Higher-Moment Risk

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Time Variation in Higher-Moment Risk

We use option prices to estimate perceived physical distribution of S&P 500 returns in real time.

**New stylized fact:** Higher-moment risk increases when variance decreases

<table>
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<tr>
<th>Low variance</th>
<th>High variance</th>
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<td>Lower skewness</td>
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⇒ Higher-order moments ”riskier” when variance is low
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⇒ Higher-order moments "riskier" when variance is low

**Emphasize two implications:**

1. Broadly inconsistent with classical disaster models
2. Tail-risk of volatility-targeted strategies spikes in good times when variance is low
Forward-Looking Probabilities from Asset Prices

- The standard asset pricing formula gives us the relation

\[
\text{price of a claim} = \sum_{s \in \text{states}} \text{payoffs}_s \times \text{SDF}_s \times \text{probabilities}_s
\]  

\[= E_t[X_{Tm_t,T}] \]  

- From options prices, we can find the product of the SDF and probabilities (risk-neutral probabilities)

- We decompose these following Martin (2017):
  - We make assumptions about the sdf \( m_{t,T} \)
The standard asset pricing formula gives us the relation

$$\text{price of a claim} = \sum_{s \in \text{states}} \text{payoffs}_s \times \text{SDF}_s \times \text{probabilities}_s$$  \hspace{1cm} (1)

$$= E_t[X_{Ttm_t,T}]$$  \hspace{1cm} (2)

From options prices, we can find the product of the SDF and probabilities (risk-neutral probabilities)

We decompose these following Martin (2017):

- We make assumptions about the sdf $m_{t,T}$

  $$m_{t,T} = kR_{t,T}^{-\gamma}$$  \hspace{1cm} (3)

- The SDF of an unconstrained and rational power utility investor who chooses to be fully invested in the market
Physical Moments from Asset Prices

- Given the stochastic discount factor and some algebra

\[ E_t[R^n_{t,T}] = \frac{E^*[R^{n+\gamma}_{t,T}]}{E^*[R^\gamma_{t,T}]} \]  

(4)

- Breeden and Litzenberger (1978):
  - Using a continuum of out-of-the-money option prices written on the market, we can evaluate the risk-neutral expectation of \( R^{n+\gamma}_{t,T} \)

- These raw moments can then be converted into standardized moments, e.g. skewness:

\[ \text{Skewness}_{t,T} = \frac{E_t[R^3_{t,T}] - 3E_t[R_{t,T}]E_t[R^2_{t,T}] + 2E_t[R_{t,T}]^3}{(E_t[R^2_{t,T}] - E_t[R_{t,T}]^2)^{3/2}} \]  

(5)
Estimating physical higher order moments

Time series plots

- We choose $\gamma = 3$ (Bliss and Pangirtzoglou, JF 2004)
Principal Components of Higher-Order Moments

- Map higher order moments into a single measure of risk

<table>
<thead>
<tr>
<th></th>
<th>Skewness</th>
<th>Kurtosis</th>
<th>5th moment</th>
<th>6th moment</th>
<th>Variation explained</th>
</tr>
</thead>
<tbody>
<tr>
<td>PC 1</td>
<td>−0.46</td>
<td>0.52</td>
<td>−0.52</td>
<td>0.50</td>
<td>91%</td>
</tr>
<tr>
<td>PC 2</td>
<td>−0.84</td>
<td>−0.04</td>
<td>0.24</td>
<td>−0.48</td>
<td>8%</td>
</tr>
<tr>
<td>PC 3</td>
<td>−0.28</td>
<td>−0.80</td>
<td>−0.03</td>
<td>0.53</td>
<td>1%</td>
</tr>
<tr>
<td>PC 4</td>
<td>0.06</td>
<td>−0.30</td>
<td>−0.82</td>
<td>−0.48</td>
<td>0%</td>
</tr>
</tbody>
</table>

- The first principal component explains 91% of variation in higher order moments

- Higher-Moment Risk Index:
  \[
  \text{HRI} = \text{PC 1}
  \]
Variance and Higher-Moment Risk are Negatively Correlated

\[ \text{cor} = -0.60 \]
CAY and Higher-Moment Risk are Negatively Correlated

\[ \text{Higher-Moment Risk Index} \]

\[ \text{Cay} \]

\[ \text{cor} = -0.50 \]
Past Returns and Higher-Moment Risk are Positively Correlated

\[
\text{Past two year return
}\]

\[
\text{Higher-moment risk index}
\]

\[
cor = 0.26
\]
Implications for Asset Pricing Models
Higher-Moment Risk in Asset Pricing Models

Log-normal models: Distribution is *positively* skewed

Disaster models: Disaster-risk makes return distribution left-skewed
- More disaster risk = more left-skewed return distribution
- But, more disaster risk = higher variance
  ⇒ Inconsistent with the empirical evidence

More generally, the disaster risk is cyclical
- Bad times (low prices) occur because of high disaster risk
- In bad times, skewness should thus be lower and kurtosis higher
- We see the opposite empirical pattern (higher-moment risk lower in bad times)
Higher-Moment Risk in Gabaix (2012)

We simulate the model and study how risk-neutral skewness and kurtosis relate to risk-neutral variance
Implications for Investors
Proposition 1: Real Time Tail-Loss Probabilities

For the unconstrained rational power utility investor who wants to hold the market, the conditional physical probability that market return from time $t$ to $T$ is lower than $\alpha$ is:

\[
P_t(R_t, T < \alpha) = \frac{R_{t,T}^f}{E_t^*[R_{t,T}^\gamma]} \left[ \alpha^\gamma \text{put}'_{t,T}(\alpha S_t - D_t, T) - \frac{\gamma}{S_t} \alpha^{\gamma-1} \text{put}_{t,T}(\alpha S_t - D_t, T) ight. \\
\left. + \int_{0}^{\alpha S_t - D_t, T} \frac{\gamma(\gamma - 1)}{S_t^2} \left( \frac{K + D_{t,T}}{S_t} \right)^{\gamma-2} \text{put}_{t,T}(K) \, dK \right]
\]

where $\text{put}'_{t,T}(\alpha S_t - D_{t,T})$ is the first derivative of the put option price with respect to the strike price.
Real-Time Tail Risk For Constant-Notional Investor

What is the probability of loosing $3\sigma \approx 24\%$

Takeaways:
- Higher moment risk constitute large fraction of tail risk
- Must variation for constant notional investors is driven by variance
Implications for Investors

**Volatility-Targeting Investors**

Many professional investors have volatility targets

- The targets help control risk
- But do not hedge all risk because higher-order moments also vary

Exposed to higher-moment risk

- Because volatility is constant, the variation in higher-moment risk shows up directly in their portfolios
- The investors lever up portfolios when volatility drops, but these are periods where higher-moment risk is high

Invest $\omega_{t,T} = \bar{\sigma}/\sigma_t$

\[
P_t(r_{t,T}^{i, \text{shock}} < -\bar{\sigma}) = P_t (\omega_{t,T} (R_{t,T} - E_t[R_{t,T}]) < -\bar{\sigma})
= P_t (R_{t,T} - E_t[R_{t,T}] < -\sigma_t)
\] (6)

(7)
Real Time Tail Risk of Volatility Targeting Investor

Takeaways:

- Despite targeting a constant volatility, risk varies substantially over time
- Relatively high before the financial crisis
- Similar results for 3sigma event (from 0.5% to 1.6%)
Alternatives to Volatility Targeting

Instead of targeting volatility, investors can target tail risk:

- Example: the probability of a 25% loss over the next quarter should be 1.17%
- Our framework allows for estimating the required market weight in real time

How does this portfolio look relative to the volatility-targeting portfolio?
Alternatives to Volatility Targeting

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Takeaways:

- Volatility-targeting ”over-invests” in the market pre crisis and post crisis
- But underinvests during crisis
Related Literature

- Large literature on tail risk can broadly be categorized into two groups:
  
  1) physical moments based on backward looking information (e.g. Bollerslev and Todorov (2011), Kelly and Jiang (2014))

      → we estimate forward-looking moments

  2) risk-neutral moments based on forward looking information in option prices (e.g. Siriwardane (2015), Bates (2000), Schneider and Trojani (2017))

      → we estimate physical moments

- Further differ by emphasizing
  
  - the relation between variance and higher-order moments
  - implications for investors and asset pricing models
Conclusions

New stylized fact: Higher-moment risk is high when variance is low

We highlight two implications:

1. Inconsistent with consumption based AP models
   - Log-normal models cannot generate the correct unconditional shape
   - Disaster models get the time variation wrong

2. Important implications for investors
   - Higher-order moments responsible for large fraction of tail risk
   - Tail risk for volatility-targeting investors spikes when variance is low