Higher-Moment Risk

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Time Variation in Higher-Moment Risk

We use option prices to estimate perceived physical distrubtion of S&P 500 returns in real time.

New stylized fact: Higher-moment risk increases when variance decreases

Low variance	High variance
Lower skewness	No skewness
Higher kurtosis	No excess kurtosis

⇒ Higher-order moments ”riskier” when variance is low

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Emphasize two implications:

1. Broadly inconsistent with classical disaster models
2. Tail-risk of volatility-targeted strategies spikes in good times when variance is low

Forward-Looking Probabilities from Asset Prices

- The standard asset pricing formula gives us the relation

$$\text{price of a claim} = \sum_{s \in \text{states}} \text{payoffs}_s \times \text{SDF}_s \times \text{probabilities}_s \tag{1}$$

$$= E_t[X_T m_{t,T}] \tag{2}$$

- From options prices, we can find the product of the SDF and probabilities (risk-neutral probabilities)

- We decompose these following Martin (2017):
  - We make assumptions about the sdf $m_{t,T}$

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$$m_{t,T} = kR_{t,T}^{-\gamma} \tag{3}$$

  - The SDF of an unconstrained and rational power utility investor who chooses to be fully invested in the market

Physical Moments from Asset Prices

- Given the stochastic discount factor and some algebra

$$E_t[R^n_{t,T}] = \frac{E^*_t[R^{n+\gamma}_{t,T}]}{E^*_t[R^{\gamma}_{t,T}]} \tag{4}$$

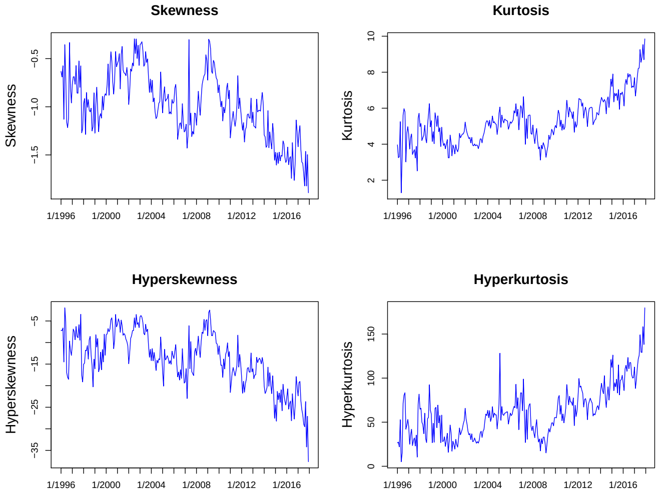
- Breeden and Litzenberger (1978):
  - Using a continuum of out-of-the-money option prices written on the market, we can evaluate the risk-neutral expectation of $R^{n+\gamma}_{t,T}$

- These raw moments can then be converted into standardized moments, e.g. skewness:

$$\text{Skewness}_{t,T} = \frac{E_t[R^3_{t,T}] - 3E_t[R_{t,T}]E_t[R^2_{t,T}] + 2E_t[R_{t,T}]^3}{(E_t[R^2_{t,T}] - E_t[R_{t,T}]^2)^{3/2}} \tag{5}$$

Estimating physical higher order moments
Time series plots

- We choose $\gamma = 3$ (Bliss and Pangirtzoglou, JF 2004)



Principal Components of Higher-Order Moments

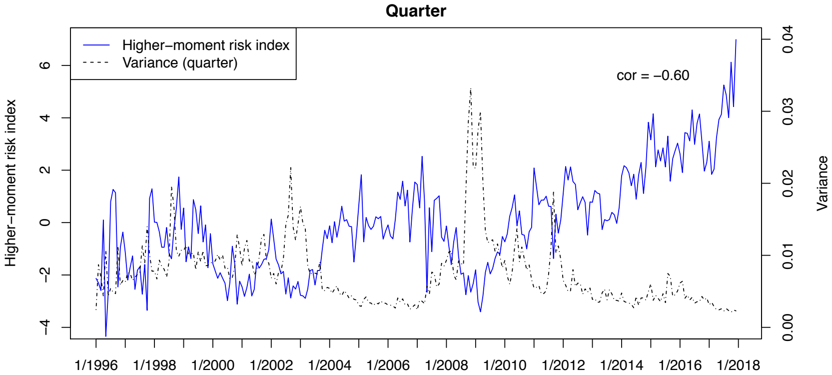
- Map higher order moments into a single measure of risk

	Skewness	Kurtosis	5th moment	6th moment	Variation explained
PC 1	−0.46	0.52	−0.52	0.50	91%
PC 2	−0.84	−0.04	0.24	−0.48	8%
PC 3	−0.28	−0.80	−0.03	0.53	1%
PC 4	0.06	−0.30	−0.82	−0.48	0%

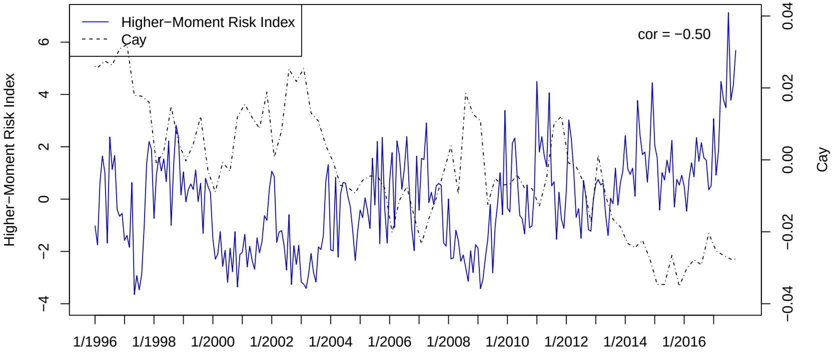
- The first principal component explains 91% of variation in higher order moments
- Higher-Moment Risk Index:

$$\text{HRI} = \text{PC 1}$$

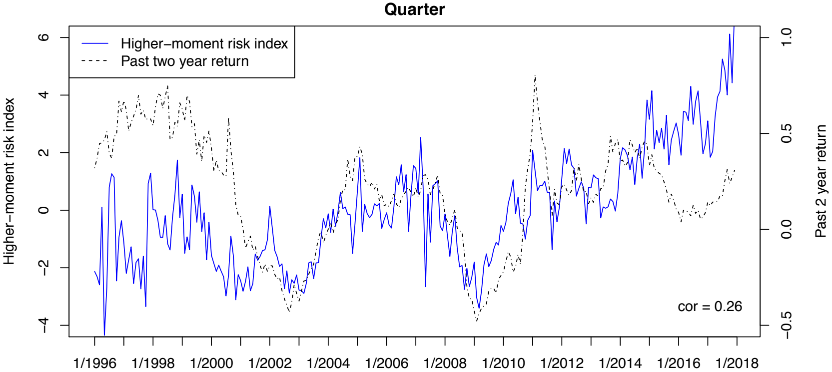
Variance and Higher-Moment Risk are Negatively Correlated



CAY and Higher-Moment Risk are Negatively Correlated



Past Returns and Higher-Moment Risk are Positively Correlated



Implications for Asset Pricing Models

Higher-Moment Risk in Asset Pricing Models

Log-normal models: Distribution is *positively* skewed

Disaster models: Disaster-risk makes return distribution left-skewed

- More disaster risk = more left-skewed return distribution
- But, more disaster risk = higher variance

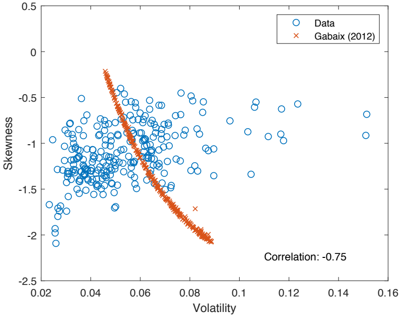
$\Rightarrow$ Inconsistent with the empirical evidence

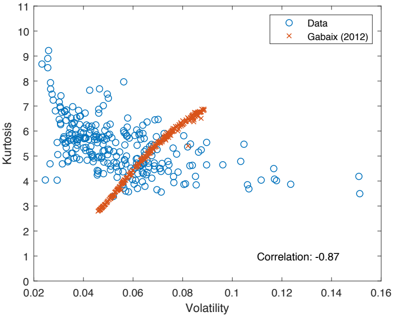
More generally, the disaster risk is cyclical

- Bad times (low prices) occur because of high disaster risk
- In bad times, skewness should thus be lower and kurtosis higher
- We see the **opposite** empirical pattern (higher-moment risk lower in bad times)

Higher-Moment Risk in Gabaix (2012)

We simulate the model and study how risk-neutral skewness and kurtosis relate to risk-neutral variance





Implications for Investors

Higher-Moment Risk Influence Tail-Loss Probabilities

Proposition 1: Real Time Tail-Loss Probabilties

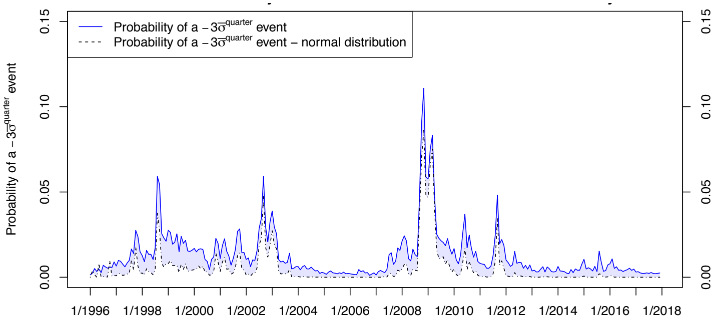
For the unconstrained rational power utility investor who wants to hold the market, the conditional physical probability that market return from time t to T is lower than α is:

$$P_t(R_{t,T} < \alpha) = \frac{R^f_{t,T}}{E^*_t[R^\gamma_{t,T}]} \Bigg[\alpha^\gamma \text{put}'_{t,T}(\alpha S_t - D_{t,T}) - \frac{\gamma}{S_t}\alpha^{\gamma-1}\text{put}_{t,T}(\alpha S_t - D_{t,T}) + \int_0^{\alpha S_t - D_{t,T}} \frac{\gamma(\gamma-1)}{S_t^2}\left(\frac{K + D_{t,T}}{S_t}\right)^{\gamma-2} \text{put}_{t,T}(K)dK \Bigg]$$

where $put'_{t,T}(\alpha S_t - D_{t,T})$ is the first derivative of the put option price with respect to the strike price.

Real-Time Tail Risk For Constant-Notional Investor

What is the probability of loosing $3\bar{\sigma} = \sim 24\%$



Takeaways :

- Higher moment risk constitute large fraction of tail risk
- Must variation for *constant notional* investors is driven by variance

Volatility-Targeting Investors

Many profesional investors have volatility targets

- The targets help control risk
- But do not hedge all risk because higher-order moments also vary

Exposed to higher-moment risk

- Because volatility is constant, the variation in higher-moment risk shows up directly in their portfolios
- The investors lever up portfolios when volatility drops, but these are periods where higher-moment risk is high

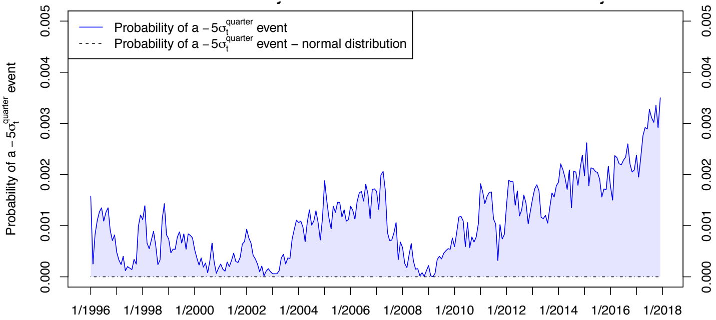
Invest $\omega_{t,T} = \bar{\sigma}/\sigma_t$

$$P_t(r_{t,T}^{\text{i, shock}} < -\bar{\sigma}) = P_t\left(\omega_{t,T}(R_{t,T} - E_t[R_{t,T}]) < -\bar{\sigma}\right) \tag{6}$$

$$= P_t\left(R_{t,T} - E_t[R_{t,T}] < -\sigma_t\right) \tag{7}$$

Real Time Tail Risk of Volatility Targeting Investor

THE PROBABILITY OF A 5SIGMA EVENT FOR A VOLAILITY-TARGETING INVESTOR



Takeaways:

- Despite targeting a constant volatility, risk varies substantially over time
- Relatively high before the financial crisis
- Similar results for 3sigma event (from 0.5% to 1.6%)

Alternatives to Volatility Targeting

Instead of targeting volatility, investors can target tail risk:

- Example: the probability of a 25% loss over the next quarter should be 1.17%
- Our framework allows for estimating the required market weight in real time

How does this portfolio look relative to the volatility-targeting portfolio?

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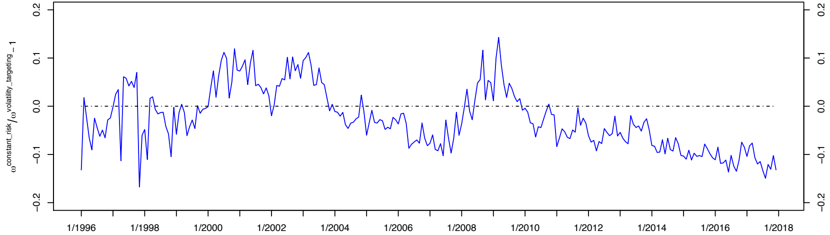
THE ω^{Mkt} FOR CONSTANT TAIL RISK RELATIVE TO VOLATILITY TARGET



Takeaways:

- Volatility-targeting ”over-invests” in the market pre crisis and post crisis
- But underinvests during crisis

Related Literature

- Large literature on tail risk can broadly be categorized into two groups:
  1) physical moments based on backward looking information (e.g. Bollerslev and Todorov (2011), Kelly and Jiang (2014))

     $\rightarrow$ we estimate forward-looking moments

  2) risk-neutral moments based on forward looking information in option prices (e.g. Siriwardane (2015), Bates (2000), Schneider and Trojani (2017))

     $\rightarrow$ we estimate physical moments

- Further differ by emphasizing
  - the relation between variance and higher-order moments
  - implications for investors and asset pricing models

Conclusions

New stylized fact: Higher-moment risk is high when variance is low

We highlight two implications:

1. Inconsistent with consumption based AP models
   - Log-normal models cannot generate the correct unconditional shape
   - Disaster models get the time variation wrong
2. Important implications for investors
   - Higher-order moments reponsible for large fraction of tail risk
   - Tail risk for volatility-targeting investors spikes when variance is low