Demand Reduction in Multi-unit Auctions with Varying Numbers of Bidders: Theory and Field Experiments

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Abstract

Auction theory has recently revealed that multi-unit uniform-price auctions, such as those used by the U.S. Treasury for debt sales, entail demand-reduction incentives that can cause inefficient allocations. Recent experimental results show that bidders do indeed strategically reduce their bids in uniform-price auctions. The present paper extends this work, both theoretically and experimentally, to consider the effects of varying numbers of bidders. We derive several theoretical predictions, including the result that demand reduction should decrease with increasing numbers of bidders, though some demand reduction remains even in the asymptotic limit. We then examine the bidding behavior of subjects in this environment by auctioning dozens of Cal Ripken, Jr. baseball cards using both uniform-price and Vickrey auction formats. The field data are broadly consistent with the theoretical predictions of our model: most notably, demand reduction on second-unit bids becomes much smaller and harder to detect as the number of bidders increases.

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1. Introduction

Uniform-price sealed-bid auctions have been used in several important applications, from Treasury bonds to transferable pollution permits to initial public offerings of stock shares. Recent theoretical work on multi-unit auctions has established a potential problem with this auction format: in a uniform-price auction, bidders have an incentive to bid lower than their true values, an effect usually referred to as "demand reduction." This theoretical effect is cause for concern because it may lead to inefficient allocations. Recent experimental evidence demonstrates that significant demand reduction does indeed occur with actual bidders, and that the effect is large enough to influence equilibrium allocations.

In a uniform-price sealed-bid auction, each bidder submits bids on one or more identical units of the same good. If there are m units of good for sale, then the m highest bids each win, and the $(m+1)$st bid becomes the price paid for each unit won.[1] Vickrey [1961, 1962] analyzed this auction for the case where each bidder demands no more than one unit of the good. Bidders have a dominant strategy of bidding equal to their values, and the auction allocates the good efficiently. It is more complicated to analyze the case where an individual bidder may win more than one unit. As before, each bidder finds it optimal to bid her true value on the first unit that she may win, but this is no longer the case with bids for additional units. In particular, an individual's bid on any unit after the first may prove to be the marginal bid that sets the price she must pay on all units won. This gives the auction a "pay-your-bid" flavor, so bidders should bid less than their values on all units except the first; we refer to this phenomenon as "demand

[1] In practice, the lowest accepted bid has often been used to determine the price instead of the highest rejected bid. Similar arguments are true for that alternative uniform-price auction.

reduction."[2] Because bidders no longer truthfully reveal demand, the uniform-price auction may fail to allocate the good efficiently.

An alternative auction mechanism is the (generalized) Vickrey auction. The pricing rule for this auction ensures that each winning bidder pays an amount equal to the sum of the losing bids that would have won had she not bid.[3] The dominant strategy is for individuals to bid their true values on all units; hence, there is no demand reduction. In theory, this multi-unit Vickrey auction results in an efficient allocation of goods. The Vickrey auction, however, does have potential drawbacks. In addition to the fact that its rules are relatively difficult to understand,[4] it also entails the possibility of an "unfair" outcome where a low-demand bidder pays more than her high-demand rival, despite the fact that her bids were lower than his. This may occur even to the point where the low-demand bidder pays more for a single unit than her rival does for multiple units.

Given these potential drawbacks of the Vickrey auction relative to the uniform-price auction, a designer of auction mechanisms might wish to know whether demand reduction in uniform-price auctions is a nontrivial problem in practice. Available empirical evidence suggests that bids in uniform-price auctions do exhibit significant demand reduction. For example, Kagel and Levin [1997] have examined the case in which one bidder may win more than one unit. They designed a laboratory experiment where a single human bidder competed for two units of a good against robot bidders with unit demand, and found significant demand reduction. They also

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[2] See Ausubel and Cramton [1997] and Engelbrecht-Wiggans and Kahn [1998] for more formal arguments.
[3] Note that this is a special case of a Groves-Clarke mechanism (Groves[1973], Clarke[1971]).
[4] Ausubel [1997] proposes an auction to solve this problem. The Ausubel auction is an ascending-bid auction that is strategically equivalent to the sealed-bid Vickrey auction for the case of private values, but the rules are much simpler to explain to bidders. However, his auction does not address the potential problem of perceived
(continued on next page)

provided the first theoretical work on the comparative statics of demand reduction, examining the special case of a single multi-unit bidder competing against *m* single-unit bidders with uniformly distributed private valuations. They derived a unique Nash equilibrium prediction that as the number of bidders increases, the multi-unit bidder should engage in less demand reduction. Their laboratory experiments show an asymmetric confirmation of this prediction: subjects moving from 3 to 5 rivals do not learn to decrease their demand reduction, but subjects moving from 5 to 3 rivals do exhibit additional demand reduction.

Evidence of demand reduction also exists in cases with two multi-unit bidders. List and Lucking-Reiley [2000] found evidence of demand reduction in a field experiment, auctioning nearly $10,000 worth of sportscards in two-unit, two-person sealed-bid auctions. They found that individuals tend to reduce bids on second units by as much as 73 percent in a uniform-price auction relative to a theoretically demand-revealing Vickrey auction. Wolfram [1998] provides evidence of the same strategic effect in uniform-price electricity supply auctions in England and Wales with two major firms competing against each other.[5]

But what about auctions with more than two bidders who can win more than one unit? Many important practical auctions — for example, those for pollution permits, FCC spectrum licenses, and Treasury bills — have more than two bidders who may win multiple units. Does demand reduction disappear as the number of bidders increases, and if so, how quickly? Katzman [1999] provides theoretical arguments that some demand reduction remains even in the limit of infinitely many bidders. However, his asymptotic arguments do not tell us how much

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"unfairness" of the pricing rule.

[5] Technically there are more than two competitors in the Welsh electricity auctions, but Wolfram explains that the market consists of two major players plus a competitive fringe of smaller players.

demand reduction, or how quickly the limit is approached. Furthermore, empirical evidence on these issues is very scant.

This paper provides some answers to these questions. We design a field experiment to explore demand reduction in auctions with three or five multi-unit bidders bidding for two units. The methodology follows that of the two-bidder auctions of List and Lucking-Reiley [2000], with participants at a sportscard trading show invited to participate in sealed-bid auctions for collectible sportscards. Note that our field experiments differ from laboratory experiments in that they involve subjects bidding for real objects in a pre-existing market. They represent a joint test of both the underlying modeling assumptions (weakly decreasing individual demand for multiple units, common knowledge, and symmetry of the distribution of bidder values) and the behavioral assumptions (Nash equilibrium bidding). By contrast, laboratory experiments typically impose the modeling assumptions, and concentrate on clean tests of the behavioral assumptions. In order to gain the ability to perform field tests of both types of assumptions, the experimenter must give up some of the controls of the laboratory. In particular, we give up being able to control (or even observe) bidders' valuations for the good, which makes it impossible to compare observed bids to bidders' true values directly. Therefore, we focus on testing theoretical predictions which compare behavior across two different auction formats: uniform-price and Vickrey.

We also derive new theoretical predictions for the uniform-price auction, predicting demand reduction's dependence on the number of bidders. Specifically, we show that in Nash equilibrium, (1) the second-unit bid strategy in the uniform-price auction increases (weakly) with the number of bidders, (2) any increase comes from a second-unit bid that was zero becoming positive, and (3) second-unit bids will be strictly less than bidders' values even in the limit of infinitely many bidders. Because of the dominant strategy of truthtelling in the Vickrey auction, changes in the number of bidders should not affect bidding behavior in this auction institution. Hence, in our experiment, we consider differences between the Vickrey and uniform-price auctions in terms of: (1) the mean second-unit bid across bidders, (2) the proportion of second-unit bids equal to zero, and (3) the proportion of auctions where both goods are allocated to the same bidder. The theory predicts that in each case the absolute difference between auction formats shrinks as the number of bidders increases, but stays bounded away from zero.

Our experimental results are generally consistent with the theory's predictions. In particular, we see continued evidence of demand reduction, but the demand reduction is much smaller and less statistically significant than it was in the case of n=2 bidders. Differences in mean bids between Vickrey and uniform-price auctions are not statistically significant, but proportions of zero bids are (and only when pooling data across all of the new experiments). Allocations of goods also differ between auction formats in the predicted direction, but not in a statistically significantly manner. In addition, also in accordance with theory, but in contrast to the results of List and Lucking-Reiley [2000], we find equality between first unit bids across the Vickrey and uniform-price auctions. Hence, whatever caused the theory to fail for first-unit bids with n=2 bidders is no longer a problem when n>2. We also note that our Vickrey and uniform-price auctions generate roughly the same revenues, independent of the number of bidders.

We organize the remainder of the paper as follows. In the next section, we develop a theory for how the amount of demand reduction varies with the number of bidders. Coupling our theoretical results with implications from other studies suggests a number of testable hypotheses.

In Section 3, we describe the design of our field experiments, designed to test these hypotheses in auctions involving three and five bidders. Section 4 describes our experimental results, and Section 5 concludes.

2. Demand Reduction in Theory

2.1 The Model

Two identical units will be auctioned (without a reserve price) to n expected-profit-maximizing bidders with independently distributed, privately known values. Specifically, bidder i has a value v_{i1} for winning one unit and a marginal value of v_{i2} for also winning a second unit. Bidder i knows the values v_{i1} and v_{i2}. The value vectors (v_{11},v_{12}), (v_{21},v_{22}) ... (v_{n1},v_{n2}) are the outcomes of n independent draws from some commonly known, bivariate distribution. Let G_1 and G_2 denote the marginal distributions, and g_1 and g_2 the corresponding densities, where g_1 has support $[0,V]$. In the special case of "independent, decreasing marginal values," v_{i1} and v_{i2} are the larger and the smaller of two independent draws from a commonly known distribution H with density h; this means that $G_1 = H^2$ and $G_2 = 1\text{-}(1\text{-}H)^2$.

In the auction, each bidder i submits two bids b_{i1} and b_{i2}, and the two highest bids win. As previously mentioned, bidders in the Vickrey auction have a dominant strategy of bidding equal to their values on each of the units. In the uniform-price auction, Engelbrecht-Wiggans and Kahn [1998] argue that any equilibrium in undominated strategies has bidders bidding equal to their true values on the first unit. This leaves only the question of how bidders should bid on the second unit in uniform-price auctions. In Appendix 1, we examine existing theory for insights into the relationship between second-unit equilibrium bids and the number n of bidders. We derive new theory suggesting that equilibria may depend on n in a very specific fashion. In particular, the equilibrium bidding strategy $b(v;n)$ for a second unit of value v in an auction with n bidders has the form $b(v;n) = 0$ if $v \leq v^*(n)$, and $b(v;n) = c(v)$ otherwise, where $c(v)$ is some function independent of n. In words, this result implies that for different n, the equilibrium bid on the second unit may change from being zero to being positive, but if the bid is positive then it is independent of n.

Our theory also allows us to compute equilibria for specific examples. Since we have been unable to establish general existence conditions, we turn to such examples. Fortunately, however, our theory does provide a systematic procedure for finding candidate equilibria and for verifying whether or not they are in fact equilibria.

2.2 Examples

We start by considering a family of examples with independent, decreasing marginal values. Specifically, let $H(x) = 1 - (1-x)^k$ for $0 \leq x \leq 1$, where $k \geq 1$. The case k=1, for example, implies that each bidders' values are two independent draws from a uniform distribution.[6] Using our theory, we find equilibria of the form described above. Table 1 shows how the cutoff $v^*(n)$ varies with n for various k. Note that the cutoff is a weakly decreasing function of n. As the number of bidders increases, the equilibrium bid for bidder i may change from $(v_{i1}, 0)$ to $(v_{i1}, c(v_{i2}))$. But that is the only possible change. Figure 1 graphically displays two possible bid functions for the distribution with k=3. For n=2 bidders, one can see that the bid function starts at $b(v)$=0 for all $v<v^*$, then jumps to $c(v^*)>0$ at v^*, and remains on the function $c(v)$ for all $v>v^*$. For n=5 bidders, one can see that the appropriate v^* moves to the left, so that there are fewer zero bids when there

[6] The case where $k = n = 2$ is example 3 in Engelbrecht-Wiggans and Kahn [1998].

are more bidders, but the n=5 bid function otherwise looks identical to the n=2 bid function.

In some cases, the equilibrium bidding strategies are entirely independent of the number of bidders. Specifically, Corollary 4.4 of Engelbrecht-Wiggans and Kahn [1998] assures that if the marginal density $g_1(x)$ of first-unit values is a non-decreasing function of x, then a Nash equilibrium results when each bidder i bids $(b_{i1}, b_{i2}) = (v_{i1}, 0)$. This condition for a "single-unit-bid" equilibrium is independent of n. Therefore, when this condition on the distribution of values is satisfied, the cutoff $v^*(n)=V$ for all n, and the equilibrium bidding strategies do not vary with n. The case of $k = 1$ in the example above illustrates this scenario. In this case, $H(x)$ is the uniform distribution, so $g_1(x) = 2x$, which satisfies the condition for a single unit bid equilibrium. Indeed Engelbrecht-Wiggans and Kahn [1998] argue that this is essentially the only equilibrium.

Another distinct possibility is that the cutoff $v^*(n) = 0$ for all n, and therefore the equilibrium is again independent of n. For example, let $H(x) = x^{1/3}$ for $0 \leq x \leq 1$. In this case, $g_1(x) = (2/3)\, x^{-1/3}$, a very sharply decreasing function of x and clearly not satisfying the condition for a single-unit-bid equilibrium. In fact, Engelbrecht-Wiggans and Kahn [1998] present the solution for $n = 2$ bidders as an example where bids equal zero with probability zero - in a sense, this is the opposite extreme from a single-unit-bid equilibrium (though it still involves second-unit bids being strictly lower than values). In Table 1, we have verified that the same solution remains an equilibrium for $n = 3, 4, \ldots, 9$. In other words, we have an equilibrium in which bidders bid positively with probability one, and their bids are independent of n.

2.3 Discussion

In each of our examples, the equilibrium bidding strategy for the second unit is a weakly increasing function of the number *n* of bidders. In some cases, the bid function is independent of *n*, and therefore the amount of demand reduction is also independent of *n*. In other cases, the amount of demand reduction decreases as *n* increases. But the bidding strategy depends on *n* in a peculiar fashion. In particular, in all of our examples, the equilibrium bidding strategy $b(v;n)$ for a second unit of value v in an auction with *n* bidders has the form $b(v;n) = 0$ if $v < v^*(n)$, and $b(v;n) = c(v)$ otherwise, where $c(v)$ is some function independent of *n*, and the cutoff value $v^*(n)$ is a non-increasing function of *n*. In English, the result is that as *n* increases, the equilibrium bid on the second unit may change from being zero to being positive, but if the bid is positive then it is independent of *n*.

This peculiar form of the bidding strategy's dependence on *n* also provides insight into the limiting case of infinitely many bidders. As *n* increases, the amount of demand reduction might strictly decrease, with $v^*(n)$ approaching zero as *n* approaches infinity.[7] However, $c(v) < v$ with probability one, so the limit of $b(v;n)$ is also less than v with probability one. In other words, some demand reduction persists even in the limit of infinitely many bidders.[8]

[7] In the examples we consider in Table 2, we have shown numerically that the value of $v^*(n)$ converges to a strictly positive value. In this case, there is even more asymptotic demand reduction than if $v^*(n)$ converged to zero.

[8] Katzman [1999] establishes the asymptotic persistence of demand reduction without first characterizing the equilibria.

3. Experimental Design and Procedures

We design a field experiment to explore demand reduction in multi-unit auctions with more than two bidders, and examine how the amount of demand reduction varies with the number of bidders. The design follows the sportscard auctions of List and Shogren [1998] and List and Lucking-Reiley [2000]. The auctioned good was three different versions of a Cal Ripken, Jr. rookie baseball card: a 1982 *Topps*, 1982 *Donruss*, and 1982 *Fleer* card. The former card had a book value of $70 while the latter cards had book values of $40. We consider the *Topps* variety to be an imperfect substitute for the *Fleer* and *Donruss* cards, and the *Fleer* and *Donruss* cards to be nearly perfect substitutes. We base our notion of the relationship between the *Fleer* and *Donruss* cards on observed market factors. First, the best estimated figures indicate they had similar production runs. Second, book values of the two cards over the past 10 years show a 1 for 1 relationship. Third, each card was independently graded as "PSA 8 near-mint" by a well-known agency, Professional Sports Authenticators (PSA).[9] All auctions displayed the same sportscards to bidders, and identical copies were sold to winning bidders.

Table 2 shows the experimental treatments based on (i) number of bidders — 2, 3, or 5; (ii) bidder experience — professional card dealers or non-dealers; and (iii) auction type — uniform-price or Vickrey auction. The table also introduces the mnemonics we will use to refer to the different treatments: D2 for two dealers per auction, D3 for three dealers per auction, ND5 for five nondealers per auction, and so on.

Each bidder in each auction went through a four-step procedure. In Step 1, the monitor invited potential subjects to participate in an auction for the two sportscards. The potential subject was told that the auction would take about five minutes. He (or she) could pick up and visually examine each card, which were sealed with the appropriate grade clearly marked. Nobody bid in more than one auction. In Step 2, the monitor gave the bidder two printed materials: (i) an auction rules sheet which included a practice worksheet, and (ii) a bidding sheet. The instructions were identical across treatments, except for auction type and number of bidders.[10] The instructions explained how the auction worked, provided examples to illustrate the auction, and asked the bidder to work through an example himself to make sure he understood the auction rules. Each bidder was told that he would be randomly grouped with two or four other bidders of the same type—dealers matched with dealers; non-dealers matched with non-dealers. The auction treatment was changed at the top of each hour. In Step 3, the participant submitted two bids on a bidding sheet. In Step 4, the monitor explained the ending rules of the auction, how the winners would be determined, and how and when the exchange of money for cards would occur.

The dealer treatments were identical to the nondealer treatments, except that the monitor visited each dealer at his/her booth before the sportscard show opened. All auctions were held at a sportscard show in Tampa, FL, in November 1998.[11] A total of 183 subjects participated in these auctions.

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[9] The 1982 *Topps* cards were also graded 8 by PSA.

[10] The instructions and bidding sheets for the n=5 treatments are displayed in Appendix 2 (uniform-price) and Appendix 3 (Vickrey). The instructions for the n=3 case are identical, except that two of the bidders were deleted from the examples.

[11] The results from the 2-person auctions are taken from List and Lucking-Reiley [2000]. These data were gathered at a different sportscard show, in June 1998. The n=2 subject instructions are virtually identical to those for n=3 and n=5, except that the uniform-price treatment used only one example for the uniform-price auction, rather than three.

4. Results

To put our results into perspective, we review the results of List and Lucking-Reiley [2000], for two-unit, two-person sealed-bid auctions. Having auctioned both low-valued cards ($3 book value) and high-valued cards ($70 book value), they found statistically significant evidence of demand reduction for the high-valued cards.[12] Second-unit bids were considerably lower in the uniform-price auctions than in the Vickrey auctions: the difference in means was between eleven and twelve dollars, or 31% to 54% of the mean second-bid level. Furthermore, the uniform-price auction treatment resulted in significantly more zero bids, and the bid reductions were large enough to cause significant changes in the allocation of goods. They also found an anomalous result that does not conform to theoretical predictions: first-unit bids were higher in uniform-price auctions than in Vickrey auctions.[13]

We present descriptive statistics for our new auction data in Tables 3 and 4. Table 3 reports sample means and standard deviations of first-unit bids, second-unit bids, and auction revenues for each auction treatment. Table 4 reports several interesting sample proportions for each treatment. For comparison purposes, we reprint data on the two-bidder auctions of List and Lucking-Reiley [2000] in the first two rows of each table.[14] The four new experimental treatments are displayed in the last four rows.

[12] The differences were statistically significant only for the two high-value-card treatments, but point estimates of the differences went the same direction in all three low-valued card treatments.

[13] There is good reason to believe that the environment changed significantly between the n=2 and the n=3, 5 experiments. Specifically, whereas the two-unit, two-person sealed-bid auctions took place in Orlando, FL in May, 1998 (in the early part of the baseball season), our 3 and 5 person auctions took place after the baseball season, during the middle of the football season in Tampa, FL. Because of Ripken's and the Baltimore Orioles' subpar 1998 seasons, consumer demand for Ripken cards was much lower in November than in May, and this change is not reflected in the book values as reported by PSA (which tend to be sticky downwards).

[14] We present only the data for relatively high-valued ($70) cards, omitting the less striking results for low-valued ($3) cards, since the high-valued cards are more comparable to those in the new experiment.

The remainder of this section is organized as follows. First, we examine first-unit bids, and show that the anomaly found by List and Lucking-Reiley [2000] disappears in auctions with more than two bidders. The next three sections look at demand reduction by comparing second-unit bids across auction formats: mean second-unit bids in section 4.2, proportions of second-unit bids equal to zero in section 4.3, and the overall distribution of second-unit bids in section 4.4. In section 4.5, we examine individual bid schedules across auction formats, comparing first-unit and second-unit bids submitted by the same individual. Section 4.6 moves from data on individual bidder strategies to data on group outcomes: auction revenues, and allocations of goods. Finally, we summarize the experimental results in Section 4.7.

4.1 Mean first-unit bids

We begin by examining first-unit bids, displayed in the first two columns of Table 3. Theory predicts the distribution of first-unit bids to be constant across auction formats, because it is always a dominant strategy to bid one's value on the first unit. The first two rows demonstrate the violation of this prediction discovered by List and Lucking-Reiley [2000] for two-bidder auctions. First-unit bids were significantly higher in the uniform-price than in the Vickrey auctions, by more than $9 in the dealer treatment (t=3.22) and $13 in the nondealer treatment (t=1.76). In three other treatments for low-valued cards (not reproduced here), List and Lucking-Reiley found every point estimate indicating higher first-unit bids in the uniform-price auctions, though these differences were not statistically significant.

By contrast, our new data show no such effect. Consistent with theory, first-unit bids are very similar across auction formats in each treatment. The mean bid is higher in the Vickrey auction for two treatments (D3, ND5), but higher in the uniform-price auction for the other two treatments (ND3, D5). None of these Vickrey-uniform differences is statistically significant (t_{3D} = -0.20; t_{ND3} = 0.22; t_{D5} = 0.34; t_{ND5} = -0.07). The first-bid anomaly identified by List and Lucking-Reiley [2000] appears to be special to the case of two bidders; with three to five bidders, the first-unit-bid data support the theory.

Our theory predicts that the distribution of first-unit bids should be constant with *n* in both the Vickrey auction and the uniform-price auction. We do not have a perfectly clean test of this hypothesis, but since the *Donruss* and *Fleer* cards are close substitutes with roughly equivalent distributions of values, we compare nondealers' bids on these cards in the *n*=3 versus *n*=5 treatments.[15] The data in Table 3 suggest that the distributions of first-unit bids in both the Vickrey and uniform-price auctions are independent of *n*. As *n* increases from 3 to 5, the mean first-unit bid moves from $20.03 to $20.77 in the Vickrey auction, and from $20.68 to $19.44 in the uniform-price auctions. Both are statistically insignificant differences, with t-statistics of –0.21 and 0.33, respectively.

Overall, first-unit bids appear to be independent of both the auction format and of the number of bidders. These results are consistent with bidders understanding the rules of the auction institutions and bidding according to their dominant strategies.

[15] Since we auctioned the identical *Topps* card in the new 3-dealer auction as in the previously reported 2-dealer auction, this might seem a better opportunity for comparison. However, as noted in footnote 10, the demand for this card decreased dramatically between the original experiment in June and the later experiment in November. Indeed, Table 3 shows that mean bids of all types (first-unit or second-unit, Vickrey or uniform-price) were considerably lower in the November auctions (3 dealers) than in the June auctions (2 dealers) for the same card.

4.2. Mean second-unit bids

Columns 3 and 4 of Table 3 generally provide support for the theoretical prediction of second-unit bids being lower on average in uniform-price than in Vickrey auctions. This occurs in three of the four new experiments; the exception is the treatment with five nondealers per auction. In the remaining three treatments, mean Vickrey second-unit bids exceed the corresponding uniform-price bids by approximately 10-23%. However, individual t-tests indicate that each of these results is statistically insignificant at conventional levels ($t_{D3} = 1.15$; $t_{ND3} = 0.44$; $t_{D5} = 0.53$; $t_{ND5} = -0.24$). By contrast, the previous data on two-bidder auctions, shown in the first two rows of Table 3, show statistically significant differences ($t_{D2} = 3.10$; $t_{ND2} = 2.82$). These results indicate that demand reduction effects on mean second-unit bids become smaller and harder to detect as the number of bidders increases beyond two.[16]

A comparison of second-unit bids between the $n = 3$ and $n = 5$ treatments provides additional evidence on changes in demand reduction with increasing numbers of bidders. Recall that our theory predicts that as n increases, the distribution of second-unit bids remains constant in the Vickrey auction while growing weakly higher in the uniform-price auction. The data in Table 3 support these predictions. As noted earlier, the only relatively clean comparison is between the ND3 and ND5 treatments, where the bidders are drawn from the same population and the card types auctioned are close substitutes for one another. In the Vickrey auctions, as n changes from 3 to 5, the mean second-unit bid increases from \$9.69 to \$9.77, a statistically insignificant

[16] Another possible explanation for the diminished demand reduction is the slightly lower stakes in the $n>2$ experiments relative to the $n=2$ experiments. The n=2 auctions reported here were for cards with book values of \$70, while three of our four n>2 auctions were for cards with book values of only \$40. Indeed, List and Lucking-Reiley [2000] reported that \$70 cards generated statistically significant demand reduction, while \$3 cards did not. While lower stakes may be part of the explanation for our results with \$40 cards, we note that \$40 stakes are still *(continued on next page)*

difference (t=0.03) consistent with the theoretical prediction of no change.[17] In the uniform-price auctions, the mean second-unit bid increases considerably more, from $8.63 to $10.48. Though not statistically significant (t=0.69), this point estimate represents an increase of nearly 20%, in the same direction as predicted by the theory.

4.3 Zero second-unit bids

Another measure of demand reduction is the proportion of zero second-unit bids (full demand reduction). The Nash equilibrium theory predicts more zero bids on second units in the uniform-price auction than in the Vickrey auction. The first two columns of Table 4 show that the proportion of zero bids increases in each treatment when we move from Vickrey to the uniform price auction treatment. Again, the *n*=2 treatments from the previous study are included for comparison in the first two rows of the table, directly above the new results for the *n*=3 and *n* =5 treatments. In each of the six treatments, the proportion of zero bids increases from the Vickrey to the uniform-price auction. None of the increases is individually statistically significant, but combining the four new treatments yields an aggregate t-statistic of 2.00, which is statistically significant at the 5% level.[18] Thus, we have statistical evidence that demand reduction persists

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relatively large, and believe our evidence indicates that changes in the number of bidders do have an important effect.

[17] By contrast, note that there *are* statistically significant differences between the bids of dealers and nondealers. In 5-bidder auctions for the identical 1982 Cal Ripken *Fleer* card, the dealers' second-unit Vickrey mean bid is $20.68 while the nondealers' is $9.77, which is statistically significant (t=3.24). The differences are similar in magnitude for second-unit bids in the uniform-price auctions (t=2.34). Both differences are consistent with dealers having higher demands for the cards on average, probably because of their greater resale opportunities.

[18] There are two different ways to consider pooling the data across treatments. The first is to compute the aggregate proportion of zero bids in the Vickrey and uniform-price auctions, and to compute the standard t-statistic to test for a difference. The aggregate proportions of zero bids across the four treatments are 19.5% for the Vickrey auction, and 30.1% for the uniform-price auction, which produces a t-statistic of 1.92 (p=0.055). However, we choose to use a more powerful aggregate test statistic that takes into account the fact that the underlying proportions of zero bids may vary across treatments, even though they might still be equal (under the null hypothesis) between the

(continued on next page)

when $n > 2$.

Next, we examine how the zero-bid data vary for different values of *n*. Again, we test this hypothesis by comparing nondealers' bids across *n*=3 and *n*=5, since the cards used in those treatments were close substitutes. Our theory makes two predictions: first, the proportion of zero bids in the Vickrey auction is independent of *n*, and second, the proportion of zero second unit bids in the uniform-price auction is weakly decreasing in *n*. Table 4 shows that the proportion of zero second unit bids decreases in both treatments: from 31% to 27% in the Vickrey auctions, and from 42 to 40% in the uniform-price auctions. Neither difference is statistically significant, so both results are consistent with our theoretical predictions. For the uniform-price data, this is consistent with a value distribution for which $v^*(n)$ either remains constant, or changes very slowly, with *n*.

4.4 Distribution of second-unit bids

Our theoretical results indicate that there are two qualitatively different types of demand reduction in Nash equilibrium. First, low-value bidders may reduce their second-unit bids from small, positive amounts in the Vickrey auction to zero bids in the uniform-price auction. Second, high-value bidders may lower their second-unit bids in the Vickrey auction to some lower, but still nonzero value in the uniform-price auction. Both types of demand reduction appear to be present in the data, at least in five of the six treatments. Table 5 displays the raw data for second-unit bids

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Vickrey and uniform-price formats. For example, the value distributions might be different across different cards, and nondealers tend to submit higher proportions of zero bids than dealers. This new test relies on the fact that the t-test statistic for each individual treatment has an approximately normal distribution with mean 0 and variance 1. Furthermore, the four tests are all independent. The sum of four independent, normally distributed random variables is itself a normally distributed random variable with mean equal to the sum of the means and variance equal to the sum of the variances. Thus, the aggregate t-statistic equals the sum of the four independent t-statistics divided by the square root of 4, which in our case equals 2.00 (p=0.045). See Bushe and Kennedy [1984] and Christie [1990] for additional information about such aggregate test statistics.

in all six treatments. The data have been sorted in ascending order, to facilitate comparing the pairs of distributions.

In section 4.3, we presented statistically significant evidence of the first type of demand reduction: bids reduced all the way to zero. In each of the six treatments, the number of zero bids is higher in the uniform-price auction than in the corresponding auction, which can clearly be seen in Table 5. We should also note that, consistent with our theory as illustrated in Figure 1, the uniform-price bid distributions tend to have a gap between zero and the first positive bid amount, relative to the Vickrey bid distributions. The sharpest example is the ND3 treatment, which has four Vickrey bids of $1, while the lowest positive uniform-price bid is $3. Comparing the distributions, the $1 Vickrey bids all appear to have become $0 bids in the uniform-price auction. Using our theory's notation, it looks as if $c(v^*)$ equals $3, so that no bids are observed in the uniform-price auction below this amount. The other treatments show a similar pattern concerning low positive bids. In particular, ND2 has Vickrey bids of $2 and $3, while the lowest positive bid in the uniform-price auction is $5. D5 had three Vickrey bids of $1 or $2, but the lowest positive uniform-price bid in the uniform-price auction is $4. And ND5 shows six Vickrey bids between $1 to $4, while the lowest positive uniform-price bid is $5. The D2 and D3 treatments' lowest positive uniform-price bids precisely equal their lowest positive Vickrey bids ($10 and $3, respectively), which is consistent with the theoretical case of $v^*=0$.

Table 5 also shows evidence of the second type of demand reduction: high bids being reduced to somewhat lower, positive bids. This clearly occurs[19] in five of the six treatments, with

[19] We do not know how to conduct an appropriate formal hypothesis test of this claim. A Kolmogorov-Smirnoff test detects differences in distributions, but it would have low power in this situation, where gaps between the distributions are small but persistent (it measures only the maximum gap). A t-test might be able to demonstrate a *(continued on next page)*

ND5 being the exception. For the D2 treatment, 29 of the uniform-price bids are positive. Comparing these 29 bids to the corresponding 29 bids in the Vickrey bid distribution, we find that 27 of the bids are lower than their Vickrey counterparts (on the same row of the table), while none are higher. Thus, the empirical distribution of uniform-price bids is first-order stochastically dominated by the empirical distribution of Vickrey bids. The other cases are similar, though not quite as clean as in the D2 treatment. In the ND2 treatment, of 26 positive uniform-price bids, 25 are higher, and only 1 is lower, than their counterparts in the Vickrey bid distribution. Of 22 positive uniform-price bids in D3, 14 are lower and only 3 are higher than the corresponding Vickrey bids. Of 21 positive uniform-price bids in ND3, 13 are lower, and only 4 are higher, than their counterparts in the Vickrey bid distribution. And of 24 positive uniform-price in D5, 14 are lower and only 1 higher than their Vickrey counterparts. As predicted, the rightmost part of the bid distribution does tend to shift from higher to lower positive bid amounts when moving to the uniform-price auction.

4.5 Individual bid schedules

Another type of evidence of demand reduction involves the slopes of the bid schedules submitted by the individual bidders. In the uniform-price auction, we expect bidders' bid schedules to be more steeply sloped than they are in the Vickrey auction. That is, the difference between a bidder's first bid and his second bid should be greater under the uniform-price than under the Vickrey auction format. The mean differences between a bidder's first and second bids show this

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significant difference between the means of the distributions once the distributions were truncated to eliminate zero bids in the uniform-price auction, but we are unaware of a statistical technique designed to compare two distributions both truncated at an endogenously determined location.

pattern in all treatments but ND5, as can be inferred from the bid statistics in Table 3. For example, in the D2 treatment from the previous paper, the uniform-price bid difference of $32.07 far exceeds the Vickrey bid difference of $7.83. The corresponding comparisons are $15.72 versus $10.48 in the D3 treatment, and $12.82 versus $10.43 in treatment. The effects are in the predicted direction (except for the case of ND5), though the effects become smaller as the number of bidders increases. In the four new treatments, none of the differences is statistically significant (the closest to statistical significance is D3, with a t-statistic of 1.14).

The middle columns of Table 4 report data on the proportion of individuals who submitted flat bid schedules, with first bid equal to second bid. As predicted, the proportions are higher for the Vickrey auction than for the uniform-price auction in each of the n=2 and n=3 treatments. For example, in the ND3 treatment the Vickrey auction exhibits 19.4% flat bid schedules, while the uniform-price auction exhibits only 11.1%. The effects are statistically significant in the n=2 treatments, but not in the new n=3 treatments. The effect disappears entirely for the n=5 treatment (equal proportions for D5, and a small difference in the wrong direction for ND5). Again, the evidence of demand reduction becomes progressively smaller as we move from 2 to 3 to 5 bidders, which is consistent with our comparative-static theory.

The data on bid schedules also illustrate an important difference between dealers and nondealers. In the five-bidder treatments, the identical *Fleer* card was auctioned to both dealers and nondealers. Dealers submitted flat bid schedules in both auction formats (approximately 30% of the time) more often than nondealers (approximately 20%). Though this difference is not quite statistically significant in isolation, a similar pattern exists for the other treatments, even though the auctioned cards in those treatments were not the same between dealers and nondealers. This confirms the intuition we had when we originally designed the experiment: dealers' demands for cards tend to be flatter than nondealers' demands. Our reasoning was that while dealers have plenty of resale opportunities for multiple identical cards, nondealers' resale opportunities are limited, and they may be more interested in just a single copy for their collections. The data in Table 3 also indicate another difference between bidder types: dealers bid larger amounts than nondealers on average. For the five-bidder treatments, where the comparison is clean, dealers bid approximately $30 and $20 on average for first and second units, while nondealers bid only $20 and $10. This difference in demand levels is also most likely due to differences in resale opportunities: if a nondealer does not desire a card for his own collection, he will find it harder to resell that card at a favorable price than a dealer.

4.6 Revenues and allocations

Having considered evidence on individual bidding decisions, we now turn to group outcomes: revenues and allocations of goods. Regarding revenues, Ausubel and Cramton [1997] predict that for many standard probability distributions of bidder values, Vickrey auction revenues dominate uniform-price auction revenues, but that there also exist distributions of values for which the opposite revenue ranking holds. The rightmost columns of Table 3 present descriptive statistics on revenues in our new auction experiments. As in the N=2 experiments reported in an earlier paper, we find no consistent revenue ranking between the auction formats. Mean Vickrey revenues were higher than mean uniform-price revenues in the D3 and ND5 treatments, while the opposite ranking held in the ND3 and D5 treatments. None of these revenue differences were statistically significant.

The presence of demand reduction can lead, in theory, to inefficient allocations of goods. In particular, when one bidder has the two highest values for each of two units, but submits a strategically low bid on the second unit, the second unit may be allocated inefficiently to another bidder. The previous experiments for *n*=2 showed that demand reduction was large enough to cause measurable changes in allocations: significantly more auctions resulted in the two units being split between two bidders in the uniform-price auctions than in the Vickrey auctions. The rightmost columns of Table 4 compare those allocation results to the new results for the *n*=3 and *n*=5 experiments. In the new experiments, the proportion of split allocations changes in the predicted direction for all but the ND5 treatment, though none of the differences are statistically significant. There are two reasons why the effect on allocations is smaller with more bidders. First, demand reduction effects are smaller. Second, the baseline level of split allocations is higher with more bidders to divide the cards among (12% to 28% when *n*=2, versus 90% to 95% when *n*=5), so there is, in a sense, less room for the uniform-price auction to increase this proportion further.

4.7 Summary of experimental results

The preceding sections have provided a variety of experimental results. Here, we summarize them for convenient reference:

1. The anomalous first-unit-bid differences discovered in the two-bidder case disappear when the number of bidders increases to three or more.
2. Demand reduction in mean second-unit bids also diminishes when the number of bidders increases, but not as quickly as the first-unit effect disappears.

3. The proportion of zero bids remains significantly higher in the uniform-price auction than in the Vickrey auction for $n > 2$.
4. There is evidence of both types of theoretical demand reduction: small positive bids become zero, and large positive bids become smaller positive bids, when moving from the Vickrey to the uniform-price auction.
5. Our data are consistent with the comparative-static predictions of our theory. As n increases from 3 to 5, first-unit mean bids remain approximately constant, as predicted. Second-unit mean bids also remain approximately constant, consistent with a theory that predicts bids to be either constant or increasing in n. These failures to reject null hypotheses might, of course, result from type-II error. However, we note that for the same bid data, our measurements are precise enough to find statistically significant differences between the mean bids of dealers and nondealers for the same card.
6. Allocations continue to differ in the predicted direction between the Vickrey and uniform-price auctions, though the effect is no longer statistically significant for $n > 2$.
7. The revenues generated are approximately equal between auction formats. This equivalence holds for all n from 2 to 5.
8. The treatment with five nondealer bidders produced results that contrasted with theoretical predictions and with the results of all the other experimental treatments. In the ND5 treatment, the distribution of positive second-unit bids was higher, rather than lower, in the uniform-price relative to the Vickrey auction. However, the difference in mean bids was not statistically significant, and the proportion of zero bids did change in the expected direction.

5. Concluding Remarks

In this paper, we have presented new theory and new experimental evidence on bidding behavior in multi-unit auctions. We have shown theoretically that holding constant the number of units to be auctioned, an increase in the number of bidders causes weakly diminished incentives for demand reduction in second-unit bids. Though diminished with increasing *n*, some demand reduction does persist in the asymptotic limit. In our field experiments, we generally confirm the predictions of this theory. We note that demand reduction remains present with three to five bidders per auction, but it becomes smaller and less statistically significant than in previous results with only two bidders per auction. We also note another victory for the Nash equilibrium bidding theory: an anomaly in first-unit bids disappears when moving from two bidders to more than two bidders.

Demand reduction is a topic of considerable practical relevance, with multi-unit auctions increasingly being utilized in the economy. Examples include recent government auctions for spectrum rights and pollution permits, auctions in recently deregulated electricity markets, and the emergence of multi-unit auctions on the Internet for objects from laser printers to Beanie Babies. The investment-banking firm OpenIPO has even begun to conduct initial public offerings of stock shares via uniform-price sealed-bid auctions on the Internet (see Lucking-Reiley [1999]). Previous economic research has shown that strategic demand reduction is a serious concern that may cause large inefficiencies, both in theory and in practice. These previous findings suggested using Vickrey auctions in place of uniform-price auctions, despite the increased complexity and the potential for perceived "unfairness" of the mechanism.

But things are not quite as clear in the case of more than two bidders. True, we see that demand reduction persists, but the amount of demand reduction seems to decrease dramatically. Indeed, we found it relatively difficult to find statistically significant evidence of demand reduction for more than two bidders. Furthermore, for any fixed amount of demand reduction in individuals' bids, the probability that a second-unit bid (rather than a first-unit bid) determines the price will decrease as the number of bidders increases. Thus, we might expect demand reduction to have a very limited effect when there are more than a few bidders.[20]

However, we have so far considered only the case of two units being auctioned.[21] In many real-world applications, such as auctions for Treasury debt or for electric power, the number of units is likely to be considerably greater than two. We might expect demand-reduction incentives to increase with the number of units, because a lower bid on the price-determining marginal unit may then provide the bidder with the benefit of a lower price on *multiple* inframarginal units. Indeed, in practical applications such as Treasury debt and electric power, the number of units may well be larger than the number of bidders. Further research remains to be done on the problem of demand reduction with varying number of units.

[20] Indeed, as Swinkels [1996] demonstrates, the uniform-price auction is efficient in the limit as the number of bidders goes to infinity.

[21] This was done for reasons of theoretical tractability. In this model, the dominant strategy of truthtelling reduces the problem to a one-dimensional strategy: that of the second-unit bid amount. This allows us to provide a general characterization of equilibria with any distribution of independent private values. With three or more units up for auction, the problem becomes much more complicated, as second-unit and third-unit bid strategies depend on each other. For this reason, we have not yet found a way to characterize equilibria in auctions for more than two units.

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Appendix 1. Calculation of uniform-price equilibria.

This appendix provides theoretical insights into how equilibria might vary with the number n of bidders. To do so, we turn to the work of Engelbrecht-Wiggans and Kahn [1998] on uniform-price auctions with privately known values. For the case of two-units, Engelbrecht-Wiggans and Kahn provide conditions characterizing equilibria in undominated strategies. Examining these conditions suggests that equilibria might change with n in a very specific manner.

We start by recalling the key definitions and results from Engelbrecht-Wiggans and Kahn. In particular, Lemma 2.2 assures that any symmetric Nash equilibrium in undominated strategies has the first bid equal to the first value and the second bid $b(v_{i2})$ satisfying $0 \leq b(v_{i2}) \leq v_{i2}$ for all v_{i2}. Therefore we focus our attention on the second bid.

To characterize the second bid, Engelbrecht-Wiggans and Kahn define

$$\Gamma(c,v;n) = \int_0^c (n-1)G_1(x)^{n-2}[G_1(x) - G_2(v) + g_1(x)(v-x)]dx$$

and

$$C(v;n) = \underset{c \in [0,v]}{\operatorname{argmax}}\, \Gamma(c,v;n)$$

where we have amended their original notation to show that Γ and C both also depend on n.

They also define $C(v;n)$ to be an "increasing correspondence" (for any fixed n) if

$x < y$, $a \in C(x;n)$, and $b \in C(y;n)$

together imply that

$a \leq b$.

Their Corollary 5.4 implies that, if $C(v;n)$ is an increasing correspondence and $c(v;n)$ is a selection from $C(v;n)$, then a Nash equilibrium in undominated strategies results from each bidder i bidding

(v_{i1}, $c(v_{i2})$). Furthermore, if the distribution of v_{i2} conditional on v_{i1} has support [0, v_{i1}], then their Theorem 3.2 assures that for any Nash equilibrium in undominated strategies, the second bid $b(v_{i2};n)$ is strictly less than the second value v_{i2} for all v_{i2} (0,V).

Now let us see what can be inferred from these definitions and results. First, note that Corollary 5.4 and Theorem 3.2 together imply that, if $C(v;n)$ is an increasing correspondence and $c(v;n)$ is a selection from $C(v;n)$, then $0 \leq c(v;n) < v$ for all v (0,V). Next note that $c(v;n)$ $C(v;n)$ and $0 < c(v;n) < v$ together imply that $c(v;n)$ satisfies the following first-order condition:

$$G_1(c(v)) - G_2(v) + g_1(c(v))\,(v - c(v)) = 0$$

Let $C'(v)$ denote the set of solutions to the first order condition; note that $C'(v)$ is independent of n. Then combining the two previous observations gives the following Lemma, which combined with the previously mentioned results of Engelbrecht-Wiggans and Kahn gives the subsequent Theorem:

Lemma: If $C(v;n)$ is an increasing correspondence and $c(v;n)$ is a selection from $C(v;n)$, then for each v (0,V), either $c(v;n) = 0$ or $c(v;n)$ $C'(v)$.

Theorem: If $C'(v)$ is an increasing correspondence, $c'(v)$ is a selection from $C'(v)$, and there exists a cutoff $v^*(n)$ such that

v (0,$v^*(n)$) implies that $\Gamma(0,v;n) < \Gamma(c'(v),v;n)$ (1)

and

v ($v^*(n)$,V) implies that $\Gamma(0,v;n) > \Gamma(c'(v),v;n)$ (2)

then a Nash equilibrium in undominated strategies results, for the auction with n bidders, from each bidder i bidding (v_{i1},$b(v_{i2})$), where $b(v_{i2}) = c'(v_{i2})$ for v ($v^*(n)$,V) and $b(v_{i2}) = 0$ otherwise.

This theorem suggests a procedure for computing equilibria. Specifically, first solve for $C'(v)$. Then, pick a $c'(v)$ from $C'(v)$; if the first order condition has a unique solution, then it determines $c'(v)$. Next fix n. Finally, compare $\Gamma(0,v;n)$ with $\Gamma(c'(v),v;n)$ for each v (0,V) and determine whether there exists a cutoff *v*(n)* satisfying the conditions (1) and (2) of the Theorem.

Ideally, we would now provide general conditions under which *C'(v)* is an increasing correspondence and there exists an appropriate cutoff *v*(n)*, and then characterize how the cutoff *v*(n)* varies with *n*. Unfortunately, we have been unable to do so. Instead we turn to the examples discussed in Section 2.

In all of our examples, the first order condition happens to have a unique solution and that solution is weakly increasing in *v*.[22] Furthermore, the necessary conditions in the Theorem are always satisfied and the cutoff is always uniquely defined. As a result, our Theorem gives one - and only one - equilibrium for each of our examples. [23]

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[22] There do exist examples without uniqueness or monotonicity. Specifically, Engelbrecht-Wiggans [1999] provides an example in which *C'(v)* = [0,*v*] for all *v*. This example has a continuum of equilibria, and this set of equilibria is independent of *n*. Engelbrecht-Wiggans and Kahn [1998] provide an example in which the first order condition has a unique solution, but it is not monotonic; they also show how a continuum of equilibria may be constructed in such cases.

[23] Uniqueness is a sticky issue. In particular, Engelbrecht-Wiggans and Kahn [1998] do not claim that their procedure finds all the equilibria. There may be others. But we know of no examples of equilibria that can not be obtained by their procedure.

Appendix 2. Subject instructions for uniform-price auction.

Welcome to Lister's Auctions. You have the opportunity to bid in an auction for two identical sportscards. There are only five bidders in this auction; the other four bidders will be randomly chosen from other participants at today's card show. (If you are a card dealer you will be paired randomly with four other card dealers in your auction. If you are a non-dealer, you will be paired with four other non-dealers.)

The cards up for auction are two copies of the following card: *Card A PSA 8*

Auction Rules:

You are asked to submit two bids — one bid for each card. If you choose to place only one bid, your second bid will be counted as a bid of zero dollars.

Since there are five bidders, there will be a total of ten bids submitted. The winning bids will be the highest two from the group of ten bids. For each card won, the purchase price is equal to the amount of the third-highest bid (that is, the highest losing bid).

I will order the ten bids from highest to lowest to determine the winners of the two items.

Example 1: if the bids are ranked highest to lowest as follows:

$C1	(first bid from bidder C)
$D1	(first bid from bidder D)
$A1	(first bid from bidder A)
$B1	(first bid from bidder B)
$B2	(second bid from bidder B)
$E1	(first bid from bidder E)
$A2	(second bid from bidder A)
$C2	(second bid from bidder C)
$D2	(second bid from bidder D)
$E2	(second bid from bidder E)

Bidder C wins one card and pays **$A1**. Bidder D wins the second card and pays **$A1**.

Example #2. If bids are rank ordered as follows:

$C1	(first bid from bidder C)
$D1	(first bid from bidder D)
$C2	(first bid from bidder C)
$D2	(first bid from bidder D)
$B1	(second bid from bidder B)
$E1	(first bid from bidder E)
$A1	(second bid from bidder A)
$A2	(second bid from bidder A)
$B2	(second bid from bidder B)
$E2	(second bid from bidder E)

Bidder C wins one card and pays **$C2**. Bidder D wins the second card and pays **$C2**.

Example #3. If bids are rank ordered as follows:

$C1	(first bid from bidder C)
$C2	(first bid from bidder D)
$D1	(first bid from bidder C)
$D2	(first bid from bidder D)
$B1	(second bid from bidder B)
$E1	(first bid from bidder E)
$A1	(second bid from bidder A)
$A2	(second bid from bidder A)
$B2	(second bid from bidder B)
$E2	(second bid from bidder E)

Bidder C wins two cards and pays **$D1** for each card.

If a **TIE** occurs between bidders, I will flip a coin to determine the winner.

Example

Before you submit your actual bids, I would like you to work through an example. Consider a couple of bids that you might submit, and write the numbers here in these two blanks.

my 1st bid______________ my 2nd bid______________

Now submit best guesses of bids that the other four bidders might submit, and fill those numbers into these blanks.

bidder 2's 1st bid_____________ bidder 2's 2nd bid_____________

bidder 3's 1st bid_____________ bidder 3's 2nd bid_____________

bidder 4's 1st bid_____________ bidder 4's 2nd bid_____________

bidder 5's 1st bid_____________ bidder 5's 2nd bid_____________

Take the three largest bids and order them from highest to lowest:

highest bid:____ ____ lowest bid:____

Now, determine how many cards you have won, how many cards the other bidders have won, and the amount each winner has to pay. Fill those numbers in here.

number of cards I won______ amount I must pay______

if applicable:

bidder____has won____card(s) and must pay_____

bidder____has won____card(s) and must pay_____

To assure that you understand how this auction mechanism operates, I will check your work after you complete this example.

Final Transaction
At 1PM I will determine the winners of each auction completed between 8AM and 12:30PM. For those auctions completed after 1PM I will determine the winners at 5PM. After the winners pay me (cash or check) for the cards, the cards will be awarded to the winners. Note, regardless of price, the cards will be awarded to the winners. In case you cannot attend the "determination of winners" sessions, please provide your name, mailing address, and phone number below:

Name_____________________

Address____

Phone#____________________

If you are unable to attend at 1PM (or 5PM), I will contact you by phone. Upon receipt of your check or cash, I will send you the cards that you won. All postage will be paid by Lister's Auctions for cards mailed to winners.

Note that while this is a real auction for real cards, I plan to use data on the bids in this auction for economic research. I guarantee to sell both of the cards listed to the winners of this five-bidder auction, no matter what the final auction prices turn out to be. Your bids represent binding commitments to buy cards you win at the prices specified by the auction outcomes.

Good luck—please write your bids on the sheets provided.

Thanks for participating.

Appendix 3. Subject instructions for Vickrey auction.

All text is identical to that of Appendix 1, with the exception of the ***Auction Rules*** *and* ***Example*** *sections, shown below:*

Auction Rules:
You are asked to submit two bids — one bid for each card. If you choose to place only one bid, your second bid will be counted as a bid of zero dollars.

Since there are five bidders, there will be a total of ten bids submitted. The winning bids will be the highest two from the group of ten bids. For each card won, the purchase price will be determined as follows.

> For the first unit you win, you pay an amount equal to the highest rejected bid which was not your own.
>
> For the second unit you win, you pay an amount equal to the second-highest rejected bid which was not your own.

I will order the ten bids from highest to lowest in order to determine the winners of the two items.

Example 1: if the bids are ranked highest to lowest as follows:

$C1	(first bid from bidder C)
$D1	(first bid from bidder D)
$A1	(first bid from bidder A)
$B1	(first bid from bidder B)
$B2	(second bid from bidder B)
$E1	(first bid from bidder E)
$A2	(second bid from bidder A)
$C2	(second bid from bidder C)
$D2	(second bid from bidder D)
$E2	(second bid from bidder E)

Bidder C wins one card and pays **$A1**. Bidder D wins the second card and pays **$A1**.

Example #2. If bids are rank ordered as follows:

$C1	(first bid from bidder C)
$D1	(first bid from bidder D)
$C2	(first bid from bidder C)
$D2	(first bid from bidder D)
$B1	(second bid from bidder B)
$E1	(first bid from bidder E)
$A1	(second bid from bidder A)
$A2	(second bid from bidder A)
$B2	(second bid from bidder B)
$E2	(second bid from bidder E)

Bidder C wins one card and pays **$D2**. Bidder D wins the second card and pays **$C2**.

Example #3. If bids are rank ordered as follows:

$C1	(first bid from bidder C)
$C2	(first bid from bidder D)
$D1	(first bid from bidder C)
$D2	(first bid from bidder D)
$B1	(second bid from bidder B)
$E1	(first bid from bidder E)
$A1	(second bid from bidder A)
$A2	(second bid from bidder A)
$B2	(second bid from bidder B)
$E2	(second bid from bidder E)

Bidder C wins two cards and pays **$D1** for the first card and **$D2** for the second card.

If a **TIE** occurs between bidders, I will flip a coin to determine the winner

Table 1. How the cutoff value *v varies with the number of bidders.**

	n							
k	2	3	4	5	6	7	8	9
1	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
2	0.94	0.82	0.80	0.80	0.79	0.79	0.79	0.79
3	0.70	0.62	0.60	0.60	0.60	0.60	0.59	0.59
5	0.46	0.41	0.40	0.39	0.39	0.39	0.39	0.39
7	0.34	0.30	0.30	0.29	0.29	0.29	0.29	0.29
9	0.27	0.24	0.23	0.23	0.23	0.23	0.23	0.23

Notes:

1. We consider the case of independent, declining marginal values, with $H(x) = 1-(1-x)^k$. Each row considers a different probability distribution, with a different value of k. Our Nash equilibrium bid function is $b_{i2}(v_{i2})=0$ for all $v_{i2}<v^*(n)$, and $b_{i2}=c(v_{i2})>0$ for all $v_{i2}\geq v^*(n)$. The above table shows how this cutoff value $v^*(n)$ varies with n, separately for each value of k.
2. Figure 1 illustrates the bid function for the case $k=3$, with $n=2$ or $n=5$ bidders.

Table 2. Summary of Experiments.

Treatment	Card auctioned	Book value	Bidders per auction	Bidder type	Vickrey auctions	Uniform-price auctions	Total subjects
D2	Ripken Topps	$70	2	Dealers	15	15	60
ND2	Sanders Score	$70	2	Nondealers	17	17	68
D3	Ripken Topps	$70	3	Dealers	9	9	54
ND3	Ripken Donruss	$40	3	Nondealers	12	12	72
D5	Ripken Fleer	$40	5	Dealers	6	6	60
ND5	Ripken Fleer	$40	5	Nondealers	6	6	60

Notes:

1. Treatments D2 and ND2 took place in May 1998, and were previously reported in List and Lucking-Reiley [2000]. The other four treatments, designed specifically for the present paper, took place in November 1998.
2. The total number of subjects includes all number of bidders who participated in either auction format. For example, in treatment ND3, there were 36 bidders in the 12 Vickrey auctions and 36 more bidders in the 12 uniform-price auctions, for a grand total of 72 subjects.

Table 3. Summary Statistics.

Treatment	Bid on Unit #1 Vickrey	Bid on Unit #1 Uniform	Bid on Unit #2 Vickrey	Bid on Unit #2 Uniform	Total Revenue Vickrey	Total Revenue Uniform
D2	49.60	62.67	41.77	30.60	72.87	76.13
	(15.19)	(15.28)	(14.46)	(13.43)	(25.26)	(21.31)
ND2	51.82	62.21	28.82	16.62	48.71	51.06
	(23.44)	(25.32)	(19.98)	(15.40)	(32.26)	(34.03)
D3	39.74	38.17	29.26	22.44	68.67	57.67
	(26.87)	(26.44)	(25.05)	(17.81)	(32.40)	(21.19)
ND3	20.03	20.68	9.69	8.63	29.25	33.92
	(13.60)	(13.46)	(10.46)	(9.94)	(14.32)	(8.76)
D5	31.12	31.45	20.68	18.63	70.17	74.67
	(22.81)	(15.91)	(14.82)	(15.00)	(16.92)	(10.71)
ND5	20.77	19.44	9.77	10.48	48.67	46.67
	(14.20)	(15.93)	(11.03)	(11.74)	(8.36)	(19.66)

Notes:

1. Treatment names (D2, ND3, etc.) are as defined in Table 2.
2. The table presents sample means, with sample standard deviations in parentheses.
3. Bid #1 and Bid #2 are the first and second bid submitted by a bidder.
4. Total Revenue is the total payment received for both cards in the auction.

Table 4. Proportions of zero bids, flat bid schedules, and split allocations.

	Zero second-unit bids		Flat bid schedules		Split allocations	
Treatment	Vickrey	Uniform	Vickrey	Uniform	Vickrey	Uniform
D2	0.0%	3.3%	36.7%	6.7%	11.7%	43.3%
ND2	5.9%	20.6%	14.7%	0.0%	27.9%	42.6%
D3	11.1%	14.8%	25.9%	18.5%	86.1%	91.7%
ND3	30.6%	41.7%	19.4%	11.1%	74.1%	77.8%
D5	6.7%	20.0%	30.0%	30.0%	90.0%	93.3%
ND5	26.7%	40.0%	20.0%	23.3%	95.0%	93.3%

Notes:

1. Treatment names (D2, ND3, etc.) are as defined in Table 2.
2. "Zero second-unit bids" indicates the proportion of second-unit bids equal to zero.
3. We omit data on first-unit bids equal to zero, because only one bidder bid zero on the first unit in any of these experiments. (This was a bid in the uniform-price auction in the ND5 treatment.)
4. "Flat bid schedules" denotes the proportion of bidders whose bid schedules are flat (first-unit bid equals second-unit bid).
5. "Split allocations" indicates the proportion of auctions for which the two goods were split between the two bidders.

Table 5. The empirical distributions of second-unit bids.

D2		ND2		D3		ND3		D5		ND5	
V	U	V	U	V	U	V	U	V	U	V	U
10	0	0	0	0	0	0	0	0	0	0	0
20	10	0	0	0	0	0	0	0	0	0	0
25	10	2	0	0	0	0	0	1	0	0	0
25	15	3	0	3	0	0	0	1	0	0	0
25	20	5	0	3	3	0	0	2	0	0	0
25	20	10	0	5	5	0	0	5	0	0	0
25	20	10	0	9	10	0	0	8	4	0	0
25	25	10	5	10	10	0	0	8	7	0	0
30	25	10	5	10	10	0	0	8	8	1	0
30	25	20	10	10	14	0	0	10	8	1	0
35	25	20	10	15	15	0	0	10	10	1	0
40	25	20	10	15	18	1	0	15	10	3	0
40	25	20	10	20	20	1	0	15	10	3	5
43	30	20	10	25	20	1	0	20	15	4	5
45	30	25	11	35	20	1	0	20	18	5	6
45	30	25	14	35	23	4	3	23	20	5	10
50	30	25	15	40	25	5	5	25	20	10	10
50	32	25	15	40	25	5	7	25	24	10	10
50	32	25	20	45	30	5	8	25	25	10	12
50	35	30	20	45	30	5	10	25	25	10	12
50	40	30	20	50	32	10	10	25	25	10	15
55	40	30	20	50	40	10	10	30	25	15	20
55	40	30	20	50	45	10	10	35	25	20	20
55	40	34	20	50	46	15	10	35	30	20	20
55	40	45	20	60	50	15	12	35	35	20	20
55	45	45	20	65	50	20	15	40	40	25	25
60	45	48	25	100	65	20	15	40	40	25	25
60	49	50	25			20	15	40	45	25	25
60	55	50	25			20	16	45	45	30	30
60	60	55	30			21	20	50	45	40	45
		60	30			25	20				
		60	30			25	20				
		68	50			25	20				
		70	75			25	20				
						30	20				
						30	45				

Notes:
1. Treatment names (D2, ND3, etc.) are as defined in Table 2.
2. V denotes a Vickrey auction, while U denotes a uniform-price auction.
3. Each column presents the raw data on second-unit bids from a given auction treatment, sorted in ascending order.

Figure 1. Graph of the equilibrium bid function b(v) for second-unit bids versus second-unit values, where values come from the k=3 probability distribution (see Table 1). Separate graphs are displayed for n=2 and n=5 bidders.