Frege’s Conception of Logic

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Frege is of course an important progenitor of modern logic. The technical advances he made were comprehensive. He clearly depicted polyadic predication, negation, the conditional, and the quantifier as the bases of logic; and he gave an analysis of and a notation for the quantifier that enabled him to deal fully and perspicuously with multiple generality. Moreover, he argued that mathematical demonstrations, to be fully rigorous, must be carried out using only explicitly formulated rules, that is, syntactically specified axioms and rules of inference.

Less clear, however, is the philosophical and interpretative question of how Frege understands his formalism and its purposes. Upon examination, it appears that Frege had a rather different view of the subject he was creating than we do nowadays. In lectures and seminars as far back as the early 1960s, Burton Dreben called attention to differences between how Frege viewed the subject matter of logic and how we do. The point has been taken up by several commentators, beginning with Jean van Heijenoort. The technical development historically required to get from a Fregean conception to our own was discussed in my “Logic in the Twenties: The Nature of the Quantifier.” Yet there is currently little appreciation of the philosophical import of these differences, that is, the role in Frege’s philosophy that his conception of logic, as opposed to ours, plays. Indeed, some downplay the differences and assign them no influence on or role in the philosophy. Thus Dummett says only that Frege was “impeded” from having the modern view by a particular way of looking at the formulas of his Begriffsschrift.

I want to urge on the contrary that Frege’s conception of logic is integral to his philosophical system; it cannot be replaced with a more modern conception without serious disruptions in that system. The reasons for this will I hope be instructive about the roots of Frege’s philosophizing.

The first task is that of delineating the differences between Frege’s conception of logic and the contemporary one. I shall start with the latter. Explicit elaborations
There are various versions; I will lay out the one formulated by Quine in his textbooks as it seems to me the clearest.

On this conception, the subject matter of logic consists of logical properties of sentences and logical relations among sentences. Sentences have such properties and bear such relations to each other by dint of their having the logical forms they do. Hence, logical properties and relations are defined by way of the logical forms: logic deals with what is common to and can be abstracted from different sentences. Logical forms are not mysterious quasi-entities, but they can be interpreted: a universe of discourse constructed in a way that this conception, the subject matter of logic consists of logical properties of sentences. Thus logic is tied to no particular subject matter because of the logical properties of sentences. Thus logic is tied to no particular subject matter.

The notion of schematization is just the converse of interpretation: to say that a sentence can be schematized by a schema is just to say that there is an interpretation under which the schema becomes the sentence. Thus, a claim that a sentence \( R \) implies a sentence \( S \), that is, that \( S \) is a logical consequence of \( R \), has two parts, each of which uses the notion of interpretation: it is the assertion that there are schemata \( R^* \) and \( S^* \) such that

1. \( R^* \) and \( S^* \), under some interpretation, yield \( R \) and \( S \); and
2. under no interpretation is \( R^* \) true and \( S^* \) false.

This is often called the Tarski-Quine definition, or (in the Tarskian formulation) the model-theoretic definition, of logical consequence. It is precise enough to allow the mathematical investigation of the notion. For example, using this notion of logical consequence, we can frame the question of whether a proposed formal system is sound and complete, and this question may then be treated with mathematical tools. Better still, though, we should say that the definition is capable of being made precise, for the definition quantities over all interpretations. This is a set-theoretic quantification; hence, complete precision would require a specification of the set theory in which the definition is to be understood. (However, it turns out that for implications between first-order schemata, the definition is rather insensitive to the choice of set theory. The same implications are obtained as long as the set theory is at least as strong as a weak second-order arithmetic that admits the arithmetically definable sets of natural numbers.)

(As an aside, let me note that this explication of logical consequence has recently come under attack in John Etchemendy’s *The Concept of Logical Consequence.* Etchemendy argues that, if \( S \) is a logical consequence of \( R \), then there is a necessary connection between the truth of \( R \) and the truth of \( S \), and the Tarski-Quine definition does not adequately capture this necessity. Of course, neither Tarski nor Quine would feel the force of such an attack, since they both reject the cogency of the philosophical modalities. Moreover, it is only the Tarski-Quine characterization of logical consequence in terms of various interpretations of a schematism that makes the notion of logical consequence amenable to definitive mathematical treatment.)

On this schematic conception of logic, the formal language of central concern is that of logical schemata. Pure logic aims at ascertaining logical properties and logical relations of these formulas, and also at demonstrating general laws about the properties and relations. Applied logic, we might say, then looks at sentences—of one or another formal language for mathematics or science or of (regimented versions of) everyday language—to see whether they may be schematized by schemata having this or that logical property or relation. Thus, there is a sharp distinction between logical laws, which are at the metalanguage level and are about schemata, and logical truths, which are particular sentences that can be schematized by valid schemata. The pivotal role in this conception of schemata, that is, of uninterpreted formulas that represent logical forms, gives a specific cast to the generality of logic. Logic deals with logical forms, which schematize away the particular subject matter of sentences. Thus logic is tied to no particular subject matter because it deals with these “empty” forms rather than with particular contents.

Such a schematic conception is foreign to Frege (as well as to Russell). This comes out early in his work, in the contrast he makes between his *begrijschrift* and the formulas of Boole: “My intention was not to represent an abstract logic in formulas, but to express a content through written signs in a more precise and clear way than it is possible to do through words.” And it comes out later in his career in his reaction to Hilbert’s *Foundations of Geometry*: “The word ‘interpretation’ is objectionable, for when properly expressed, a thought leaves no room for different interpretations. We have seen that ambiguity [Vieldeutigkeit] simply has to be rejected.” There are no parts of his logical formulas that await interpretation. There is no question of providing a universe of discourse. Quantifiers in Frege’s system have fixed meaning: they range over all items of the appropriate logical type (objects, one place functions of objects, two place functions of objects, etc.). The letters that may figure in logical formulas, for example, in “\( \exists \text{p} \text{q} \rightleftharpoons p \)” are not schematic: they are not sentence letters. Rather, Frege understands them as variables. Here they are free variables, and hence in accordance with Frege’s general rule the formula is to be understood as a universal closure.
that is, as the universally quantified statement "(\(\forall p)(\forall q)(p\&q \to p\)."
Similarly, logical formulas containing one-place function signs are to be understood not
schematically, but as generalizing over all functions.

On Frege's conception the business of logic is to articulate and
prove the principle of making all true general statements, the logical laws. "(\(\forall p)(\forall q)(p\&q \to p\)." It states a law,
might say, about all objects. Similarly, "(\(\forall p)(\forall q)((\forall x)(\forall y)(\forall z)(\forall w)(x+y=x+w)\to (y+z=w+y)\) is a law about all quantifiers. 

The business of pure logic is to arrive at such laws, just as the business of physics is to arrive at physical
laws. Logical laws are as descriptive as physical laws, but they are more
general. Indeed, they are supremely general; for, aside from variables, all that
figure in them are the all-signs, the conditional, and other signs which are not
specific to any discipline, but which figure in discourse on any topic whatsoever.

Notions of the special sciences first appear when we apply logic. In applied logic,
we infer claims that contain more specialized vocabulary on the basis of the laws
of pure logic. For example, in applied logic we might demonstrate, "If Cassius is
lean and Cassius is hungry, then Cassius is lean"; or, "If all whales are mammals
and all mammals are vertebrates, then all whales are vertebrates." These state­
ments may be inferred from the logical laws given at the beginning of this para­
graph. Here we also see a typical situation, that these specialized statements are
inferred from the logical law by instantiation of universal quantifiers.

On Frege's universalist conception, then, the concern of logic is the articula­
tion and proof of logical laws, which are universal truths. Since they are uni­
versal, they are applicable to any subject matter, as application is carried out
by instantiation. For Frege, the laws of logic are general, not in being about
nothing in particular (about forms), but in using topic-universal vocabulary to
state truths about everything.

The question arises immediately of how different these conceptions actually are.
They can look very close. Both take pure logic to be centrally concerned with gen­
erality. Generality is captured in the schematic conception by definitions that in­
voke all interpretations of the given schemata, and in the universalist conception
by universal quantifiers with unrestricted ranges. In the schematic conception,
logic is applied by passing from schemata to sentences that are particular inter­
pretations of them; in the universalist conception, applications are made by
instantiating the quantified variables of a general law. Given these close parallels,
it is no wonder that many logicians and philosophers would be inclined to mini­
imize the distinction between the two conceptions.

Parallels are not identities, however, and there are philosophically important
ways that the conceptions differ. First and most obviously, the schematic conception
is metalinguistic. The claims of logic are claims about schemata or about sen­
tences, and thus logic concerns features of discourse. In contrast, on the
universalist conception logic sits squarely at the object level, issuing laws that are simply
statements about the world. What logical laws describe are not phenomena of
language or of representation. As Russell put it, "Logic is concerned with the real
world just as truly as zoology, though with its more abstract and general
features." This difference will have consequences for the philosophical character­
ization of logic. For example, the universalist conception leaves no room for the

notion that logic is without content: the laws of logic, although very general, have
to be seen as substantive. Indeed, in the Tractatus, Wittgenstein breaks with the
universalist conception in order to arrive at a view in which the propositions
of logic are empty. Even if Wittgenstein's characterization of logic is rejected,
the metalinguistic conception will inevitably make the nature of discourse, or
of our representations, the focus of any account of logic. A sharp sense of this
can be obtained by contrasting the remark of Russell's just cited with this one
of Dummett's, made unself-consciously and with no argument at all, at the start
of laying out his own metaphysics: "Reality cannot be said to obey a law of logic;
it is our thinking about reality that obeys such a law or flouts it." On Frege's
view, as on Russell's, it is precisely reality that obeys the laws of logic.

Indeed, the universalist conception is an essential background to many of
Frege's ontological views. Frege took not just proper names but also sentences
and predicates to be referring expressions, that is, to have Bedeutung; in the
latter case, the referents were of a different logical sort from those of proper names
and sentences. From many contemporary viewpoints, it is odd to think of
sentences as names at all, and if predicates are thought to refer, it would be to pro­
erties or sets or some other entities that need not be sharply distinguished in logical
character from the referents of singular terms.

It should be clear that the universalist conception demands that sentences and
predicates refer. As we have seen, for Frege the truth-functional laws look like
"(\(\forall p)(\forall q)(p\&q \to p\)." and they are applied by instantiating the quantifiers with
sentences. For "If Cassius is lean and Cassius is hungry then Cassius is lean" to
count as a genuine instance of the law, the expressions which instantiate the
quantified variables have to refer, to things that are values of the variables, just
as to count as a genuine instance of "(\(\forall x)(x\to x)\" the name replacing "x" has to refer, and what it refers to must be among
the values of "x." (To be is to be the value of a variable as much for Frege as for
Quine.) Similarly, since the laws of logic include many that generalize in pred­i­
cates places, and their application requires instantiating those quantified variables
with predicates, here too we are driven to take predicates as referring expressions.

In the case of sentences, it requires a further argument, based on inter­
substitutivity phenomena, to conclude that what sentences refer to are their truth-values, and
it requires yet other considerations to support taking the truth-values to be of the
same logical type as ordinary objects. The former is pretty compelling: the latter
has elicited heated objections.

For predicates, however, support for the sharp distinction in logical type of the
referred to comes from the structure of applications of logic, on the universalist
conception. If the position occupied by a predicate in a statement is taken to be
generalized on directly, the distinction in logical type is apparent, since the pred­i­
cate position has argument places: and if an expression has an argument place
and so can be used in an instantiation of a quantified predicate variable, then it
cannot be used to instantiate a singular term, without yielding expressions that
violate the most basic rules of logical (and grammatical) syntax. Thus we see that
the universalist conception demands second-order logic. Indeed, it was one of
Quine's avowed motivations, in developing the schematic conception, to show that
logic did not require us to take there to be anything designated by the predicates in our statements.

Logic, as construed on the universalist conception, is also in back of a doctrine of Frege's that many have found puzzling, namely, that all functions be defined everywhere; for the special case of concepts, this is the requirement that concepts "have sharp boundaries." For Frege, all quantified variables have unrestricted domain. Given this, and given that "(\forall x)(x \land \neg x)" is a logical law, Frege's requirement follows at once. If something is a concept, then for expression for it can instantiate the quantifier in this law; thus we can logically derive that, for every object, either the concept holds of it or the concept does not. This is just what Frege means by "sharp boundaries."

II

A second important difference between the two conceptions concerns the role of a truth predicate. Clearly, the schematic conception employs a truth predicate: the definitions of validity and logical consequence talk of the truth under all interpretations of schemata. Since the predicate is applied to an infinite range of sentences, it cannot be eliminated by disquotation. On the universalist conception, in contrast, no truth predicate is needed either to frame the laws of logic or to apply them. Moreover, although Frege sometimes calls logical laws the "laws of truth," he does not envisage using a truth predicate to characterize the nature of those laws.

On the schematic conception, logic starts with the definitions of validity and consequence and goes on to pronounce that a given schema is valid or is a consequence of other schemata. Formal systems may be introduced as a means to establish such facts, but this then requires a demonstration of soundness to show that what the system produces are, in fact, validities and consequences. The introduction of a formal system also raises the (less urgent) question of completeness, of whether all validities and all implications can be obtained by means of the system. Thus it is the overarching notions of validity and consequence that set the logical agenda and provide sense to the question of how well a system for inference captures logic. On this conception, the notion of logical inference rule is posterior to that of consequence: a logical inference rule is one whose premises imply its conclusion or, in the context of a system for establishing validities only, is one that always leads from valid premises to a valid conclusion.

In Frege's universalist conception, there is no analogous characterization of what is a logical law or what follows logically from what. Frege's conception of logic is retail, not wholesale. He simply presents various laws of logic and logical inference rules, and then demonstrates other logical laws on the basis of these. He frames no overarching characteristic that demarcates the logical laws from others. Consequently, the only sense that the question has of whether the laws and rules Frege presents are complete is an "experimental" one—whether they suffice to derive all the particular results that we have set ourselves to derive. For example, at one point, Frege entertains the possibility that a failure to obtain established results while developing an area of mathematics axiomatically could lead us to recognize a new logical inferential principle. The closest Frege comes to providing a notion of logical consequence occurs in "On the Foundations of Geometry," where he defines one truth's being logically dependent on another. The definition is: when the one can be obtained by logical laws and inferences from the other (Frege 1906, p. 423). No further characterization of logical laws and inferences is made. Thus, in direct contrast to the situation in the schematic conception, Frege's notion here rests on the provision of the logical laws and inference rules.

Now Frege does say, "Logic is the science of the most general laws of truth." But he does not intend this as a demarcation of logic, only as a "rough indication of the goal of logic." As we have seen, generality and absence of vocabulary from any specialized science are, on the universalist conception, features of the logical. Frege does not attempt to give any specification of the vocabulary allowable in logic; moreover, there is no reason to think that he would take truth and absence of specialized vocabulary as sufficient for logical status. Yet there is a deeper reason that his phrase gives only a "rough indication," and that there is to do with the anomalous status of "true" when used as a predicate.

Frege repeatedly calls attention to that anomalous status. In "Der Gedanke," he presents a regress argument to show that any attempt to define truth must fail, and concludes that "the content of the word 'true' is sui generis and indefinable." Both the argument and his subsequent considerations show that he does not mean simply that the notion of truth is a primitive notion, not to be defined in terms of anything more basic. After reflecting that I smell the scent of violets" and "It is true that I smell the scent of violets" have the same content, so that the ascription of truth adds nothing, he concludes: "The meaning of the word 'true' seems to be altogether sui generis. May we not be dealing here with something which cannot be called a property in the ordinary sense at all?" (Frege 1918, p. 61). In "Introduction to Logic," Frege goes further in suggesting that truth is not a property at all: "If we say 'the thought is true' we seem to be ascribing truth to the thought as a property. If that were so, we should have a case of subsumption. The thought as an object would be subsumed under the concept of the true. But h ere we are misled by language. We don't have the relation of an object to a property" (PW, p. 194). In "My Basic Logical Insights," he connects the use of "true" in characterizing logic with the idea that the ascription of truth to a thought adds nothing:

So the sense of the word "true" is such that it does not make any essential contribution to the thought. If I assert "it is true that sea-water is salty," I assert the same thing as if I assert "sea-water is salty." This enables us to recognize that the assertion is not to be found in the word "true" but in the assertoric force with which the sentence is uttered. "That" makes only an abortive attempt to indicate the essence of logic, since what logic is really concerned with is not contained in the word "true" at all but in the assertoric force with which a sentence is uttered, (Frege 1915, pp. 251-252).

Thus, rubrics like "general laws of truth" cannot serve to give a real characterization of logic or a demarcation of the realm of the logical. The notion of truth is
unavailable for the role of setting the agenda for logic. Moreover, if we take Frege's scruples seriously, it follows that the schematic conception of logic is simply unavailable to him. To formulate it, as we have seen, use has to be made of a truth predicate. That predicate figures not as a suggestive way of talking, nor as a term whose usefulness arises only from the "imperfection of language," as Frege puts it (Frege *1915), but as a scientific term in the definitions of the most basic concepts of the discipline. Clearly, Frege would not think that legitimate.

The question then arises of whether Frege's scruples are well-posed, or whether they can be dismissed as merely peripheral phenomena, with no deep systematic connections. Addressing this question requires a careful examination of the arguments Frege adduces. I shall not attempt this here; for a detailed treatment, see Thomas Ricketts' "Logic and Truth in Frege." I limit myself to mentioning the philosophical outlook which I take to be expressed in Frege's scruples about a truth predicate. It is that objective truth is not to be explained or secured by an ontological account. Such an account would take us to have a conception of things "out there" and of their behaviors or configurations that exist independent of our knowledge, and it would depict those behaviors or configurations as being that which renders our thoughts true or false. Such an account is often ascribed to Frege, for it is just what is involved in ascribing a truth-conditional semantics to him. But this ascription is incompatible with Frege's remarks on truth. To take Frege's scruples seriously is to appreciate that there is no general notion of something's making a truth true—that is, that there is no theory of how the thoughts expressed by sentences are determined to be true or false by the items referred to in them. It is thus to put us in a position to appreciate the extraordinarily subtle view Frege can be read as unfolding. On this reading, Frege is not a realist, on the usual philosophical characterisations of that position. He is committed to the objectivity of truth and its independence of anyone's recognition of that truth, but the conception of truth here is immanent within our making of judgments and inferences, our recognitions of truth.

Earlier I noted that the most obvious difference between the universalist and the schematic conceptions is that in the former logic operates at the object level, whereas in the latter it operates at the meta-level. Even this by itself has consequences, and it can be used to get at an important role the universalist conception has in Frege's system.

Of course, it is important to avoid anachronism here. At the time Frege was writing, a distinction between object level and meta-level could hardly have been drawn; in fact, it was not to become clear until the 1920s. Nonetheless, we can see a precursor of the distinction as being at issue. Many traditional logicians spoke of logic as being about the forms of judgment, which were to be obtained by abstraction from judgments. Although this conception was far from precise and traditional logic lacked the machinery to work it out, it seems clear that forms of judgment were invoked as a way of capturing the generality of logic and lack of tie to the content of individual judgments. Thus we can see here a proto-schematic conception. (This is particularly visible in Bolzano.) Frege rarely speaks of forms of judgment. It is not hard to surmise some reasons.

First, talk of forms of judgment and of abstracting from individual judgments has a dangerously psychological ring to it. The very location "forms of judgment" suggests that the forms are of mental acts and so are prime material for psychologistic treatment. Moreover, Frege argued vigorously against any notion of abstraction as needed to get from particulars to general notions. Indeed, elimination of any role for abstraction is central not just to Frege's antipsychologism, but also to his anti-Kantianism. To eliminate abstraction is to eliminate the question. How do we attain the general? Frege replaces it with the question of the relation between the (already given) general and the particular, a question to be answered by logic.

This leads us to the second reason Frege has for discounting talk of the forms of judgment. He has no need of such talk, precisely because his devising of the quantifier gives him a rigorous tool to capture the generality that "forms of judgment" gestures toward. The generality is directly expressed by the quantifier. The relation of general to particular is given by the logical rule of instantiation from former to latter, not by some imprecise, psychological notion of abstraction from the former to the latter.

These two reasons are not relevant to the modern schematic conception, which has found precise nonpsychological notions to replace "forms of judgment" and "abstraction" and which uses quantification (in the metalanguage) to capture the desired generality. There is, however, another consideration at work in Frege that is not simply obsolete.

Frege's conception of logic is intertwined with his notion of justification. A cornerstone of Frege's thinking is the sharp distinction between the rational basis of a claim—the truths that it presupposes or depends upon—and what we might call concomitants of thinking or making the claim: the psychological phenomena that occur when a person thinks of the claim, or believes it, or comes to accept it. The empirical conditions someone must satisfy in order to know the claim, the history of the discovery of the claim, and so on. The distinction is emphasised throughout Frege's writings, and particularly vividly in the Introduction to the Foundations of Arithmetic. Remarks like these abound: "Never let us take a description of the origin of an idea for a definition, or an account of the mental and physical conditions on which we become conscious of a proposition for a proof of it." The point is more general than antipsychologism, or a distinction between objective and subjective, as the following shows:

A delightful example of the way in which even mathematicians can confuse the grounds of proof with the mental or physical conditions to be satisfied if the proof is to be given is to be found in E. Schröder. Under the heading "Special Axiom" he produces the following: "The principle I have in mind might well be called the Axiom of Symbolic Stability. It guarantees us that throughout all our arguments and deductions the symbols remain constant in our memory—or preferably on paper," and so on. (Frege 1884, p. viii)
Clearly, we would not be able to arrive at correct mathematical arguments if our
inkblots were constantly to change. Yet that does not imply that mathematics
presupposes the physical laws of inkblots, that those laws would figure in the
justifications of mathematical laws.

It is important to note that something must give content to the distinction
between rational basis and mere concomitant; something must provide a means for
saying what counts as showing that one proposition is the rational basis for anoth­
er, and showing when one proposition presupposes another. It is Frege's logic
that plays this role. Logic tells us when one claim is a ground for another, namely,
when the latter can be inferred, using logical laws, from the former. Explanation
and justification are matters of giving grounds. For Frege, then, the explanation
of a truth is a logical proof of that truth from more basic truths; the justification
of a truth is a logical proof of that claim from whatever first principles are its ut­
imate basis. Thus the laws of logic are explicable of explanation and justification:
on this rests their claim to the honorific title "logic."

Given this role for logic, it should occasion no surprise that Frege's concep­tion
of logic and the demands he puts on the notion of justification are closely linked.
Now the notion of justification plays a philosophically very important role for
Frege, as it is key to his argument for the logicist project. Although we might start
off thinking that arithmetical discourse is completely understood, transparent,
and poses no problem, Frege urges that we lack knowledge of the ultimate jus­
tification of the truths of arithmetic. In order to "afford us insight into the de­
pendence of truths upon one another," we must analyze the seemingly simple concept
of number and find the "primitive truths to which we reduce everything" (Frege 1884, p. 2).
Frege also brings up "philosophical motives" for the logicist project, asking what looks to be the traditional philosophical question of whether
arithmetical truths are analytic or synthetic. But actually he redefines these notions (as well as those of a priori and a posteriori) so that they concern "not the content of
the judgment but the justification for making the judgment" (Frege 1884, p. 3). Here
too it is the notion of justification that is doing the work.

Essential to the role of this notion of justification in supporting the logicist project,
and to the plausibility of Frege's redefinitions of traditional philosophical terminology,
is the applicability to all knowledge of the standards of justification. The canons
of justification must be universal in their purview: "Thought is in essentials the
same everywhere; it is not true that there are different kinds of laws of thought to suit
the different kinds of objects thought about" (Frege 1884, p. iii). Another im­
portant feature of justification is explicitness: a justification must display everything
on which the truth of the claim being justified depends. To ensure that "some other type
of premise is not involved at some point without our noticing it," a justification
must provide "a chain of inferences with no link missing, such that no step in
it is taken which does not conform to some one of a small number of principles of
inference recognized as purely logical" (Frege 1884, p. 102).

Obviously, these demands are met when logic, as invoked in Frege's notion of
justification, is taken on the universalist conception. That the canons of justifica­tion
must extend to all areas of knowledge requires utmost generality and uni­
versal applicability of the logical principles. Explicitness is vouchsafed by the di­rect applicability of logic: there are no presuppositions, no implicit steps, in the
application of logical laws. To illustrate this, let us examine how, on Frege's pic­
ture, logic would be used to justify the conclusion that all whales are vertebrates
on the basis of the claims that all whales are mammals and that all mammals are
vertebrates. We start with the assertions:

(1) All whales are mammals.

(2) All mammals are vertebrates.

We then provide a logical demonstration from first principles that ends with:

(3) \( (\forall x)(x \text{ is a whale} \rightarrow x \text{ is a mammal}) \rightarrow (\forall x)(x \text{ is a mammal} \rightarrow x \text{ is a vertebrate} \rightarrow (\forall x)(x \text{ is a whale} 
\rightarrow x \text{ is a vertebrate}). \)

Or, in ordinary English:

(4) If all whales are mammals, then if all mammals are vertebrates then
all whales are vertebrates.

By modus ponens from (4) and (1) we obtain:

(5) If all mammals are vertebrates then all whales are vertebrates.

Finally, by modus ponens from (5) and (2), we arrive at:

(6) All whales are vertebrates.

Taken together, all these assertions, including those in the logical proof of (3),
constitute the justification of the assertion of "All whales are vertebrates" on
the basis of the assertions of "All whales are mammals" and "All mammals are
vertebrates."

The requirement of explicitness and the need for the logical laws to be directly
applicable can be highlighted by consideration of an argument against logicism
devised by Henri Poincaré. The version I summarize here is formulated by
Charles Parsons. In order to show that arithmetic is logic, one must devise a
formal system of logic and show how the theorems of arithmetic can be obtained
in that formal system. Now, to give a formal system is to specify, first, the class
of formulas and, second, the class of derivable formulas. The usual form of specifi­
cation is this: certain basic expressions are stipulated to be formulas; other for­
mulas are specified as those and only those expressions obtained from the basic
expressions by finitely many applications of certain syntactic operations. Similarly,
certain formulas are stipulated to be axioms; the derivable formulas are specified as those and only those formulas obtained from the axioms by finitely many applications of certain inference rules. Thus these specifications are inductive in nature: the notion of a finite number of applications of given operations is essential to them. Therefore, number is presupposed in the logistic foundation for arithmetic. This is a petitio principii. Thus there is a logical circle in the logistic reduction.

I believe Poincare's objection fails, and it is important to see why. The objection would succeed if Frege construed the justification of arithmetic to involve, for one or another arithmetical claim, the following assertion: "This claim is provable in such-and-such formal system." That assertion is a metatheoretic one. It is about the formal system; since Poincare is quite right that inductive definitions are used to specify the formal system, it follows that the assertion relies on number theory. That is not, however, how Frege conceives of justification. To give a justification of an arithmetical claim is to give the claim with its grounds. It is not to assert that the claim is provable; it is to give the proof. Now, of course, one might want to verify that what has been given is, in fact, a proof by the lights of the formal system. Such a verification would proceed by syntactic means, and does presuppose the specification of the system. The verification is not constitutive of the argument's being a justification; it is just a means for us to ascertain that it is. In order for us to be psychologically sure that what we are giving are justifications, we have to use our knowledge of the formal system, that is, our metatheoretic knowledge which is of an inductive nature. But that is different from what the justification of the claim actually is.

Here Frege is relying precisely on the distinction between what we might have to do, in fact, by our natures, in order to be in a position to do mathematics, and what the justification of mathematics is. That we need to set out a formal system to be sure of our justifications is no more relevant to the rational grounds of mathematics than our need to write down proofs because otherwise we will not remember them.

The Fregean rebuttal to Poincare requires that in what Frege would call a justification, say of an arithmetical truth, everything that is presupposed by the truth does play a role. This lies in back of his demand for "gap-free" deductions.

To gain an appreciation of the role of the universalist conception of logic in this, it is instructive to contrast how a justification abiding by the Fregean requirement of explicitness would have to proceed if logic were taken on the schematic conception. Let us once again undertake a justification of "All whales are vertebrates" on the basis of "All whales are mammals" and "All mammals are vertebrates." We can't simply pass from the latter to the former, with a note ("off to the side," so to speak) that the latter two jointly imply the former, since this does not make explicit what is involved in the inference. Rather, matters have to be laid out as follows. As before, we start by asserting:

(1) All whales are mammals.

(2) All mammals are vertebrates.

We then assert, along with whatever grounds needed to show it from first principles:

(3) There is an interpretation of "(\forall x)(Fx \rightarrow Gx)," "(\forall x)(Gx \rightarrow Hx)," and "(\forall x)(Fx \rightarrow Hx)," under which these schemata become (regimented versions of) the sentences "All whales are mammals," "All mammals are vertebrates," and "All whales are vertebrates," respectively.

We now adduce a mathematical proof culminating in:

(4) Any interpretation that makes "(\forall x)(Fx \rightarrow Gx)" and "(\forall x)(Gx \rightarrow Hx)" true also makes "(\forall x)(Fx \rightarrow Hx)" true.

Using some logical laws and intermediate steps for making the transition, we can assert on the basis of (3) and (4):

(5) If "All whales are mammals" and "All mammals are vertebrates" are true, then "All whales are vertebrates" is true.

To apply (5), we must adduce the Tarski paradigms:

(6) "All whales are mammals" is true if and only if all whales are mammals.

(7) "All mammals are vertebrates" is true if and only if all mammals are vertebrates.

(1), (2), (5), (6), (7), and truth-functional laws will allow us to obtain:

(8) "All whales are vertebrates" is true.

Finally, adducing

(9) "All whales are vertebrates" is true if and only if all whales are vertebrates.

we obtain:

(10) All whales are vertebrates.

 Needless to say, from Frege's point of view this outline already looks terribly circuitous, and the amount that has to be filled in to provide justifications for (3) and (4) will make matters worse. Even ignoring Frege's scruples about a truth predicate, the status of the disquotational biconditionals is also troublesome, for, in what is outlined, those biconditionals figure among the grounds of "All whales are mammals" as much as do assertions (1) and (2). If, for example, they are meant to be consequences of a substantial semantic theory, then we are in the position of re-
quiring that theory in the justification of "All whales are vertebrates" on the basis of "All whales are mammals" and "All mammals are vertebrates." Matters look less peculiar if the truth predicate is meant to come merely from a Tarski-style definition; but even here an oddly large body of mathematics must figure in order to justify what is, after all, a rather simple logical inference. All this is to say that the schematic conception of logic fits poorly with the Fregean picture of justification.\(^\text{12}\)

This lack of fit comes out in another difficulty as well. In the justification as just outlined, various transitions, like that from (3) and (4) to (5), will be made by applying logical rules. On the schematic conception, logical rules are justified only on the basis of their soundness, that is, their yielding logical consequences. But then it looks like the justification we have presented is not fully explicit; there is something left unsaid that it presupposes.

It might be objected, however, that there is a similar problem in the justification given on the universalist conception. In it, inferences are made in accord with certain inference rules. Shouldn't the demand of explicitness be invoked further, to require that whatever principles lie behind the correctness of the inference rules be made explicit and considered part of the justification? In general, the only way of stating these principles are as the soundness or truth-preservingness of the rules and involve semantic ascent and a truth predicate. Thus the "directness" alleged for the universalist conception papers over an elision.\(^\text{33}\)

Now I believe Frege would reject the idea that inference rules rest on or presuppose the principles expressing their soundness. Rather, our appreciation of the validity of the rules is not the recognition of the truth of any judgment at all; it is manifested in our use of the rule, in our making one assertion on the basis of another in accordance with the inference rule.\(^\text{34}\) There is nothing more to be made explicit, although of course individual instances of the inference rule can always be conditionedized and asserted as logical truths.

To some this may appear to be an evasion. But let us investigate the question we left hanging with respect to the schematic conception. There, the justification looked inexplicit because it omitted a demonstration of the soundness of the logical rules it employed, and, on the schematic conception, logical rules are justified only on the basis of their soundness. Of course, one could adjoint a demonstration of soundness. Naturally, that demonstration will use logical rules. Usually the soundness of those rules will not be vouchsafed by the adjointed demonstration, because the quantified variables in the demonstration will have to range over a larger class than any of the universes of discourse of the interpretations covered by the soundness proof. For example, an everyday soundness proof shows that the usual logical rules are sound with respect to all interpretations whose universes of discourse are sets. The reasoning in that proof involves variables ranging over all sets; hence, the universe of discourse of that reasoning is a proper class. A soundness proof for the logical rules used in the everyday proof would therefore have to show something stronger than everyday soundness, namely, that the rules were sound with respect to interpretations whose universes of discourse were proper classes. This would require a stronger set-theoretic language yet, in which collections of proper classes existed, and the reasoning in the stronger soundness proof would involve variables ranging over such collections. This process continues with no end. To avoid a vicious regress, we have to be able to take the logical rules used in the justification for granted. Yet, on this conception, it has to be admitted that a fuller justification, one amplified by a further soundness proof, is always possible. In passing to that fuller justification, we also pass to a larger universe of discourse. The upshot is that at no level can one think of the quantifier as ranging over everything; there is no absolutely unrestricted quantifier. All the while, though, in enunciating the claims at any level, one is not (yet) in a position to specify how the quantifiers are restricted; they range over everything that at that point one can have. This is a curious position, one which goes far more against Frege's demand for explicitness than our acceptance of a rule of inference without an explicit semantic principle to back it up.\(^\text{45}\)

This last argument has brought us rather far afield. My central aims in this paper have been to delineate Frege's universalist conception of logic and contrast it with a more familiar one, to show that this conception connects with many other points in Frege's philosophy, and to suggest that the conception is a well-motivated one, given the nature of Frege's project. Of course, today most of us would find the schematic conception (or some variant of it) far more natural, if not unavoidable. But I hope to have caused us to reflect on how much else has to shift in order to make it so.\(^\text{36}\)

NOTES


4. Elementary Logic (Boston: Ginn, 1941) and Methods of Logic (New York: Holt, 1950).

5. Tarski's formulation in "On the Concept of Logical Consequence" (in <i.AD Tarski, Logic, Semantics, Metamathematics</i>, Oxford: Oxford University Press, 1956, originally published 1935), pp. 409–420) does not introduce schemata, but obtains the same effect for the formalized languages he treats by disinterpreting the nonlogical vocabulary so as to allow for arbitrary reinterpretations.

6. That the arithmetical sets are enough for implications between schemata was shown in David Hilbert and Paul Bernays, Grundlagen der Mathematik, vol. 2 (Berlin: Springer, 1939), p. 252. The same proof shows that, for an infinite set of schemata, we need no more than the sets arithmetically definable from that set, but that we may introduce more than just the arithmetical sets themselves; however, this was noted in George Boolos, "On Second-Order Logic," Journal of Philosophy 72 (1975): 509–527.


10. Throughout this paper I use modern logical notation rather than Frege's.

11. Here, in using modern notation, I am eliding a nicely required by Frege's quantifying over all functions, not just all concepts, namely, his use of the horizontal.


15. For example, see Dummett, Frege: Philosophy of Language (London: Duckworth 1973), p. 180 and following. Dummett calls Frege's view on the matter a "gratuitous blunder."

16. Charles Parsons canvasses an objection to Frege's conclusion in his "Frege's Theory of Number" (in C. Parsons, Mathematics in Philosophy [Ithaca: Cornell University Press, 1982, original date of publication 1965]), pp. 150–172, [hereafter cited as Parsons 1965]), on pp. 499–501. Suppose we take it that predicates are generalized not directly but only via "nominalization," that is, only once they are transformed into names of qualities, properties, sets, or the like; for example, only once "x is malleable" is transformed into "x has the property of malleability." Since "malleability" lacks argument places, it may be taken as generalizable using a variable of the same logical type as those over objects. Parsons notes a problem with this: since the assertions with which one starts and ends will contain the unnominalized predicate, one needs a principle to underwrite the transformation, that is, a general principle a particular instance of which will be that "(forall x) (x has the property of malleability)." But any such principle, it seems clear, will have to contain a generalization directly in the predicate place occupied in this instance by "is malleable." Hence the Fregean conclusion stands. As Parsons notes (1965 p. 503), there is one way around this conclusion. That is to use the method of semantic ascent and understand the principle underwriting the nominalization not as a direct generalization over what predicates refer to, but as an assertion that any assertion of a certain form may be transformed into another form salva veritate. To adopt this strategy simply is to give up the universalist conception, as it requires a metalinguistic principle that makes ineliminable use of a truth predicate.


18. The truth predicate needed is a predicate of sentences. For Frege, it was not sentences but rather thoughts (senses of sentences) that were true or false. Consequently, those who take Frege to have a form of the schematic conception treat implication as a relation among thoughts. In this version, the definition would require a truth predicate of thoughts.

19. In this Frege differs from Russell, who from the first tries to formulate necessary and sufficient conditions for a truth to be a logical truth. Russell tries to use generality as the key element of the characterization. Wittgenstein takes up the problem, but criticizes Russell's invocation of generality. He takes himself to have solved the problem with the notion of tautology; that notion gives him a general conception of the logical.


22. Richard Heck suggests that Frege may have formulated the principle of countable choice to himself and found himself unable to derive it. "The Finite and the Infinite in Frege's Grundgesetze der Arithmetik," in Philosophy of Mathematics Today, ed. M. Schirn (Oxford: Oxford University Press, 1998). If so, Frege may have wondered whether that principle is an example of a truth statable without using nonlogical vocabulary, but not itself a truth of logic.


29. See Frege, Begriffsschrift, eine der arithmetischen nachgelassene Formelsprache des reinen Denkens (Halle: L. Nebert, 1879), §23.


32. Not that it was meant to Tarski's and Quine's views of justification are different from Frege's.

33. This line of objection is suggested by a remark by Charles Parsons in "Objects and Logic," Monist 65 (1982): 491–516, on p. 503.


35. The position that there can be no such thing as a truly unrestricted quantifier is due to Parsons in "The Liar Paradox" (in C. Parsons, Mathematics in Philosophy [Ithaca: Cornell University Press, 1982, originally published 1974]), pp. 221–250. Parsons's argument is based on phenomena associated with the Liar Paradox. The use of the idea in the current context is due to Ricketts.

36. I am greatly indebted to Thomas Ricketts for countless conversations and comments, as well as for access to his unpublished works. Needless to say, he does not agree with all the formulations given in this paper.