic function which maps from subsets of agents to the value they can independently obtain: \( v : 2^{\mathbb{N}_n} \to \mathbb{R} \), with \( v(\emptyset) = 0 \). Any coalition, \( S \subseteq \mathbb{N}_n \), which does not include the owner of the firm does not have access to the firm’s underlying production process and therefore \( v(S) = |S|w \). When \( S \) does include the owner of the firm, the value

\[18. \] As an alternative to deriving the solution to the continuous-labour game by solving (9), we could instead derive the limit of the solution to the discrete game obtaining,

\[
\lim_{h \to 0} \tilde{\pi}(n) = \frac{1}{n+h} \sum_{i=0}^{n/h} \pi(ih)h = \frac{1}{n} \int_0^n \pi(s) ds.
\]
of the coalition is \( v(S) = F(|S| - 1) \). Theorem 4 states that the Shapley value for this cooperative game is equivalent to the bargaining outcome of Theorem 3.

**Theorem 4.** Suppose that there are \( n \) employees hired by the firm and the solution to (2) is feasible. Then the equilibrium wages and profit are given by the Shapley values of the underlying cooperative game \( (\mathbb{N}_n, v) \).

**Proof.** Following Myerson (1980) and Hart and Mas Collell (1989), it is sufficient to show that equilibrium wages and profits, given by (7) and (8), are part of a payoff structure over all subsets that induces balanced contributions and efficiency. That is,

\[
\phi_i(v, S) - \phi_i(v, S - j) = \phi_j(v, S) - \phi_j(v, S - i), \quad \forall S \subseteq \mathbb{N}_n, \forall i \in S, \forall j \in S,
\]

and

\[
\sum_{i \in S} \phi_i(v, S) = v(S), \quad \forall S \subseteq \mathbb{N}_n,
\]

where \( \phi_i(v, S) \) is the payoff to \( i \) when the game \( v \) is restricted to the coalition \( S \). It is straightforward to check that (12) is satisfied when \( 0 \in S \) by the payoffs in Theorem 1. Letting,

\[
\phi_i(v, S) = w \quad \text{whenever } 0 \notin S,
\]

(12) is satisfied for all such sets as well. With symmetric employees, balanced contributions are trivially guaranteed for \( 0 \notin S \); when instead \( 0 \in S \), (11) requires that,

\[
\bar{w}(|S| - 1) - \bar{w}(|S| - 2) = \bar{w}(|S| - 1) - \phi_i(v, S \setminus \{0\}),
\]

which follows from equations (1) and (13).

Theorem 4 implies that the stable outcome of the non-cooperative bargaining process satisfies very reasonable axioms regarding the division of surplus. Additionally, Theorem 4 underscores the irrelevance of randomization in obtaining a non-cooperative foundation to the Shapley value in our setting. In contrast, the games in Gul (1989) and Hart and Mas Collell (1992) use randomizations to determine player ordering, and then take expectations of payoffs over all orderings. In our setting, Shapley values are obtained for any ordering of the employees, and not just in expectation over all orderings.

It important to recognize that we are only able to obtain the Shapley value without randomization in this manner because, in our game, the induced cooperative game between employees absent the firm is inessential, and because all negotiations take place in a

19. We are implicitly assuming that when working in the firm, the outside reservation wage \( w \) is not obtainable; i.e., the firm cannot hire out its employees for the outside wage. Despite the fact that this characteristic function is not super-additive, equilibrium conditions ensure that agents obtain at least their reservation values.

20. Alternatively, we could proceed by demonstrating that equations (5) and (6) are first differences of the potential function defined in Hart and Mas Collell (1989). Their paper demonstrates that there exists a unique function, called the potential, \( P: (S, v) \rightarrow \mathbb{R}_+ \), mapping from all games to the positive reals, such that for all games \( (S, v) \),

\[
\sum_{i \in S} (P(S, v) - P(S \setminus \{i\}, v)) = v(S),
\]

and furthermore, the first difference of the potential \( P \) yields Shapley values. In our context, one can show that insisting that equilibrium wages (e.g., employees' payoffs) are given by the first difference of this potential function and consequently by Shapley values, is equivalent to imposing our non-cooperative difference equation (2) on wages.

21. In our game, splitting-the-difference within pairwise matches (via BRW) takes the place of randomization in which one of the two parties is randomly chosen to make a take-it-or-leave-it offer. Loosely speaking, this is sufficient for generating the Shapley value for any ordering under our extensive form, because the potential for renegotiations in our game is sufficient to ensure that an employee's position in the ordering does not affect her ability to hold up the firm.
pairwise fashion through the essential agent (i.e., the firm). To understand why this inessentiality of the subgame is needed to obtain Shapley values, one must recognize a subtle distinction between our split-the-difference notion which follows from BRW bargaining in pairwise meetings (i.e., equation (1)) and Myerson’s balanced contributions notion (i.e., in the present context, equation (14)). In particular, balanced contributions prescribes that what the firm loses in coalition with the other employees when an employee quits is equivalent to what an employee loses in coalition with the other employees when the firm quits. In contrast, the split-the-difference criterion that derives from our extensive-form game instead equates what the firm loses in coalition with the other employees when an employee quits with what an employee loses when he quits and instead realizes his outside wage. These two criterion will be equivalent, and our game will imply balanced contributions and hence yield Shapley values, if and only if each employee’s outside wage is equivalent to what they receive in coalition with any other set of employees when the firm is absent—i.e., if and only if the game without the firm is inessential.

3. GENERALIZATIONS

3.1. Generalizations of bargaining power

Suppose for generality that in any pairwise meeting, the net surplus is not divided equally between the bargaining parties, but rather is split according to the bargaining parameter $\lambda(n)$ in the following sense

$$\Delta \pi(n) = \lambda(n)[\bar{v}(n) - \bar{w}],$$

where we allow for $\lambda$ to depend upon $n$.\footnote{To illustrate, consider the following simple example. Fix a firm with two employees, with a production function given by, $F(0)=0$, $F(1)=4$, $F(2)=6$. Let the worker’s outside option to be given by 1 each, but let the two employees be able to obtain 3 in coalition with one another (i.e., the game without the firm is essential). Then our result prescribes that $\pi(2)=\frac{1}{2}$, $\pi(2)=\frac{3}{2}$, while in contrast, Shapley values to the game are given by 2 for the firm and 2 for each employee as well. Intuitively, the employees do worse in our game than their Shapley values, because our game presumes that when negotiations between an employee and the firm break down, it is always the employee who leaves the firm, and not the firm who leaves the employees. As such, employees cannot benefit from any synergies between themselves that only occur without the firm being around, though this can, of course, add to their Shapley value. Even if we changed our game to allow employees upon quitting to join with other already separated employees in order to realize synergies which exceed their outside option, in general their equilibrium payoff would still fall short of the Shapley value. This occurs because in equilibrium each worker understands that even if he left, all other employees would remain with the firm, and hence there would be no such synergies to be realized. Note that the separation of worker from the firm rather than from the firm workers following a bargaining breakdown is in line with the standard property rights theory of the firm notion of Grossman and Hart (1986) and Hart and Moore (1990), whereby ownership of a firm confers the residual right to exclude others from this firm’s capital.

23. In terms of our previously defined extensive-form game, it is straightforward to construct a BRW subgame such that a fixed $\lambda \in (0, \infty)$ emerges as a bargaining parameter, providing we allow for different probabilities of breakdown for the firm and workers. Specifically, let $q_{f}$ be the probability that negotiations break down following a rejection by the firm and let $q_{w}$ be the corresponding probability for each worker. Then it is straightforward to demonstrate that without taking limits on $q_{f}$, every $\lambda \in (0, \infty)$ is consistent for some pair of probabilities:

$$\lambda = \frac{q_{w}}{q_{f}(1 - q_{w})}.$$}

When $q_{f}=q_{w}=q$ and $q \rightarrow 0$, we obtain our familiar result that $\lambda = 1$. This extension to the BRW game, of course, does not explain how $\lambda$ might depend upon the worker configuration, $n$. We are agnostic as to how or why pairwise bargaining power should depend upon $n$, but provide a few possible reasons. First, variations in $\lambda$ with $n$ may reflect labour assignments and corresponding outside options implicit in the production function. For example, as the firm adds more employees, implicit in the production function may be the assignment of employees to tasks to control an outsider for such marginal tasks, or the cost of having an outsider imperfectly perform such tasks, is lessened. Alternatively, fixed bargaining costs for each pairwise meeting or a probability of bargaining breakdown that depends on the number of bargaining sessions could also give rise to bargaining power that varies with the number of workers.
non-negative, continuous function over \( n \) with \( \lambda(0) > 0 \). A higher \( \lambda \) is associated with greater surplus going to the firm. In the case where \( \lambda = 1 \), we have our original bargaining framework with equal division of surplus. Since \( \hat{w}(n) \equiv w + (\pi(n) - \hat{\pi}(n))/n \), the above expression is equivalent to

\[
n\Delta \hat{\pi}(n) = \lambda(n)[\pi(n) - \hat{\pi}(n)].
\]

Alternatively,

\[
\hat{\pi}(n) = \theta(n)\pi(n) + [1 - \theta(n)]\hat{\pi}(n - 1),
\]

where \( \theta(n) = \lambda(n)/(n + \lambda(n)) \in (0, 1] \). Note that \( \theta(n) \) is the firm’s surplus in a simple bargaining game in which 1 unit of surplus is realized with \( n \) employees and the firm present, 0 is realized if anyone is absent, and the firm’s bilateral relative bargaining power is given by \( \lambda \). Consequently, it is natural to interpret \( \theta(n) \) as the firm’s marginal bargaining power; that is, given \( n \) employees, \( \theta(n) \) gives the firm’s share of the marginal output for which the presence of all \( n \) employees is necessary.

From the basic first-order difference relation, equation (15), we demonstrate that for any \( n \), there exists a probability measure \( \mu(i) \) on \( \{0, 1, \ldots, n\} \) such that \( \hat{\pi}(n) = \sum_{i=0}^{n} \mu(i) \pi(i) \). Because \( \mu(i) \) generally depends upon the support \( n \), we index this family of probability measures by \( n \): \( \mu(i|n) \).

**Theorem 5.** Suppose that there are \( n \) units of labour in the firm and the solution to the recursive equation is feasible. Then there exists a unique probability measure, \( \mu(i|n) \), with support \( \mathbb{N}_n \), such that

\[
\hat{\pi}(n) = \sum_{i=0}^{n} \mu(i|n)\pi(i),
\]

where

\[
\mu(i|n)\equiv \frac{\lambda(i)}{n} \prod_{j=i}^{n} \left(1 + \frac{\lambda(j)}{j}\right)^{-1}, \quad \text{for } i \in \{0, 1, \ldots, n\}.
\]

With this result, the at-will firm’s profits can be written as an expectation over neoclassical profits for the probability measure \( \mu \), where \( \mu \) arises from the underlying bargaining game. It is significant that this result also holds in the limit as labour becomes infinitely subdivisible.\(^{24}\)

**Theorem 6.** Suppose that there are \( n \) units of divisible labour in the firm and the solution to the recursive equation is feasible. Then there exists a unique probability measure, \( \mu(s|n) \), with support \( [0, n] \) such that

\[
\hat{\pi}(n) = \int_{0}^{n} \mu(s|n)\pi(s)ds,
\]

where

\[
\mu(s|n)= \frac{\lambda(s)}{s} \exp \left(-\int_{s}^{n} \frac{\lambda(t)dt}{t} \right), \quad \text{for } s \in [0, n].
\]

\(^{24}\) To this end, we assume that \( \lambda(\cdot) \) is continuously differentiable, \( \int_{s}^{n} (\lambda(t)/t)dt \) exists for all \( s \in (0, n] \), and \( \lambda'(0) \) is finite.
Both Theorems 5 and 6 indicate that the at-will firm's profits can be characterized as a probabilistic value—the expectation of the neoclassical profits over the possible employee configurations using the probability measure $\mu$. Such a notion is similar to the probabilistic values studied by Weber (1988), who finds that in monotonic cooperative games, all resulting values that satisfy Shapley's linearity and dummy axioms (but not necessarily the symmetry axiom) have a probabilistic structure. Specifically, a player's value is the expectation (using some probability measure) over the player's marginal contribution to the various possible coalitions. In our setting, the marginal value of the firm to the coalition of $n$ workers is $\pi(n)$. Of additional importance, we have provided an analytic characterization of the underlying probability measure in our game, $\mu$, which is a function of bargaining power, $\lambda(s)$.

It follows immediately from differentiating equations (17) and (19) that the underlying probability measures in Theorems 5 and 6 satisfy a significant elasticity condition.

**Theorem 7.** Suppose that there are $n$ units of divisible labour in the firm and the solution to the recursive equation is feasible. Then the unique probability measure, $\mu(\cdot|n)$, satisfies (depending upon whether the environment is discrete or continuous)

$$\frac{n\Delta_n \mu(i|n)}{\mu(i|n)} = -\lambda(n) \quad \text{or} \quad \frac{d \log \mu(s|n)}{d \log n} = -\lambda(n). \tag{20}$$

Conversely, for any family of probability measures $\{\mu(\cdot|k)\}_{k \leq n}$ which satisfies (20), there exists bargaining weights (given by the corresponding function $\lambda(\cdot)$) that generate (16) or (18) as the unique stable outcome in our bargaining game for all $k \leq n$.

Hence, the elasticity of $\mu$ with respect to $n$ is constant over its support. The essence of this result is that the addition of an extra worker to the support does not affect the relative weights attached to two different inframarginal labour sizes. That is, while a firm can affect the weight placed on an inframarginal unit of neoclassical profit $\pi(i)$ by altering its number of employees $n$, where $n > i$; such changes do not change the relative weights between different inframarginal units. Instead, the firm's choice to add a marginal unit of labour decreases proportionally the weight on all previous units of (neoclassical) profit, and in return, puts this weight on the neoclassical profits under the new labour configuration. In the discrete case, the weight on the neoclassical profits under the new labour configuration is given precisely by $\theta(n)$, the firm's marginal bargaining power.

Note that if we assume that $\lambda$ is independent of $n$, then we can simplify our characterization of $\mu(i|n)$, obtaining $\mu(i|n) = (\lambda/i) \prod_{j=1}^{n} (j + \lambda)$. In particular, when $\lambda = 1$ we have $\mu(i|n) = 1/(n+1)$; that is, the discrete uniform distribution found in Theorem 1. Alternatively, in the continuous case, if $\lambda$ is independent of $s$, we have

$$\mu(s|n) = \left(\frac{\lambda}{s}\right)^\lambda.$$ 

And when $\lambda = 1$, we have a continuous uniform distribution: $\mu(s|n) = 1/n$.

Also note that under either the discrete or continuous formalization, as $\lambda \to \infty$, $\mu$ converges to an atom on $n$, and thus $\bar{\pi}(n) \to \pi(n)$. The neoclassical outcome is thus a
special case in which the firm has all bargaining power vis-à-vis its employees. Not surprisingly, in either case, as \( \lambda \to 0 \), \( \mu \) converges to an atom at 0 and the firm’s profit converges to \( \nu(0) \).

Finally, following Theorem 4, it is straightforward to show that when \( \lambda \) is constant over labour size, the resulting payoffs to the firm are identical to the weighted-Shapley value of the corresponding cooperative game. As such, our characterization of payoffs in Theorem 5 when \( \lambda \) depends on \( n \) can be seen to be a generalization of weighted-Shapley values.

3.2. A generalization to multi-asset environments

We now consider the problem of a firm which has several different “assets” or “tasks” for which to hire employees, captured by multiple arguments to its production function. We assume that employee reassignment is not possible: firms can only choose between retention of a trained employee or dismissal. To the extent that there are productive externalities (negative or positive) between the two assets, the ultimate payoffs to the firm and the employees in each group may vary. When no such externalities exist, the bargaining environment is no different from a multiple application of the simple bargaining game presented in Section 2.

Note that although we interpret results as applying to a multi-asset or multi-task firm throughout this section, we could as easily interpret different arguments in the production function to represent different “types” of employees. As such, this section demonstrates that our model generalizes to allow for any arbitrary heterogeneity in employees (provided we restrict ourselves to considering a finite number of different types). Note further that under this alternative interpretation, worker reassignment is not an issue; by definition, workers of a given type affect a firm’s production through the argument of the production function that corresponds to their type.

We denote the production function for \( M \) underlying assets \( m = 1, \ldots, M \) on which \( n_1, \ldots, n_M \) workers are employed as \( F(n) : \mathbb{N}_+^M \to \mathbb{R}_+ \), where \( n \equiv (n_1, \ldots, n_M) \). Let \( \Delta_m \) be the first partial-difference operator over the \( m \)-th argument; i.e.,

\[
\Delta_m F(n) = F(n) - F(n_{(m)}),
\]

where \( n_{(m)} = (n_1, \ldots, n_{m-1}, n_{m+1}, \ldots, n_M) \). Finally, for greater generality, let \( \omega_i \) be the reservation wage available to an employee assigned to the \( i \)-th asset and \( \omega = (\omega_1, \ldots, \omega_M) \). In such a setting, the neoclassical profit resulting from an allocation of \( n \) is given by

\[
\pi(n) = F(n) - \sum_{i=1}^M n_i \omega_i.
\]

Note that within this framework we allow for the possibility that each asset is highly specialized and designed for a single worker with his or her own unique outside option.

Following previous inductive arguments, a set of first-order partial difference equations uniquely determines the non-cooperative equilibrium of the bargaining game, providing that the solution is feasible (i.e., \( \tilde{w}_m(i) \geq \omega, \forall i \in \mathbb{N}_i \times \mathbb{N}_m, \forall m \)). For any \( i \leq n \), and \( m = 1, \ldots, M \),

\[
\Delta_m F(i) - \tilde{w}_m(i) - (i_m - 1) \Delta_m \tilde{w}_m(i) - \sum_{k \neq m} n_k \Delta_m \tilde{w}_k(i) = \tilde{w}_m(i) - \omega_m,
\]

or more simply, for any \( i \leq n \) and \( m = 1, \ldots, M \),

\[
\Delta_m \tilde{\pi}(i) = \tilde{w}_m(i) - \omega_i.
\]
Similar to the single equation case, we demonstrate that the unique solution to this system of first-order partial difference equations also yields the Shapley value of the underlying cooperative game.

**Theorem 8.** Suppose that the labour-asset allocation is given by the vector \( \mathbf{n} \). If the solution to the partial differential equation system in (22) is feasible, then the equilibrium to the non-cooperative game has wages and profit given by the Shapley values of the underlying cooperative game, where profits are given by

\[
\hat{\pi}(\mathbf{n}) = \frac{1}{N+1} \sum_{i_1=0}^{n_1} \cdots \sum_{i_M=0}^{n_M} \left( \frac{n_1}{i_1} \cdots \frac{n_M}{i_M} \right) \pi(i_1, \ldots, i_M),
\]

and where \( N = \sum_{m=1}^{M} n_m \).

Theorem 8 indicates that Shapley values are the solution to a system of partial difference equations given by the system of equations in (22). As with the case of a single asset, we find it useful to characterize the firm’s profits given divisible labour. Analogous to Theorem 3, we proceed by solving the limiting differential equations.\(^{25}\)

**Theorem 9.** Suppose that the labour-asset allocation is given by \( \mathbf{n} \). If labour is infinitely divisible, then

\[
\hat{\pi}(\mathbf{n}) = \frac{1}{N} \int_{0}^{N} \pi(sa)ds,
\]

providing that \( \int_{0}^{N} (\pi(sa) - \pi(Na))ds \geq 0 \), where \( \pi(sa) = F(sa) - sa \cdot w' \), \( N = \sum_{m=1}^{M} n_m \), \( \alpha_i \equiv n_i/N \), and \( \alpha \equiv (\alpha_1, \ldots, \alpha_M) \).

The result looks very similar to the continuous version of the single-asset profit function in equation (10) except that the integral now proceeds along the diagonal of the hyper-cube from \( 0 \) and \( aN \).\(^{26}\) Equation (24) is the Aumann–Shapley value (for transferable utility games) of the firm (Aumann and Shapley (1974)). One can consider the heuristic of the firm getting its marginal contribution over random orderings of the firm and all employees. As labour becomes infinitely divisible, with probability approaching 1, the firm will be preceded in a random ordering by employees of types \( i \) and \( j \) in proportion with their population. While the proportion of different employees will thus be fixed in the limit, the total weight of employees preceding the firm will be uniformly distributed over \( [0, N] \). Thus the firm’s expected marginal contribution is consequently found by integrating

\(^{25}\) Alternatively, we could proceed by examining the limit of the discrete-game solution as labour becomes continuous. In notes available from the authors, Serge Resnick has demonstrated that the discrete probability distribution in equation (23) (i.e., the coefficient of \( \pi \) in the summation) converges in distribution to a uniform distribution along the diagonal of a hypercube defined by the vertices \( 0 \) and \( aN \), thereby obtaining the same result as in Theorem 9.

\(^{26}\) The proviso regarding the integral of the profit function guarantees the feasibility of the solution and is always satisfied at the firm’s optimum when neoclassical profits are quasi-concave.
its marginal contribution along the diagonal of the hypercube of employees assigned to the assets, as in equation (24) above.

Equation (24) yields a nice interpretation for how the nature of the problem of optimally allocating a given number of employees across different groups differs between an at-will and a neoclassical firm. For the latter, given $N$ total employees, the firm must choose $a$ to maximize

$$\pi(a_1 N, \ldots, a_M N) = F(a_1 N, \ldots, a_M N) - N \sum_{j=1}^{M} a_j w_j.$$ 

That is, the neoclassical firm compares final profit over all employee assignments. Equation (24), however, indicates that the at-will firm compares average profit along the vector connecting the origin to an assignment over all possible employee assignments:

$$\pi(a_1 N, \ldots, a_M N) = \frac{1}{N} \int_{0}^{N} \pi(a_1 s, \ldots, a_M s) ds.$$ 

Insofar as the optimal mix of employees (from the neoclassical firm's point of view) varies with firm scale, the allocational decision of the at-will firm will diverge from that of the neoclassical firm.

It is important to note that going from a discrete number of employees to a continuum, one economic effect not present in the single-asset case is lost. In the continuum, the firm's profit for a given employee assignment is precisely given by the weighted production (net wages) along the diagonal. With a finite number of employees, off-diagonal profit levels contribute to the firm's negotiated profit as well. While the weights given to the off-diagonal proportions fall off rapidly as one moves off the diagonal, in general, it is only in the limit that their contribution can be ignored. This limitation may be particularly important in some environments. For example, if near-diagonal elements vary greatly from the diagonal element with the same total number of employees—such as for Leontief technologies in which the employee proportions match the required input ratio—going to a continuum obscures the impact of this characteristic of the technology on bargaining outcomes. Thus, for the Leontief case, an employee has less scope in using his essentiality in the production function to hold up the firm in the continuum.

4. OPTIMAL HIRING, INPUT AND TECHNOLOGY DECISIONS OF THE FIRM

4.1. Single-asset case

We now turn to the question of optimal-labour decisions for our wage-negotiable firm. In particular, given our characterization of profits in Theorems 5 and 6, we examine how the hiring decision by an at-will firm differs from that of a neoclassical firm. We demonstrate that the distortion that is introduced by the bargaining is both significant and easily interpretable. In order to generalize results, in this section we augment the neoclassical profit function with a $k$-dimensional vector of non-labour "inputs" $x$. Thus, we consider the neoclassical profit function $\pi: \mathbb{N} \times \mathbb{R}_+^k \rightarrow \mathbb{R}_+$ for the discrete case and $\pi: \mathbb{R}_+ \times \mathbb{R}_+^k \rightarrow \mathbb{R}_+$ when labour is continuous. A simple interpretation of these inputs is that they represent different types of capital, with the $i$-th unit of capital costing $r_i$ per unit. In such an event, $\pi(n, x) = F(n, x) - u - x \cdot r'$, where $r$ is a vector given by $r \equiv (r_1, r_2, \ldots, r_k)$. 
Taking these inputs as fixed and allowing wages to be determined through bargaining, the at-will firm will choose inputs to maximize one of the following programmes,

$$\max_{n,x} \int_0^\prime \pi(s, x) \mu(s|n)ds, \text{ or } \max_{n,x} \sum_0^\prime \pi(i, x) \mu(i|n),$$  \hspace{1cm} (25)

where $\mu(s|n)$ and $\mu(i|n)$ are measures which satisfy the continuous and discrete elasticity conditions in (20), respectively, and are generated by our bargaining game.

In order to simplify notation in the discrete case, we define the asymmetric binary relation, $y(n)\doteq z(n)$ to indicate that $y(n) - z(n) > 0 \geq y(n+1) - z(n+1)$. When such a relationship holds, we say that $y$ equals $z$ at $n$ to within integer rounding over its argument. 

With this definition, we proceed with our characterization of the optimal employment levels. We denote the optimal labour and input decision for the neoclassical firm by $n^*$ and $x^*$, respectively, and for the at-will firm $\bar{n}^*$ and $\bar{x}^*$, respectively.

**Theorem 10.** Suppose that $\pi(n, x)$ is quasi-concave and $\bar{\pi}(n, x)$ has an interior optimum over $(n, x)$:

1. When labour is discrete, $\{\bar{n}^*, \bar{x}^*\} \in \arg \max_{n,x} \bar{\pi}(n, x)$ satisfies

$$\pi(\bar{n}^*, \bar{x}^*) = \bar{\pi}(\bar{n}^*, \bar{x}^*),$$

$$\sum_{j=0}^n \frac{\partial \pi(i, \bar{x}^*)}{\partial x_j} \mu(i|\bar{n}^*) = 0, \hspace{1cm} j = 1, \ldots, k.$$

2. When labour is continuous, $\{\bar{n}^*, \bar{x}^*\} \in \arg \max_{n,x} \bar{\pi}(n, x)$ satisfies

$$\pi(\bar{n}^*, \bar{x}^*) = \bar{\pi}(\bar{n}^*, \bar{x}^*),$$

$$\int_0^\prime \frac{\partial \pi(s, \bar{x}^*)}{\partial x_j} \mu(s|\bar{n}^*) ds = 0, \hspace{1cm} j = 1, \ldots, k.$$

Several remarks are in order. First, it is instructive to consider a simple case, where there are no non-labour input choices, $x$. Note that if bargaining power is evenly distributed (i.e., $\lambda = 1$), and therefore $\mu$ is the uniform density, then $\pi(\bar{n}^*) = \bar{\pi}(\bar{n}^*)$ is a restatement of an elementary result of microeconomics: Just as the marginal cost function of a standard firm cuts its average cost function at the latter’s minimum, here the marginal function (in this case $\pi$) intersects it corresponding average function (in this case $\bar{\pi}$) at the latter’s peak. Furthermore, given the quasi-concavity of the neoclassical profit function, the marginal function must cut the average function from above, and therefore the point of intersection lies to the right of the maximum of the marginal function. That is, $\bar{n}^* \geq n^*$: The at-will firm overhires relative to the neoclassical firm. Our result states that these

27. Note that $\mu(s|n)$ does not depend on $x$, as the previous section demonstrated that bargaining weights are determined by exogenously given bargaining power which is taken to be independent of non-labour inputs.

28. Over the infeasible range, the firm would actually have to pay $y > \bar{n}(n)$, and therefore these expressions overstate profits. However, the maximization problem is still correct: since the firm would not hire in the infeasible range even if wages were given by $\bar{n}(n)$ over this range, it certainly will choose not to do so with the higher wage of $y$. Such considerations are handled immediately in our extensive-form game by breakdown when net surplus is negative.

29. When we consider function with continuous arguments (e.g., $x$) in addition to labour it is understood that $\doteq$ applies only to the integer labour argument. Intuitively, if $y$ and $z$ were continuously and monotonically extended to $\bar{y}$ and $\bar{z}$ over the interval $[n, n+1]$, $y(n) \doteq z(n)$ or $z(n) \doteq y(n)$ if and only if $\bar{y}(m) = \bar{z}(m)$ for some $m \in (n, n+1]$. The first element of a pair which satisfies the binary relationship, $\doteq$, denotes the element which is greater at the argument $n$. 

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conclusions are true given any bargaining power (including those varying with \( n \)). In fact, it is precisely the elasticity condition of equation (20), which characterizes the weights generated by our bargaining game, that is needed to obtain the result that the marginal function cuts the weighted average function from above at its peak.

Additionally, the theorem implies that the at-will firm hires until employees' wages are driven down to the neoclassical wage, given by the outside option, \( w \) (or to within integer roundings on the number of employees for the discrete case). In the continuous case, because \( \pi(n^*, x) = \pi(n, x) \), we have \( F(n^*, x) - w n^* = F(n, x) - \tilde{w}(n^*) n^* \), and consequently, \( \tilde{w}(n^*) = w \). In the discrete case, \( \pi(n^*, x) \neq \pi(n, x) \), which implies \( F(n^*, x) - w n^* \neq F(n, x) - \tilde{w}(n^*) n^*, \) and consequently, \( \tilde{w}(n^*) \neq w \). The above theorem states that at the at-will firm's optimal level of labour (for any corresponding levels of \( x \)), the at-will and the neoclassical profits are equated (to within integer roundings on \( n \) when labour is discrete). It then follows that at this point wages must also be the same (once again to within integer roundings on \( n \) when labour is discrete), since in both cases profits are simply given by \( F(n, x) \) net wages. The neoclassical wages are fixed at \( w \) for all \( n \), so at this at-will optimal level of labour, the firm's wages must also equal \( w \) (to within integer roundings when labour is discrete). Summarizing, we have the following result.

**Corollary 1 (Over-employment.)** Under the conditions of Theorem 9, there is over-employment (i.e., \( n^* > n^* \)) and the internal wage is bid down to the outside option (i.e., \( \tilde{w}(n^*) = w \) for the discrete-labour setting or \( \tilde{w}(n^*) = w \) for the continuous-labour setting).

The second expression in both parts of Theorem 10 also has an interesting interpretation. The firm chooses to increase each asset until the marginal profits from that asset are 0. However, given the nature of the bargaining relationship with employees, the marginal profits that a firm realizes from increasing an asset, \( x_j \), is no longer given simply by marginal profits, \( \partial \pi(n, x)/\partial x_j \). Rather, the firm must take into account that due to bargaining, its profits (at any level of assets) are given by a weighted average of neoclassical profits from 0 to \( n \) workers (at the same asset level). Consequently, the firm's marginal profits (with respect to changes in assets) are also a weighted average of neoclassical marginal profits. We return to considerations of optimal capital choice briefly in Section 4.4.

Finally, the assumption of quasi-concavity in Theorem 10, while useful to eliminate problems of feasibility, is by no means necessary for the above results. Consider a simple example of non-quasi-concave profit, \( \pi(n) \), as an illustration of how one would otherwise proceed.

\[
\pi(n) = \begin{cases} 
100 - n & \text{if } n \geq 10, \\
-n & \text{if } n < 10.
\end{cases}
\] (26)

The underlying reservation wage is 1 and the production function yields a return of 100 whenever at least 10 workers are employed. Although quasi-concavity is not present, we can nonetheless solve for the stable outcome if we take care to consider the firm's optimal firing decisions. Certainly whenever the firm has fewer than 10 workers left, it pays the firm to shut down and fire the remaining workers. Thus, for a firm with 10 workers, the pairwise-bargaining equation is

\[
(100 - 10w(10)) - 0 = \tilde{w}(10) - 1,
\]
or \( \hat{w}(10) = \frac{101}{11} \). Over the region \( n > 10 \), the profit function is quasi-concave, and so the previous arguments apply with the exception that the initial conditions for \( \hat{w} \) are given at \( n = 10 \).

\[
\hat{w}(n) = \frac{(n-1)\hat{w}(n-1) + w}{n+1}.
\]

It is straightforward to calculate that the optimal hiring decision for the firm is \( \hat{n}^* = 43 > n^* = 10 \). More generally, whenever faced with a non-quasi-concave profit function, we can proceed as usual, providing we make allowances for jumps in labour demand over the regions that fail quasi-concavity. The stable wage profiles can then be pieced together from the component stable paths.

4.2. Frontload factors

The extent of the bargaining distortions under a given production technology can be characterized in a rather simple manner with a statistic which we call the frontload factor. Here and throughout this section, we suppress notationally the dependence of profits on capital decisions, \( x \), as this is not important for the main points we presently make. For analytic simplicity, we consider the case with even bargaining power, where \( \mu \) takes on the uniform distribution. It is straightforward to see that our concept of frontload factors generalizes to arbitrary bargaining powers by use of the appropriate weighting function, \( \mu \).

**Definition 4.** When labour is discrete, the frontload factor for technology \( F \) at \( n \) is defined by the statistic

\[
\gamma(F, n) \equiv 1 - \frac{1}{\pi(n)} \sum_{i=0}^{n} \frac{i}{n+1} \Delta \pi(i);
\]

When labour is continuous, the frontload factor for technology \( F \) at \( n \) is defined by,

\[
\gamma(F, n) \equiv 1 - \frac{1}{\pi(n)} \int_{0}^{n} \frac{s}{n} \pi'(s) ds.
\]

Intuitively, the frontload factor indicates to what extent margins of production are realized "up front" in early units of production, rather than as later units for a given \( n \). Noting that \( \Delta \pi(i) = \Delta F(i) - w \), it is clear that this statistic simply weights the margins of production in the manner that they enter into wages of the \( n \) combined employees, as in equation (3), with more weight on later margins of production. All other components of this statistic (i.e., netting out \( w \), normalizing by profits at \( n \), and subtracting all this from 1) serve to translate this statistic into a convenient and (as shown below) insightful form. Under such normalizations, note that the frontload factor at a given scale of production, \( n \), increases as more productivity is moved from later to earlier margins of production. Intuitively, one technology is frontloaded relative to the other when its higher marginal products are distributed more up "front" and less on the marginal employees; literally, the margins are frontloaded. As noted above, the introduction of asymmetric bargaining power affects the notions of frontloading in the obvious way.\(^{30}\) When \( \pi(n) > 0 \) and the underlying solution to (2) is feasible, we have \( \gamma \in [0, 1] \). For technology that exhibits constant returns to scale, the frontload factor is given by \( \frac{1}{2} \) at all levels of production.

\(^{30}\) Specifically, in the discrete case, \( i/(n+1) \) is replaced with \( \sum_{i=0}^{n} \mu(\ell n) \); in the continuous case, \( s/n \) is replaced with \( \int_{0}^{s} \mu(\ell n) dt \).
Theorem 11. In both the discrete and continuous settings,
\[ \hat{\pi} = \pi(n)\gamma(F, n). \]  

When labour is discrete, at the optimal choice of labour (for an interior optimum)
\[ 1 = \gamma(F, \hat{n}^*) ; \]
when labour is continuous, at the optimal choice of labour (for an interior optimum)
\[ \gamma(F, \hat{n}^*) = 1. \]

Proof. This result is easily seen by taking \( \hat{\pi}(n) \) and summing by parts for the discrete case, obtaining,
\[ \hat{\pi}(n) = \frac{1}{1 + n} \sum_{i=0}^{n} \pi(i) = \frac{1}{1 + n} \sum_{i=0}^{n} i\Delta\pi(i) = \pi(n)\gamma(F, n). \]

If instead labour is continuous, integrating \( \hat{\pi}(n) \) by parts implies that,
\[ \hat{\pi}(n) = \frac{1}{n} \int_{0}^{n} \pi(s)ds = \pi(n) - \frac{1}{n} \int_{0}^{n} s\pi'(s)ds = \pi(n)\gamma(F, n). \]

The result that at the optimal choice of labour the frontload factor equals 1 (to within integer rounding for the discrete case) follows immediately from Theorem 10, where the direction of the integer-rounding equality in the discrete case follows from recalling that for \( n < \hat{n}^* \), we have \( \hat{\pi}(n) < \pi(n) \).

As a consequence, profits from equally efficient technologies differ only in the degree to which they affect the frontload factor, \( \gamma \). Furthermore, this theorem indicates that at the optimal level of labour, the firm chooses labour precisely at the level which raises the frontload factor to 1 (or as close as possible in the discrete case). Another immediate consequence of the above theorem (i.e., of equation (27)) is the following.

Corollary 2 (Preference for frontloading). Fix \( n \) and suppose that \( F^1(n) = F^2(n) \). The at-will firm strictly prefers \( F^1 \) to \( F^2 \) at \( n \) if and only if \( F^1 \) is frontloaded relative to \( F^2 \) at \( n \).

A consequence of this corollary is that, for a given \( n \), an at-will firm will choose an inefficient technology, \( \pi^1 \), rather than another, \( \pi^2 \), (i.e., \( \pi^1(n) < \pi^2(n) \)) if the technology has sufficient frontloading relative to the efficient technology (i.e., \( \gamma(F^1, n)/\gamma(F^2, n) > \pi^2(n)/\pi^1(n) \)). Note that concave transformations imply frontloading and hence are preferred ceteris paribus; specifically, if \( F^1(0) = F^2(0) \), \( F^1(n) = F^2(n) \) and \( F^1 = g(F^2) \) for some increasing, concave function, \( g \), then \( F^1 \) is frontloaded w.r.t. \( F^2 \) at \( n \). However, frontloading is a weaker concept than concavity as it by no means implies the existence of a concave transformation.\(^{31} \)

31. The point that underlying concavity or convexity may greatly affect an individual's payoff from a firm has been noted in several papers. Skillman and Ryder (1993) demonstrate in a two-worker model that a firm may rationally choose a less efficient technology if the first margin is sufficiently high relative to the total return. In a related paper, Horn and Wolinsky (1988) show that a firm's preference for two unions rather than one grand union depends upon the concavity of the technology; Bolton and Scharfstein (1993) argue that an important determinant as to whether a firm will want to borrow from one creditor or two depends upon the returns to scale in the technology evaluated over two points. All papers insightfully note that the curvature of the underlying technology is key; by considering more general technologies, however, we are able to identify precisely the manner in which this curvature is important (i.e., frontloading).
4.3. **Multi-asset case**

We now turn to the labour decision of the firm under the environment of Section 3.2, where there is more than one asset to assign workers to (or identically, if there is more than one type of worker). As we showed above, in such a setting, the firm’s profits are given by Theorems 8 and 9 for the discrete and the continuous cases, respectively. For analytic simplicity, we confine our analysis below to dealing with the continuous case. For a number of illustrative discrete examples, which demonstrate the nature of some of the distortions that follow in such a multi-asset at-will environment, we refer the reader to our companion paper, Stole and Zwiebel (1996).

From Theorem 9, when labour is continuous, the firm in a multi-asset environment will choose total labour, \( N \equiv \sum_{m=1}^{M} n_m \), and labour allocation across assets, \( a \equiv (a_1, \ldots, a_M) \), where \( a_i \equiv n_i/N \), in order to maximize,

\[
\max_{N,a \in S^{M-1}} \tilde{\pi}(a_1 N, \ldots, a_M N) \equiv \frac{1}{N} \int_{0}^{N} \pi(a_1 s, \ldots, a_M s) ds,
\]

where \( S^{M-1} \) represents the \( M - 1 \) dimensional simplex. The following theorem characterizes the solution to this program.\(^{32}\)

**Theorem 12.** Suppose that \( \pi(aN) \) is quasi-concave and \( \tilde{\pi}(aN) \) has an interior optimum over \((a, N)\). Then the optimal \( \{\hat{a}^*, \hat{N}^*\} \) for the at-will firm satisfies

\[
\pi(\hat{a}_1^* \hat{N}^*, \ldots, \hat{a}_M^* \hat{N}^*) = \tilde{\pi}(\hat{a}_1^* \hat{N}^*, \ldots, \hat{a}_M^* \hat{N}^*),
\]

and

\[
\int_{0}^{\hat{N}^*} s \frac{\partial \pi}{\partial n_j} (\hat{a}_1^* s, \ldots, \hat{a}_M^* s) ds = \int_{0}^{\hat{N}^*} s \frac{\partial \pi}{\partial n_k} (\hat{a}_1^* s, \ldots, \hat{a}_M^* s) ds, \quad \forall j, k = 1, 2, \ldots, M. \quad (32)
\]

The result follows from the standard first-order conditions. Thus, just as in the single-asset case, labour is employed up to the point where profits for the at-will and the neo-classical firm are equated, and consequently, wages are given by the reservation wage. As in the single-asset case, the quasi-concavity of \( \pi \) guarantees that \( \pi \) intersects \( \tilde{\pi} \) once from above and, as a consequence, our solution is feasible.

Similarly, differentiating equation (24) with respect to \( a \), taking into account the constraint that \( a \in S^{M-1} \), immediately yields equation (32). This first-order condition in \( a \)

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32. It might at first appear that the outcome to our extensive-form game can depend on the ordering of workers in the multi-asset environment, if the firm begins with a labour configuration, \( a \), which is infeasible with respect to \( \psi \). Specifically, one might conjecture that the order in which employees line up will determine which types of workers are eliminated due to negative net surplus. This, however, is not true. To see this, first note that since there is a unique outcome in each bargaining subgame and a finite number of such subgames, it follows that there is a unique equilibrium for any ordering. From here, the independence of this outcome on the ordering is guaranteed by our bargaining-session timing: Future potential renegotiation sessions ensure that when in a bargaining session with an employee, the firm effectively calculates its outside option to take into account the optimal choice of workers to retain given the set of remaining workers, and subsequently fires any worker outside this optimal configuration (through breakdown in negotiations). After the last such breakdown, the firm faces a feasible configuration of employees, which is independent of the initial ordering, and renegotiates a final time through this remaining set of employees.
indicates that the allocation of employees to assets is chosen in order to equate the front-load factors of each asset. If one frontload factor is lower than the others, that asset’s labour allocation is increased in order to reduce the rents accruing to labour.

Integrating the objective function by parts reveals that

\[
\hat{\pi}(a_1N, \ldots, a_MN) = \pi(a_1N, \ldots, a_MN) - \frac{1}{N} \sum_{j=1}^{M} \int_{0}^{N} s a_j \frac{\partial \pi}{\partial n_j} (a_1s, \ldots, a_ms) ds. \tag{33}
\]

Combining equations (31)–(33), it is clear that \( N \) and \( a \) are chosen such that each term in the summation of equation (33) must integrate to 0. This is equivalent to stating that the frontload factor for each asset must equal one, just as in the single-asset case.

Note that because the first condition is identical to the condition stated for the single-asset case, we obtain qualitatively similar results for the choice of \( N \). In particular, if the underlying capital, \( x \), is held fixed, overemployment results for any given \( a \). While we obtain qualitatively similar results for the questions of hiring and technology choice, \( a \) represents a new dimension of potential distortion.

The neoclassical firm allocates labour to equate the marginal profits of labour across assets,

\[
\frac{\partial \pi}{\partial n_j} (a_1N, \ldots, a_MN) = \frac{\partial \pi}{\partial n_k} (a_1N, \ldots, a_MN),
\]

while the at-will firm chooses \( a \) so as to satisfy the weighted expectation of this expression with labour varying from 0 to \( N \). These choices for \( a \) generally are different as we demonstrate in the following section. Considering the discrete case instead of the continuous case, one obtains analogous results. In particular, to within integer roundings on the amount of labour hired, labour should be hired to the point at which the neoclassical and the at-will firms’ profits are equated, just as in the single-asset case. Thus, to within such roundings, wages will be driven down to reservation wages. Similarly, frontload factors across assets should be equalized and driven to 1, to within integer roundings on the number of employees hired for each asset.

4.4. Application to labour determination

We now turn briefly to a few simple applications of the preceding methodology in order to emphasize the usefulness of this approach. For extensive applications to other economic settings, see our companion paper, Stole and Zwiebel (1996). In particular, we investigate a firm’s labour choices prior to the intra-firm bargaining game in both the single-asset and multi-asset settings. In the latter environment, this can also be interpreted as the organizational decision of how to allocate labour across productive assets or the choice of which types of heterogenous employees to hire.

33. The notion of frontload factor is generalized to the multi-asset case in the obvious manner: for the continuous case, the frontload factor for asset \( m \) is given by,

\[
\gamma_m(F, aN) = 1 - \frac{1}{\pi(a_1N, \ldots, a_MN)} \int_{0}^{a_mN} s a_mN \pi(a_1N, \ldots, a_mN, s, a_{m+1}N, \ldots, a_MN) ds.
\]

A corresponding expression holds for the discrete case where integrals are replaced with summations.
Consider the following simple illustration of a discrete-labour setting where the asset yields a total output of 6 for one unit of labour and 7 for two or more units of labour:

\[ F(0) = 0, \quad F(1) = 6, \quad F(2) = F(3) = \cdots = 7, \]

and suppose the outside reservation wage is \( w = 2 \). The neoclassical firm would choose to hire \( n^* = 1 \) and profits would be \( \pi = 4 \). If a firm with non-binding contracts chose \( n = 1 \), the wage would be \( w(1) = \frac{1}{2}(6 + 2) = 4 \), and profits would be \( \bar{\pi}(1) = 6 - w(1) = 2 \). If this latter firm instead chose \( n = 2 \), wages would be driven down to \( w(2) = \frac{1}{3}(1 + 2) + \frac{2}{3}(6 + 2) = \frac{7}{3} \), and so profits would be \( \bar{\pi}(2) = 7 - 2w(2) = \frac{7}{3} > 2 \). It also straightforward to check that \( \pi(2) \neq \bar{\pi}(2) \), and so \( n^* = 2 \) is indeed the optimal hiring decision. Over-employment results, and the internal wage differs from the reservation wage by only integer rounding on the number of employees hired: \( \frac{7}{3} = w(2) = w = 2 \).

In the continuous setting, similar implications emerge. Let \( F(n) = 10\sqrt{n} \) and \( w = 1 \). A neoclassical firm would choose \( n^* = 25 \) units of labour and obtain a profit of \( \pi(25) = 25 \). An at-will firm instead chooses \( n^* = 44 \frac{5}{6} \), resulting in a profit of \( \bar{\pi}(45) = 22 \frac{2}{3} \). Graphically, we illustrate both \( \pi(n) \) and \( \bar{\pi}(n) \) in Figure 2. At the optimum for the wage-negotiating firm, \( \bar{\pi}(n^*) = \pi(n^*) \), the internal wage is bid down to the external wage, \( \hat{w}(n^*) = w \), through the firm's over-employment decision, \( n^* > n^* \).

In both examples, we have ignored the possible presence of additional input variables such as (at least partially) irreversible capital, \( k \), which the firm would also presumably choose before the intrafirm bargaining begins. As Theorem 10 indicates, the marginal product of such an input from the neoclassical firm's viewpoint is quite different from the marginal product of the input to the at-will firm; the latter derives a strategic effect as well as a direct effect from the choice of the input. When capital increases the marginal product of labour, \( \pi_{mk} > 0 \), the weighted average (over labour) of the neoclassical marginal product of capital, \( \sum_{k} \pi_k(i, k)\mu(i|n) \) or \( \int_{0}^{w} \pi_k(s, k)\mu(s|n)ds \), is less than the neoclassical

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34. It is important to note that the following implications for additional "capital" inputs rely on the assumption that such capital is in place and fixed when negotiations with employees occur. If instead capital could be adjusted after bargaining occurred (say due to a liquid capital market in which a unit of type \( t \) capital could be bought or sold any time at \( r_t \)), our results would be altered. In particular, all bargaining participants would realize that, upon an agreement in bargaining with \( n \) workers, the firm would always choose to adjust capital to satisfy \( x^*(n) = \arg \max_x \pi(n, x) \), and bargain accordingly. Consequently, we could replace \( \pi(n, x) \) with \( \bar{\pi}(n, x^*(n)) \) and proceed as in the case where there is no capital (or it does not adjust at all).
marginal product of capital, \( \pi_k(n, k) \). This in turn implies that for a given level of employment, the at-will firm underutilizes capital relative to the neoclassical firm due to the ability of labour to appropriate some of the returns from capital investment. When the reverse is true and capital is a substitute for labour (i.e., increases in capital lower the marginal return from labour due to automation, etc.), the at-will firm prefers a higher capital stock relative to the neoclassical firm for a given level of employment.\(^{35}\)

In the multi-asset setting, in addition to our standard over-hiring results, the at-will firm generally alters the levels of employment across assets relative to the neoclassical firm. To see this point, consider an at-will firm’s choice of its scale, \( N \), and its inter-asset allocation \( \alpha \) over two assets, asset 1 and asset 2. When technology is non-homothetic, there is generally inter-asset misallocation. That is, at the at-will firm’s optimal labour choice,

\[
\frac{\partial \pi}{\partial n_1} (\alpha_1 N, \alpha_2 N) \neq \frac{\partial \pi}{\partial n_2} (\alpha_1 N, \alpha_2 N).
\]

The at-will firm chooses \( \alpha \) so as to satisfy the weighted expectation of this expression with labour varying from 0 to \( N \). These choices for \( \alpha \) are generally different.

Consider a weaker variant of homotheticity which we refer to as \( \alpha \)-homotheticity.

\textit{Definition 5.} \( \pi(n) \) is \( \alpha \)-homothetic iff \( \pi_n(\alpha N)/\pi_n(\alpha N) \) is constant along \( \alpha \), for all \( N \).

Note that a homothetic function is \( \alpha \)-homothetic at \textit{every} vector, \( \alpha \). Using this definition, we have an immediate result following Theorem 12.

\textbf{Theorem 13.} Suppose that \( \bar{\pi}(n) \) is strictly concave in \( n \) and that \( \pi(n) \) is \( \alpha \)-homothetic at \( \bar{\alpha} \). Then \( \bar{\alpha} = \bar{\alpha} \). Furthermore, considering the optimal labour-allocation decision for a fixed number of employees, \( N \), \( \alpha^*(N) = \bar{\alpha}(N) \), \( \forall N \), only if \( \pi(n) \) is \( \alpha \)-homothetic at \( \bar{\alpha} \).

\textit{Proof.} \( \bar{\alpha} \)-homotheticity (together with the neoclassical optimality of \( \bar{\alpha} \)) implies that

\[
\frac{\partial \pi}{\partial n_j} (\bar{\alpha} s) = \frac{\partial \pi}{\partial n_k} (\bar{\alpha} s), \quad \forall s.
\]

This in turn satisfies equation (32) in Theorem 12 for any \( \bar{N}^* \). Given that \( \bar{\pi} \) is a strictly concave function, this characterizes the unique optimal \( \bar{\alpha} \). And conversely, equation (32) can only be satisfied at \( \bar{\alpha}^*(N) \) for all choices of \( N \) if the neoclassical labour-expansion path is linear in \( N \). \( \square \)

Intuitively, when a neoclassical firm’s labour-expansion path is linear through the origin, the at-will firm’s first-order condition in \( \alpha \) is satisfied pointwise in labour at the same proportional allocation. Thus, an \( \bar{\alpha} \)-homothetic firm does not introduce a labour-allocation distortion. However, the lack of \( \bar{\alpha} \)-homotheticity in underlying profits is likely to be common, and can yield distortions with important economic consequences. For

\(35\). Capital distortions and investment reversibility are considered rigorously in Stole and Zwiebel (1996). The above statement is about relative capital utilizations for a \textit{fixed} level of employment. Because the at-will firm typically employs more labour than the neoclassical firm, there is an additional labour effect. As such, a formal comparative statics treatment must consider the joint maximization problem more carefully.
example, upper-level and mid-level managers may affect production in a non-homothetic manner, leading to additional distortions in hiring across levels. In our companion paper, Stole and Zwiebel (1996), we show in such a setting that a lack of \( \alpha^* \)-homotheticity is natural and can lead to top-heavy hierarchies.

5. CONCLUDING REMARKS

We have presented a new methodology for studying the bargaining problem between the firm and its employees within a firm’s boundaries where labour contracts are unenforceable. In the context of a non-cooperative bargaining game which we develop, we have shown that the firm’s payoffs can be characterized either as an average of the neoclassical (i.e., non-bargaining) firm’s profits or the Shapley value of the corresponding cooperative game. Furthermore, we generalize our setting over several dimensions, yielding a weighted average of neoclassical profits as the firm’s payoff and underscoring the robust properties of our bargaining outcome. Given the peculiar contractual incompleteness of labour contracts, the resulting wages and profits under this large class of complete-information bargaining games distort the technological and organizational decisions facing the owner of the firm’s capital. Additionally, we are able to highlight the nature of these distortions, demonstrating that they may yield economic decisions that are both important and significantly distinct from those of the standard neoclassical firm.

We want to stress that the applicability of this new methodology is extensive. In our companion paper, we demonstrate the broad applicability of these results to questions of the organization. This paper serves both to demonstrate the utility of our framework, and to extend our results in the context of a number of potentially interesting applications. In particular, first we examine several examples of how employee bargaining affects labour and capital input choice. Here, among other things, we show that under specific production functions, input distortions take a particularly simple form, linear in the neoclassical firm’s decisions. We also demonstrate that a pre-hiring market game over the division of “hiring costs” can be explicitly added to our bargaining game in a straightforward and appealing manner. We then proceed to consider a number of applications in labour economics. These applications include an examination of firm-specific human capital, hiring decisions under reversible capital, the preference for unionization, and the manner in which competition (on the product market) can affect the magnitude of the labour distortion. While these applications deal with very distinct economic questions, they are related insofar as they all serve to highlight the utility of the frontload factor in describing and understanding the distortions due to bargaining. In particular, these examples all demonstrate that the magnitude and direction of the distortions depend crucially on the curvature of the production function, as measured by the frontload factor. Finally, we turn to several applications regarding organizational-design decisions of the “multi-asset” firm. Central to these applications are both distortions attributable to economies of scope across assets and to non-homotheticity in the neoclassical profit function. These applications include hierarchical design and capital budgeting. We also show that our framework is well suited to consider the important question of cross-training, whereby a firm may choose to train some of its employees on multiple tasks to lessen the hold-up ability of others. As a whole, this companion paper emphasizes how general results which follow from our bargaining game imply simple straightforward distortions to a wide range of organizational decisions. Taken together, these papers suggest that when temporarily-irreplaceable employees cannot be contractually tied to their firm, a careful consideration and analysis of the bargaining process implies important economic effects on the operation and organization of the firm.
APPENDIX A: PROOFS TO THEOREMS

Proof to Theorem 2. The proof proceeds via induction over the number of workers. First, suppose \(n=1\). Then our game reduces to the breakdown-risk game of BRW, where the breakdown value for the firm and worker are given by \(F(0)\) and \(w\), respectively. The unique subgame-perfect outcome for such a game is the Nash solution, which is precisely how the stable outcome is defined for only one employee.

Now suppose that for all \(n \leq k-1\), the bargaining game has a unique subgame-perfect equilibrium which coincides with the stable outcome, which is assumed to be feasible, and let there be \(k\) employees. Consider first the initial bargaining session between worker 1 and the firm. The pair are once again engaged in a BRW bargaining game, this time with breakdown values for the worker and the firm given by \(w\) and \(\bar{\pi}(n-1)\), respectively, where the firm’s breakdown value follows from the induction hypothesis. Consequently, given a positive joint surplus—which follows from feasibility—the unique subgame-perfect outcome to this bargaining session is for the joint surplus above these outside options to be split by the two, exactly as stability prescribes. Similarly, consider the bargaining session between any of the other \(k\) employees and the firm, presuming that all prior employees in the ordering have reached an agreement. Here, once again, the two sides are engaged in BRW bargaining. The employee’s outside option is still \(w\). If breakdown occurs, however, regardless of what the prior employees have already agreed to, our game specifies that all the remaining \(k-1\) employees renegotiate with the firm, starting with the first one. Once again, by the induction hypothesis, this implies that the firm’s breakdown value is given by \(\bar{\pi}(n-1)\). Splitting the difference relative to these outside options is once again identical to what stability prescribes.

Consequently, the unique subgame-perfect equilibrium is for all workers to reach an agreement with the firm immediately in their bargaining session, with precisely the wages prescribed by the criterion of stability. 

Proof to Theorem 5. We proceed by induction. Because \(\theta(0)=1\), we have \(\bar{\pi}(0)=\pi(0)\), and so \(\mu(0)=1\). Suppose that \(\bar{\pi}(k) = \sum_{i=0}^{k} \mu(i)\pi(i)\) where \(\mu(i)\) is a probability measure over \(\{0, 1, \ldots, k\}\). Then by induction, using (15),

\[
\bar{\pi}(k+1) = \theta(k+1)\pi(k+1) + [1 - \theta(k+1)] \sum_{i=0}^{k} \mu(i)\pi(i).
\]

Hence, we can write, \(\bar{\pi}(k+1) = \sum_{i=0}^{k+1} \mu(i|k+1)\pi(i)\), where \(\mu(\cdot|k+1)\) is given by the following probability measure, defined over \(\{0, 1, \ldots, k+1\}\):

\[
\mu(i|k+1) = \begin{cases} 
\theta(k+1) & \text{for } i = k+1, \\
(1 - \theta(k+1))\mu(i|k) & \text{for } i < k+1.
\end{cases}
\]  

(34)

From this expression, the definition of \(\theta\), and \(\theta(0)=1\), we can immediately deduce \(\mu(i|n)\) in (17).

Proof to Theorem 6. Letting labour units be of size \(h\), the first-order difference equation becomes:

\[
\bar{\pi}(n) - \bar{\pi}(n-h) = \lambda(n)(\bar{\pi}(n) - w)h.
\]

Taking the limit as \(h \to 0\) yields

\[
\bar{\pi}'(n) = \lambda(n)(\bar{\pi}(n) - w).
\]

Using the fact that \(\bar{\pi}(n) \equiv \pi(n) - n(\bar{\pi}(n) - w)\), we have

\[
\bar{\pi}'(n) = \frac{\lambda(n)}{n} (\pi(n) - \bar{\pi}(n)).
\]

To prove a unique solution exists, it suffices to prove \(\|\lambda(n)[\pi(n) - \bar{\pi}(n)]/n\|\) is bounded. The problem in satisfying this Lipschitz condition occurs at \(n=0\). Using L'Hôpital's rule, we need only show the expression

\[
\|\lambda'(0)[\pi(0) - \bar{\pi}(0)] + \lambda(0)[\pi'(0) - \bar{\pi}'(0)]\|
\]

is finite, which follows from our assumptions on \(\lambda\) and \(\pi\). As such, any solution to the differential equation such that \(\bar{\pi}(0) = \pi(0)\) provides the unique solution to our game.
It is straightforward to check that the \( \mu(s|n) \) given in equation (19) satisfies this equation. The continuity and boundedness of \( \lambda \) guarantees that \( \lambda(t)/t \) is integrable, and so \( \mu(s|n) \) is well-defined. Also note that

\[
\int_0^\infty \mu(s|n) ds = \lim_{r \to 0^+} \int_r^\infty \mu(s|n) ds = \lim_{r \to 0^+} \int_r^\infty \frac{\lambda(s)}{s} \exp\left(\int_r^s \frac{\lambda(t)}{t} dt\right) ds
\]

\[
= \lim_{r \to 0^+} \exp\left(-\int_r^\infty \frac{\lambda(t)}{t} dt\right) - 1 + \exp\left(-\int_r^\infty \frac{\lambda(t)}{t} dt\right).
\]

Because \( \lambda(0) > 0 \) and \( \lambda \) is continuous, there exists a \( \delta > 0 \) such that for all \( n \in [0, \delta] \), \( \lambda(n) > a > 0 \). Thus, \( \lim_{r \to 0^+} \int_r^\infty \frac{\lambda(t)}{t} dt \geq \lim_{r \to 0^+} \int_r^\infty \frac{(a/t)dt}{dt} = \infty \), and so \( \lim_{r \to 0^+} \int_r^\infty \frac{\lambda(t)}{t} dt = \infty \). Thus, \( \int_0^\infty \mu(s|n) ds = 1 \) and so \( \mu(s|n) \) is a probability measure.

Proof of Theorem 8. The proof proceeds as in Theorem 4 by showing that the non-cooperative equilibrium payoffs are equivalent to the Shapley value, and then using the Shapley formula to determine the actual non-cooperative payoffs.

First note that the partial difference equations which determine the non-cooperative equilibrium satisfy efficiency; that is, equation (12). Hence, we need only demonstrate balanced contributions to prove the equivalence with Shapley values. Because the solution to the equations exhibits symmetry for employees within a group, the contributions are balanced across any two workers on the same asset. Additionally, balanced contributions for any pairwise comparison involving the firm follows in a manner similar to the single-asset case. In particular, given that the game without the firm is inessential, assign payoffs to the employees of their outside options for any coalition that does not include the firm. This assignment, together with equation (22) (our split-the-difference criterion), ensures that the payout structure satisfies balanced contributions for any pair of agents that includes the firm.

Lastly, we must show that contributions are balanced between one worker assigned to asset \( i \) and another assigned to asset \( j \neq i \). This is equivalent to showing for any such pair and any \( n \), \( \Delta_i \tilde{v}_i(n) = \Delta_j \tilde{v}_j(n) \). For assets \( m = i, j \), differencing equation (22) yields

\[
\Delta_i \Delta_i \tilde{v}(n) = \Delta_i \tilde{v}_i(n),
\]

\[
\Delta_i \Delta_j \tilde{v}(n) = \Delta_j \tilde{v}_j(n),
\]

which in turn implies (using the distributive property of \( \Delta_n \)) that \( \Delta_i \tilde{v}_i(n) = \Delta_j \tilde{v}_j(n) \), as desired.

Having established the equivalence of Shapley values and the equilibrium payoffs of the game, it is straightforward to verify that equation (23) is the Shapley value of the firm.

Proof to Theorem 9. We first ignore issues of feasibility and show that the limit of the equilibrium partial difference equations generates the result in equation (24). Then we demonstrate that this solution is feasible.

Let the labour assigned to asset \( m, n_m \), be subdivisible into amounts of size \( h: n_m \equiv ih \). Taking the limit of equation (22) as \( h \to 0 \) yields

\[
\frac{\partial \bar{x}(n)}{\partial n_m} = \tilde{v}_m(n) - \bar{w}_m, \quad i = 1, \ldots, M.
\]

By definition \( \bar{x}(n) \equiv \bar{x}(n) - \sum_{m=1}^M n_m \tilde{w}_m(n) - \bar{w}_m \), and so substitution with our system of partial differential equations yields

\[
\bar{x}(n) = \pi(n) - \sum_{m=1}^M n_m \frac{\partial \bar{x}(n)}{\partial n_m}.
\]  

We first check that our proposed solution satisfies this partial differential equation; we then demonstrate uniqueness. Let \( N = \sum_{m=1}^M n_m \) and \( a(n) \equiv (n_1/N, \ldots, n_M/N) \). Then partial differentiation yields

\[
\frac{\partial \bar{x}(n)}{\partial n_m} = \frac{1}{N} (\pi(n) - \bar{x}(n)) + \frac{1}{N} \sum_{s=1}^N \nabla \bar{x}(a(n)s) \cdot \left[ \frac{\partial a_s(n)}{\partial n_m} \right]_{s=1}^M ds.
\]
Simplifying,
\[ \frac{\partial \tilde{\pi} (a)}{\partial n_m} = \frac{1}{N} \left( \pi(n) - \tilde{\pi}(n) \right) + \frac{1}{N} \sum_{s=0}^{N} \left( \frac{\partial \tilde{\pi}(a(n) s)}{\partial n_m} - \tilde{\pi}(a(n) s) \cdot a(n) \right) ds. \]

Multiplying by \( n_m \) and summing up across all assets, \( m = 1, \ldots, M \), and simplifying yields
\[ \sum_{m=1}^{M} \frac{\partial \tilde{\pi}(n)}{\partial n_m} = \pi(n) - \tilde{\pi}(n), \]
satisfying the required partial differential equation.

To show uniqueness, suppose that both \( \tilde{\pi}^1 \) and \( \tilde{\pi}^2 \) are solutions to equation (35) above and \( \tilde{\pi}^1(0) = \tilde{\pi}^2(0) = \pi(0) \). Define \( \phi(n) \equiv \tilde{\pi}^1(n) - \tilde{\pi}^2(n) \). Then \( \phi(0) = 0 \) and substituting out \( \pi(n) \) yields
\[ \phi(n) = -n \overline{\nabla} \phi(n). \] (36)

Certainly \( \phi(n) \equiv 0 \) is a solution. We prove uniqueness by converting our problem to an ordinary differential equation and demonstrating that there can be no other solution. Select any point \( n' \neq 0 \). Define for all \( t \in \mathbb{R}, \ t > 0 \), the function
\[ \gamma(t) \equiv \phi(t n'). \]

Differentiating with respect to \( t \),
\[ \gamma'(t) \equiv n' \overline{\nabla} \phi(t n'), \quad \forall t > 0. \]

Using equation (36) evaluated at \( tn' \) we have
\[ t \gamma'(t) \equiv -\phi(t n'), \quad t > 0, \]
or simplifying with our definition of \( \gamma(t) \),
\[ t \gamma'(t) \equiv -\gamma(t), \quad t > 0. \]

This is an ordinary differential equation in \( t \) with solution, unique up to a constant \( k \), \( \gamma(t) \equiv k/t \). At \( t = 1 \), \( \gamma(1) = \phi(n') \), and so the particular solution is given by \( \gamma(t) \equiv \phi(n')/t \). Substituting into equation (36) yields
\[ t \phi(t n') \equiv \phi(n'), \quad \forall t > 0, \quad \forall n'. \]

Suppose that \( \phi(n) \neq 0 \). Then there exists a point, \( n' \), such that \( |\phi(n')| > \varepsilon \). By continuity of \( \phi \) and our initial condition that \( \phi(0) = 0 \), there exists a sufficiently small \( t > 0 \) such that \( t \phi(t n') < \varepsilon \), yielding a contradiction. Thus, \( \phi(n) \equiv 0 \) is the only solution, which implies that our original solution is unique.

We now demonstrate feasibility of equation (24). For a given mesh of size \( h \), a vector, \( n \), on this lattice and an underlying production function, \( F \), define \( \tilde{\pi}_A(n, F) \) from the discrete solution of (23) to the system of partial difference equations in (21), unconstrained by feasibility. Let \( \tilde{w}_0(n, F) \) be the corresponding wage (which may be less than \( \overline{w} \)). (We have shown above that \( \lim_{h \to 0} \tilde{\pi}_A(n, F) \) is given by the expression in (24).) Define the set of labour for which \( \tilde{w}_0(n, F) \geq \overline{w} \):
\[ \mathcal{A}_0(F) \equiv \{ n \in \mathbb{R}^m : \tilde{w}_0(n, F) \geq \overline{w} \}. \]

Now construct a new production function, \( F_A^0(\cdot) \), as follows. Let \( F_A^0(n) = F(n) \) on the set \( \mathcal{A}_0(F) \), and elsewhere choose \( F_A^0(n) \) to be as small as possible while maintaining \( \tilde{w}_0(n, F_A^0) \geq \overline{w} \). (It is straightforward to show that such a function is uniquely defined by proceeding inductively on the lattice in a North-East direction from the origin.) Lastly, let \( \tilde{\pi}^0 \) and \( \tilde{w}^0 \) represent the true solution to the bargaining game, explicitly taking into account feasibility issues.

By construction \( \tilde{w}_0(n, F_A^0) \geq \overline{w} \) for all \( n \in \mathbb{R}^m \). Thus, \( \tilde{w}_0(n, F_A^0) = \tilde{w}^0(n, F_A^0) \) for all \( h \). Because \( F_A^0 \) favours workers over \( F \) by raising the unconstrained solution to the partial difference equations outside the set \( \mathcal{A}_0(F) \), we also have \( \tilde{w}^0(n, F_A^0) \geq \tilde{w}^0(n, F) \) for all \( h \). And because the constraint of feasibility can only increase the worker's wages we have \( \tilde{w}^0(n, F) \geq \tilde{w}_0(n, F) \) for all \( h \). Thus, for all \( h \),
\[ \tilde{w}_0(n, F_A^0) = \tilde{w}^0(n, F_A^0) \geq \tilde{w}^0(n, F) \geq \tilde{w}_0(n, F). \] (37)

Now recall from equation (23) that \( \tilde{\pi}(n) \) is an expectation of \( \pi(n) \) over the hypercube defined by the vertices \( 0 \) and \( n \). From our initial argument above, we know that as \( h \to 0 \), this probability distribution converges to a uniform distribution along the hypercube's diagonal. The assumption that \( \int_0^1 (\pi(s a) - \pi(N a)) ds \geq 0 \) and the quasi-concavity of \( \pi \), implies that \( n \in \lim_{h \to 0} \mathcal{A}_0(F) \). Thus, in the limit the distribution's support is contained in the set \( \lim_{h \to 0} \mathcal{A}_0(F) \), on which \( F_A^0 = F \). As a consequence, \( \lim_{h \to 0} \tilde{w}_0(n, F_A^0) = \lim_{h \to 0} \tilde{w}_0(n, F) \). Together with
equation (37), this in turn implies that \( \lim_{h \to 0} \tilde{w}^s(n, F) = \lim_{h \to 0} \tilde{w}_i(n, F) \). Therefore, the wage profile \( \tilde{w}_i(n, F) \) satisfies feasibility, and the associated profit function given by the expression in (24) for \( \lim_{h \to 0} \tilde{w}_i(n, F) \) provides the correct equilibrium payoff to the firm.

**Proof to Theorem 10.** First we consider the continuous case. Differentiating the first expression in equation (25) with respect to \( n \) yields the first-order condition,

\[
\pi(\tilde{n}^s, x)\mu(\tilde{n}^s|\tilde{n}^s) + \int_0^{\tilde{n}^s} \pi(s, x)\mu_s(s|\tilde{n}^s)ds = 0.
\]

Now since we have assumed that the measure \( \mu(s|n) \) satisfies our elasticity condition (20) that makes it consistent with our bargaining game, i.e., that the elasticity of \( \mu(s|n) \) with respect to \( n \) is given by \(-\lambda(n)\), which is independent of \( \mu \)'s first argument \( s \), it follows that,

\[
\pi(\tilde{n}^s, x) - \int_0^{\tilde{n}^s} \pi(s, x)\frac{\lambda(\tilde{n}^s)\mu(s|\tilde{n}^s)}{\tilde{n}^s\mu(\tilde{n}^s|\tilde{n}^s)}ds = 0.
\]  \( (38) \)

Furthermore, since \( \mu(\cdot|n) \) is a measure for all \( n \), differentiating with respect to \( n \) implies that

\[
\int_0^n \mu_s(s|n)ds + \mu(n|n) = 0.
\]

Solving for \( \mu(n|n) \) and imposing the elasticity condition (20), it follows that for any family of measures \( \{\mu(\cdot|n)\}_n \) which satisfy our elasticity condition, we have

\[
\mu(n|n) = \frac{\lambda(n)}{n}, \quad \forall n.
\]

Substituting this expression into (38) immediately implies the first expression to be proven in the continuous case.

The second expression for the continuous case follows immediately from the first-order conditions for \( x \). Furthermore, quasi-concavity of the neoclassical production function implies that, for all levels of labour less than \( \tilde{n}^s \), neoclassical profits exceed at-will profits, implying that the at-will wages must exceed \( \psi \) for all such profiles; as a consequence, feasibility is satisfied at \( \tilde{n}^s \). Finally, our assumption of quasi-concavity, together with our assumption of an interior solution, ensures that satisfying these first-order conditions is sufficient to provide a maximum to the programme.

Turning to the discrete case, note that the at-will firm's profits (the second expression in equation (25)) is once again given by the weighted average of \( \pi(n) \) over \( i = 0, 1, \ldots, n \), where weights are given by \( \mu(\cdot|n) \), whose elasticity with respect to \( n \) is taken to be independent of \( i \), satisfying elasticity property (20). From this, as in the continuous case, given that \( \pi(0, x) = \pi(1, x) \) and \( \pi \) is quasi-concave (and therefore single-peaked in \( n \)), it follows that the maximum of \( \pi \) occurs uniquely where the marginal function (\( \pi \)) intersects the average function (\( \bar{\pi} \)); i.e., where \( \pi(\tilde{n}^s, x) = \bar{\pi}(\tilde{n}^s, x) \). The second expression is once again simply the first-order conditions with respect to \( x \), which are sufficient for a maximum. \( \| \)

**APPENDIX B: WAGE NEGOTIATIONS WHEN OUTSIDE OPTIONS ACT AS CONSTRAINTS**

In this Appendix we address an alternative specification of the bargaining game in which outside options do not directly enter as threat points, only as constraints on equilibrium outcomes. Shaked and Sutton (1984) were one of the first to present a game with outside options that highlight this property. As an example, the standard Rubinstein bargaining game of alternating offers with discounted payoffs (but without any probability of breakthrough) gives rise to a unique equilibrium which satisfies this condition. With equal discount factors and arbitrarily short periods of offers, the equilibrium outcome is to split the gross surplus, assuming the gross surplus exceeds the sum of the outside options, unless such a split would result in a payoff below one player's option. In such a case, this player gets precisely his outside option while the other receives the residual surplus. This outcome is different from the central model of our paper where the players split the net surplus, taking outside options directly into account, rather than treating them solely as a constraint.
For the purposes of this Appendix, we assume equal bargaining power between the firm and the workers and we make the following technological assumptions: labour is continuous; \( F \) is non-decreasing; \( F'(0) \geq 2w \). For notational simplicity, we introduce the outside wage into our previous definitions. Thus,

\[
\pi(n, w) = F(n) - wn,
\]

\[
\hat{\pi}(n, w) = \frac{1}{n} \int_0^n \pi(s, w) ds,
\]

\[
\hat{\nu}(n, w) = w + \frac{1}{n} [\pi(n, w) - \hat{\pi}(n, w)].
\]

As a preliminary result, note that for any \( w \) our assumptions guarantee that \( \hat{\nu}(n, 0) \) is continuous, \( \hat{\nu}(0, 0) = \frac{1}{4} F'(0) > w \) and \( \lim_{n \to \infty} \hat{\nu}(n, 0) = 0 < w. \) By the intermediate value theorem, there exists a level of labour \( \hat{n} \in (0, \infty) \) such that \( \hat{\nu}(\hat{n}, 0) = w. \) We additionally assume that this solution is unique and let \( \hat{\nu}(w) \) characterize its value as a function of \( w. \) As a consequence, for \( n < (>) \hat{n}(w) \) we have \( w < (>) \hat{\nu}(n, 0). \) This allows us to restrict our attention to two regions. For region 1, \( n \leq \hat{n} \), the reduced form of the at-will firm’s profit function is simply \( \hat{\pi}(n, 0). \) For region 2, \( n > \hat{n} \), the outside option is a binding constraint, and so the bargained wage is equal to \( w. \) In this region the firm’s negotiated profit is the same as the neoclassical firm’s: \( \pi(n, w). \) As a consequence, over the entire domain, profits are given by

\[
\hat{\pi}(n, w) = \min\{\hat{\pi}(n, 0), \pi(n, w)\},
\]

where the “hat” notation indicates the negotiated profits of a firm in our alternative outside-option-constraint setting. Intuitively, the firm operating in this environment behaves as an at-will firm facing outside wage of 0 until the actual outside option of \( w \) binds (i.e., when \( n = \hat{n} \)), and then operates as a neoclassical firm, unable to alter wages through additional hiring. Whether the firm wishes to add labour at this outside option of \( w \) once it binds, depends on whether \( \hat{n} \) exceeds the neoclassical optimum \( n^*. \) In particular, two cases are possible, as Figure 3 demonstrates.

Case 1: No Distortion on Employment. If \( \hat{n} \) is less than the neoclassical optimum, \( \hat{n}(w) < n^*(w) \), the neoclassical optimum lies on \( \hat{\pi} \), and so \( \hat{n}^* = n^* \). This is indicated in Figure 3. Here the underlying production function is \( F(n) = 25n - n^2 \) for \( n \leq \frac{5}{2} \), and \( F(n) = \frac{25}{4} n^2 \) for greater levels of labour (i.e., labour has no additional value). An outside wage of \( w = 8 \), the outside option starts to bind at \( \hat{n}(8) \), labelled by point A. This is less than the neoclassical optimum \( n^* \) given by point B, to which point the firm would choose to continue to hire. Consequently, the optimal choice of labour is \( \hat{n} = n^* \).

Case 2: Over-employment. If \( \hat{n} \) exceeds the neoclassical optimum, \( \hat{n}(w) > n^*(w) \), the firm optimally chooses \( \hat{n}^* \) at the point \( \hat{n}(w) \) where \( \hat{\pi}(n, 0) = \pi(n, w) \). This optimality condition, \( \hat{\pi} = \pi \), is analogous to the central setting of our paper (i.e., outside-option-threat-points), with most of our results in this setting ensuing. The only difference is that now the condition no longer represents a first-order condition; rather it is an equation defining a kink. Most importantly, note that in this case, over-employment emerges. Returning to Figure 3 below, when the outside wage is instead \( w = 2 \), the value of \( \hat{n} \) given by point D exceeds the neoclassical optimum given by point C, and therefore the firm chooses to over-employ at point D.

The conditions for strict over-employment are difficult to ascertain (i.e., whether the firm is in case 1 or 2). Fundamentally, they must depend upon the underlying technology \( F \) and outside option \( w \). As the above example indicates, even simple quadratic production with reasonable parameters can give rise to both cases, depending upon the outside wage. Note, however, that weak over-employment (i.e., over-employment in case 2 and optimal employment in case 1) is robust to the specification of the outside option.

Because the underlying \( \hat{\pi} \) function depends on the infra-margins over the region \( [0, \hat{n}] \), similar intuition for frontloading continues to hold in the outside-option-constraint framework. An additional effect, however, is that for sufficiently frontloaded technology, the firm’s negotiated profit function is curved to case 1 and no over-employment results. Thus, when facing two technologies, the firm may choose the more frontloaded (albeit less efficient) technology in order to be in case 1 rather than case 2. When technological choice is a continuous variable (such as with the choice of capital), and the firm finds itself in region 2 when evaluating its choices at the neoclassical levels, we generally expect a firm to distort on each input dimension in order to maximize

36. This translates to equal discount factors between the firm and the workers in the pairwise subgames, and letting the time interval for offers and counteroffers shrink to zero.

37. If we wish to relax the uniqueness assumption concerning \( \hat{n} \), we would have more than two regions of the domain to consider. The results will have a similar flavour, but with multiple intervals where the outside option binds. The technical aspects of this problem are closely related to the central model of this paper when feasibility is not guaranteed and the outside option can subsequently bind over multiple unconnected intervals.
negotiated profit. As a consequence, except for our result concerning the Shapley value, the intuitions developed in this paper are robust across the two outside-option settings.

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