The Tambara Structure of the Trace Ideal for Cyclic Groups

Maxine Calle and Sam Ginnett

Reed College, Portland OR
supervised by K. Ormsby and A. Osorno
Collaborative Mathematics Research Group, 2019

callem@reed.edu and ginnetts@reed.edu

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Overview

Preliminaries
  The Basic Ingredients

The Trace Ideal
  The Dress Map
  The Trace Ideal

The Main Results
  Example Result
    Generators for cyclic extensions
  Other Results

The End
G-Sets

G-Action
Given a group $G$ an (left) action of $G$ on a set $X$ is function
\[ \cdot : G \times X \rightarrow X \]
such that for all $g, h \in G$ and $x \in X$

1. $e \cdot x = x$

2. $(gh) \cdot x = g \cdot (h \cdot x)$

G-Set
A $G$-set is a set $X$ along with a group action of $G$ on $X$.

Example
Given a group $G$ and a subgroup $H \leq G$ we can make $G/H$ a $G$-Set by giving the action $g \cdot (g'H) = (gg')H$. 
**G-Set Category**

**Equivariant Map**
Given two $G$-sets $X$ and $Y$ an $G$-equivariant map $f : X \to Y$ is a function on the underlying sets satisfying $f(g \cdot x) = g \cdot f(x)$

**G-Set Category**
The $G$-Set Category for a *finite* group $G$ is the category whose objects are finite $G$-Sets and whose morphisms are $G$-equivariant Maps.

**Important Remark**
Any $G$-set is isomorphic to a $G$-set formed by taking a finite direct sum of $G$-sets of the form $G/H$ with action defined as in the previous slide.
Mackey Functors (Abstract Version)

A Mackey Functor $M$ on a finite group $G$ is a pair of a covariant functor $M_*$ and a contravariant functor $M^*$ from $G$-Set to the category of abelian groups such that.

1. $M_*$ and $M^*$ agree on objects
2. $M(X \sqcup Y) \cong M(X) \oplus M(Y)$
3. Given a pullback diagram in $G$-Set.

We have that $M_*(\beta)M^*(\alpha) = M^*(\gamma)M_*(\beta)$
Mackey Functors (Concrete Version)

A Mackey Functor $M$ on a finite group $G$ is an abelian group $M(G/H)$ for each $H \leq G$ and for each $H \leq K \leq G$ and $g \in G$ group homomorphisms $\text{res}_H^K : M(G/K) \to M(G/H)$, $\text{tr}_H^K : M(G/H) \to M(G/K)$ and $c_g : M(G/H) \to M(G/gH)$

1. $\text{res}_H^H = \text{tr}_H^H = c_e = \text{id}$
2. $\text{res}_H^K \circ \text{res}_K^L = \text{res}_H^L$ and $\text{tr}_K^L \circ \text{tr}_H^K = \text{tr}_H^L$
3. $c_g \circ c_h = c_{gh}$, $c_g \circ \text{res}_H^K = \text{res}_{gH}^gK \circ c_g$ and $c_g \circ \text{tr}_H^K = \text{tr}_{gH}^gK \circ c_g$
4. $\text{res}_J^H \text{tr}_K^H = \sum_{x \in J \setminus H/K} \text{tr}_{J \cap x \cap K}^J \circ c_x \circ \text{res}_{J \cap x \cap K}^K$

Translation

Given $H \leq K$ we have a quotient map $q : G/H \to G/K$ and $\text{con}_g : G/H \to G/gH$. We define $\text{tr}_H^K = M_*(q)$, $\text{res}_H^K = M_*(q)$ and $c_g = M_*(\text{con}_g)$
Tambara Functor (Abstract Version)

A Tambara functor $T$ on a finite group $G$ is a collection of a contravariant functor $T^*$, and two covariant functors $T_+$, $T_-$ from the burnside category on $G$ to the category of sets which satisfies

1. $(T^*, T_+)$ is a Mackey Functor
2. $(T^*, T_-)$ is a Semi-Mackey Functor (functor into abelian monoids instead of abelian groups)
3. Given an exponential diagram in $G$-Set

\[
\begin{array}{ccc}
X & \xleftarrow{p} & A & \xleftarrow{\lambda} & Z \\
\downarrow f & & \downarrow \lambda & & \downarrow \rho \\
Y & \xleftarrow{q} & B
\end{array}
\]

We have $T_-(f) T_+(p) = T_+(q) T_-(\rho) T^*(\lambda)$
Tambara Functors (Concrete)

A Tambara Functor on a group $G$ is given by,

1. A ring $T(G/H)$ for each $H \leq G$
2. For each $H \leq K \leq G$ and $g \in G$ the maps
   2.1 $\text{res}^K_H : T(G/K) \to T(G/H)$
   2.2 $\text{tr}^K_H : T(G/H) \to T(G/K)$
   2.3 $\text{N}^K_H : T(G/H) \to T(G/K)$
   2.4 $c_g : T(G/H) \to T(G/gH)$

Where $\text{res}$ is a ring homomorphism, $\text{tr}$ is a homomorphism on the additive abelian group and $\text{N}$ is a homomorphism on the multiplicative abelian monoid.

3. Satisfying a bunch of compatibility conditions on the three maps from which we can recover the original definition.
Examples

Constant Tambara Functor
Given a ring $R$ and a group $G$ we define the constant Tambara functor $R$ at $R$ on $G$ by

1. $R(G/H) = R$
2. $\text{res}_H^K$ and $c_g$ are the identity maps
3. $\text{tr}^K_H(a) = |K:H|a$
4. $\text{N}^K_H(a) = a|K:H|$

Galois Correspondence
Given a field extension $K/F$ with Galois group $G$. The Galois Correspondence which maps $G/H$ to $K^H$ along with inclusion maps, field trace and field norm maps gives a Tambara Functor. This is a subcase of a more general ”fixed point Tambara Functor”. 
The Burnside Tambara Functor

The Burnside Ring

Given a finite group $G$ the Burnside ring $A(G)$ on $G$ is the ring of finite $G$-Sets with addition given by direct sum and multiplication given by cartesian product.

The Burnside Tambara Functor

Given a finite group $G$ the burnside Tambara functor $A(G)$ is given by

1. $A(G/H) = A(H)$
2. restriction is given by restricting the action
3. $tr^K_H(H/J) = K/J, \ c_g(H/K) = c_g(gH/gK)$
4. $N^K_H(X) = \text{Map}_H(K, X)$
Example: $C_p$

$$A(C_p/C_p) \cong \mathbb{Z}[t_p]/(t_p^2 - pt_p)$$

$$A(C_p/e) \cong \mathbb{Z}$$
Grothendieck-Witt Ring

**Quadratic forms**

A quadratic form is a function of the form
\[ q(X_1, X_2, \ldots, X_n) = aX_1X_1 + bX_1X_2 + \cdots + dX_nX_n \]
where the input/output is taken to be in some field \( F \). Two quadratic forms are considered equal if there is a linear coordinate transformation taking one to the other. Any quadratic form is equivalent to one of the form \( q = a_1X_1^2 + \cdots + a_nX_n^2 \) which we write as \( \langle a_1, \ldots, a_n \rangle \)

**Grothendieck-Witt Ring**

Given a field \( F \) the Grothendieck Witt ring \( GW(F) \) is the ring of all quadratic forms over \( F \) where addition is given by
\[ \langle a_1, \ldots, a_n \rangle + \langle b_1, \ldots, b_m \rangle = \langle a_1, \ldots, a_n, b_1, \ldots, b_n \rangle \]
and multiplication by
\[ \langle a_1, a_1 \ldots, a_n \rangle \langle b_1, \ldots, b_m \rangle = \langle a_1b_1, a_1b_2 \ldots a_nb_n \rangle \]
Given a field extension $K/F$ with finite Galois group $G$ we define the Grothendieck Witt Tambara Functor via

1. For each $H \leq G$, $GW_F^K(G/H) = GW(K^H)$
2. Restriction is given by inclusion maps
3. transfer and norm are ..... complicated (part of the project was figuring out how to actually calculate them)
Finite Fields and Cyclic Groups

Let $\mathbb{F}_q$ denote the finite field of order $q$ (power of odd prime).

$$\mathbb{F}_q \subseteq \mathbb{F}_r \iff q \mid r \iff r = q^N$$

Then $\text{Gal}(\mathbb{F}_{q^N}/\mathbb{F}_q) = C_N$. 
Example

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Then $\text{Gal}(\mathbb{F}_q^N/\mathbb{F}_q) = C_N$.

Prime Extensions

For $\mathbb{F}_q \subseteq \mathbb{F}_q^p$ with $p$ prime, only two ‘levels.’
Example: $\mathbb{F}_q \subseteq \mathbb{F}_{q^p}$ for an odd prime $p$

$$GW(\mathbb{F}_q) \cong \mathbb{Z} \oplus \langle \alpha \rangle$$

restriction: $\langle 1 \rangle \mapsto \langle 1 \rangle$

$\langle \alpha \rangle \mapsto \langle \beta \rangle$

transfer: $\langle 1 \rangle \mapsto p\langle 1 \rangle$

$\langle \beta \rangle \mapsto (p - 1)\langle 1 \rangle \oplus \langle \alpha \rangle$

norm: $n\langle 1 \rangle \mapsto n^p \langle 1 \rangle$

$$(n - 1)\langle 1 \rangle \oplus \langle \beta \rangle \mapsto (n^p - 1)\langle 1 \rangle \oplus \langle \alpha^n \rangle$$
Tambara Functor Morphisms

Tambara Functor Morphism (Abstract)

ϕ : T → S is a collection of ring homomorphisms T(X) → S(X) that forms a natural transformation with respect to each of the component Functors of the Tambara functor.

Tambara Functor Morphism (Concrete)

A Tambara functor morphism ϕ : T → S is a ring homomorphism ϕ_H : T(G/H) → S(G/H) for each H ≤ G such that

\[
\begin{array}{ccc}
T(G/K) & \xrightarrow{\varphi_K} & S(G/K) \\
\downarrow \text{res}_H^K & & \downarrow \text{res}_H^K \\
T(G/H) & \xrightarrow{\varphi_H} & S(G/H)
\end{array}
\]

commutes. Similar diagrams for tr, N, c_g
The Dress Map

Definition

- For rings, trace homomorphism (A. Dress [2], 1971)
  \[ D : A(G) \to GW(F) \]
  maps \( G/H \mapsto \text{tr}_{F}^{K^{H}}\langle 1 \rangle \).

- For Tambara functors, Dress map
  \[ D : A_{G} \to GW_{K/F} \]
  is given by trace homomorphism at each level.
Example

\[
\begin{align*}
t_p & \mapsto p\langle 1 \rangle \\
1 & \mapsto \langle 1 \rangle \\
A(C_p/C_p) & \xrightarrow{D_{C_p}} GW(\mathbb{F}_q) \\
A(C_p/e) & \xrightarrow{D_{e}} GW(\mathbb{F}_q^p) \\
1 & \mapsto \langle 1 \rangle
\end{align*}
\]
Example

\[ A(C_p/C_p) \xrightarrow{D_{C_p}} GW(\mathbb{F}_q) \]

\[ tr(N) \xrightarrow{\text{res}} \]

\[ A(C_p/e) \xrightarrow{D_e} GW(\mathbb{F}_{q^p}) \]
Example

\[ A(C_p/C_p) \xrightarrow{\mathcal{D}_{C_p}} GW(\mathbb{F}_q) \]

\[ A(C_p/e) \xrightarrow{\mathcal{D}_e} GW(\mathbb{F}_q^p) \]

1 \xrightarrow{\text{tr}_{\mathbb{F}_q^p}} \text{tr}_{\mathbb{F}_q^p} \langle 1 \rangle = \langle 1 \rangle
Example

\[ \text{tr}^{\mathbb{F}_{q^p}}_{\mathbb{F}_q} \langle 1 \rangle = p \langle 1 \rangle \]

\[ A(C_p/C_p) \xrightarrow{D_{C_p}} GW(\mathbb{F}_q) \]

\[ A(C_p/e) \xrightarrow{D_e} GW(\mathbb{F}_{q^p}) \]

\[ 1 \xrightarrow{\text{tr}^{\mathbb{F}_{q^p}}_{\mathbb{F}_q}} \langle 1 \rangle = \langle 1 \rangle \]
Example

\[ \text{tr}_{C_p}^{e}(1) = t_p \quad \text{tr}_{F_q}^{F_{q^p}} \langle 1 \rangle = p \langle 1 \rangle \]

\[ A(C_p/C_p) \xrightarrow{\mathcal{D}_{C_p}} GW(F_q) \]

\[ A(C_p/e) \xrightarrow{\mathcal{D}_e} GW(F_{q^p}) \]
Example

\[ \text{tr}_{e}^{C_{p}}(1) = t_{p} \rightarrow \text{tr}_{e}^{\mathbb{F}_{q_{p}}} \langle 1 \rangle = \text{tr}_{\mathbb{F}_{q}} \langle 1 \rangle = p \langle 1 \rangle \]
Example

\[ \text{tr}_{e}^{C_{p}}(1) = t_{p} \quad \text{if} \quad \text{tr}_{e}^{F_{q}}\langle 1 \rangle = \text{tr}_{e}^{F_{q}}\langle 1 \rangle = p\langle 1 \rangle \]

\[
\begin{align*}
\mathcal{A}(C_{p}/C_{p}) & \xrightarrow{\mathcal{D}_{C_{p}}} \mathcal{GW}(F_{q}) \\
\text{tr} \left( N \right) \xrightarrow{\text{res}} & \\
\mathcal{A}(C_{p}/e) & \xrightarrow{\mathcal{D}_{e}} \mathcal{GW}(F_{q}^{e}) \\
1 & \xrightarrow{\text{tr}_{e}^{F_{q}}\langle 1 \rangle = \langle 1 \rangle} \\
\end{align*}
\]

Surjectivity?
The Trace Ideal

Definition
The **trace ideal** is the kernel of the Dress map,

\[ \mathcal{TI}_{K/F} = \{ \ker(\mathcal{D}_H) \}_{H \leq G} \]
The Trace Ideal

**Definition**
The **trace ideal** is the kernel of the Dress map,

\[ \mathcal{T}\mathcal{I}_{K/F} = \{ \ker(\mathcal{D}_H) \}_{H \leq G} \]

**Who cares?**
Via isomorphism theorems,

\[ GW_{K/F} \cong A_G / \mathcal{T}\mathcal{I}_{K/F} \]

when the Dress map is surjective.
Hang on, what are Tambara ideals?

**Definition**
An ideal $I$ of a Tambara functor:

1. A ring-theoretic ideal $I(G/H) \subseteq T(G/H)$ for all $H \leq G$,
2. Closed under structure maps.
Hang on, what are Tambara ideals?

**Definition**

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1. A ring-theoretic ideal $I(G/H) \subseteq T(G/H)$ for all $H \leq G$,
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**Generators**

- The ideal **generated** by a subset $S$ is the intersection of all ideals of $T$ containing $S$, denoted $(S)$.
- An ideal is **principal** if $I = ((a))$ for some $a \in T$.
- An ideal is **strongly principal** if there exists $H \leq G$ and $a \in T(G/H)$ such that $I = ((a))$. 
The Agenda

Main Example

Finite extensions of finite fields $\mathbb{F}_q \subseteq \mathbb{F}_{q^p}$ with Galois group $C_p$ (odd prime $p$).

Goal

Describe the trace ideal for cyclic extensions and find generators.
The Agenda

Main Example
Finite extensions of finite fields $\mathbb{F}_q \subseteq \mathbb{F}_{q^p}$ with Galois group $C_p$ (odd prime $p$).

Goal
Describe the trace ideal for cyclic extensions and find generators.

Spoiler: It’s strongly principal!
First Thoughts

The bottom level
Since $D(n) = n\langle 1 \rangle$,

$$n \in T I_{F_{q^p}/F_q} \iff n = 0.$$  

So $T I_{F_{q^p}/F_q}(C_p/e) = (0)$. 
First Thoughts

The bottom level
Since $\mathcal{D}(n) = n\langle 1 \rangle$,

\[ n \in \mathcal{T}\mathcal{I}_{\mathbb{F}_q/p/\mathbb{F}_q} \iff n = 0. \]

So $\mathcal{T}\mathcal{I}_{\mathbb{F}_q/p/\mathbb{F}_q}(C_p/e) = (0)$.

The top level
Have $\mathcal{D}(nt_p + m) = np\langle 1 \rangle + m\langle 1 \rangle = (np + m)\langle 1 \rangle$, so

\[ nt_p + m \iff np - m = 0 \iff \text{multiples of } t_p - p \]
In general...

**Theorem**
If \( n \) is odd, then

\[
\mathcal{T}I_{\mathbb{F}_{q^n}/\mathbb{F}_q}(C_n/C_m) = (t_{p^k} - p^k : p \text{ prime, } p^k \mid m)
\]

is generated as a Tambara ideal by \( t_p - p \in A(C_n/C_p) \) for each prime divisor \( p \) of \( n \).
In general...

**Theorem**

If $n$ is odd, then

$$\mathcal{II}_{\mathbb{F}_q^n/\mathbb{F}_q}(C_n/C_m) = (t_{p^k} - p^k : p \text{ prime, } p^k \mid m)$$

is generated as a Tambara ideal by $t_p - p \in A(C_n/C_p)$ for each prime divisor $p$ of $n$.

**Main Theorem for Cyclic Groups**

There is one generator!
Other Results and Future Work

1. Arbitrary cyclic extensions (non-finite fields)
   - For $\mathbb{C}/\mathbb{R}$, e.g., the trace ideal is 0, implying $GW \cong A$
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2. Profinite extensions of finite fields
   - Quadratic closure $\overline{\mathbb{F}}_{q^{2\infty}}$ and the algebraic closure $\overline{\mathbb{F}}_q$
Other Results and Future Work

1. Arbitrary cyclic extensions (non-finite fields)
   - For $\mathbb{C}/\mathbb{R}$, e.g., the trace ideal is 0, implying $GW \cong A$

2. Profinite extensions of finite fields
   - Quadratic closure $\mathbb{F}_{q^{2\infty}}$ and the algebraic closure $\overline{\mathbb{F}}_q$

3. Prime ideals of $A$
   - In progress...
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Thank You!
(Questions?)