Putting the “k” in Curvature:
$k$-Plane Constant Curvature Conditions

Maxine Calle

Reed College

with Dr. Corey Dunn
at California State University, San Bernardino
REU 2018

callem@reed.edu

November 10, 2018
Overview

Splashing in the Shallow End
Preliminaries
$k$-Plane Curvature

Jumping in the Deep End
$k$-Plane Constant Sectional Curvature
Main Result / Corollaries
$k$-Plane Constant Vector Curvature
General Approach
Example
Motivation:
We study curvature to generate representative numbers that can characterize model spaces.
Who Cares?

**Motivation:**
We study curvature to generate representative numbers that can characterize model spaces.

**Goal:**
We generalize the conditions known as constant sectional curvature and constant vector curvature.
Okay, what do I need to know?

- An **Algebraic Curvature Tensor (ACT)** is a multilinear function \( R : V \times V \times V \times V \to \mathbb{R} \) with the following properties:
  - Skew-symmetry in the first two slots, interchange symmetry, and the Bianchi Identity.

We say \( R \) is a **canonical ACT** if
\[
R_\phi(x, y, z, w) = \phi(x, w)\phi(y, z) - \phi(x, z)\phi(y, w)
\]
for some symmetric, bilinear form \( \phi \).
Okay, what do I need to know?

- **An** **Algebraic Curvature Tensor (ACT)** is a multilinear function $R : V \times V \times V \times V \to \mathbb{R}$ with the following properties:
  - *Skew-symmetry in the first two slots, interchange symmetry, and the Bianchi Identity.*

We say $R$ is a **canonical ACT** if

$$R_{\phi}(x, y, z, w) = \phi(x, w)\phi(y, z) - \phi(x, z)\phi(y, w)$$

for some symmetric, bilinear form $\phi$.

- A **model space** $\mathcal{M} = (V, \langle \cdot , \cdot \rangle, R)$ is defined as a vector space $V \subseteq \mathbb{R}^n$, a non-degenerate inner product on $V$, and an algebraic curvature tensor.
Okay, what do I need to know?

- **An Algebraic Curvature Tensor (ACT)** is a multilinear function $R : V \times V \times V \times V \to \mathbb{R}$ with the following properties:
  - *Skew-symmetry in the first two slots, interchange symmetry, and the Bianchi Identity.*

We say $R$ is a **canonical ACT** if

$$R_{\phi}(x, y, z, w) = \phi(x, w)\phi(y, z) - \phi(x, z)\phi(y, w)$$

for some symmetric, bilinear form $\phi$.

- A **model space** $\mathcal{M} = (V, \langle \ , \ \rangle, R)$ is defined as a vector space $V \subseteq \mathbb{R}^n$, a non-degenerate inner product on $V$, and an algebraic curvature tensor.

- Let $\mathcal{M}$ be a model space and let $x, y \in V$ be tangent vectors. Let $\pi = \text{span}\{x, y\}$ be a non-degenerate 2-plane. The **sectional curvature** is a function $\kappa : V \times V \to \mathbb{R}$, where

$$\kappa(\pi) = \frac{R(x, y, y, x)}{\langle x, x \rangle \langle y, y \rangle - \langle x, y \rangle^2}.$$
You’ve heard of 2-plane curvature...

**Constant Sectional Curvature**

A model space $\mathcal{M}$ has **constant sectional curvature** $\varepsilon$, denoted $\text{csc}(\varepsilon)$, if $\kappa(\pi) = \varepsilon$ for all non-degenerate 2-planes $\pi$. 
Constant Sectional Curvature
A model space $\mathcal{M}$ has **constant sectional curvature** $\varepsilon$, denoted $\text{csc}(\varepsilon)$, if $\kappa(\pi) = \varepsilon$ for all non-degenerate 2-planes $\pi$.

Constant Vector Curvature
A model space $\mathcal{M}$ has **constant vector curvature** $\varepsilon$, denoted $\text{cvc}(\varepsilon)$, if for every $v \in V$, there is some 2-plane $\pi$ where $v \in \pi$ and $\kappa(\pi) = \varepsilon$ for all non-degenerate 2-planes $\pi$. 
Let $M = (V, \langle \cdot, \cdot \rangle, R)$ with $V \subseteq \mathbb{R}^n$ and non-degenerate inner product. Let $L$ be a $k$-plane spanned by some orthonormal basis $\{f_1, \ldots, f_k\}$. Define the $k$-plane scalar curvature of $L$, $\mathcal{K} : L \to \mathbb{R}$, by

$$\mathcal{K}(L) = \sum_{j>i=1}^{k} \kappa(f_i, f_j).$$
...but get ready for \( k \)-plane curvature!!

- Let \( M = (V, \langle \cdot, \cdot \rangle, R) \) with \( V \subseteq \mathbb{R}^n \) and non-degenerate inner product. Let \( L \) be a \( k \)-plane spanned by some orthonormal basis \( \{f_1, \ldots, f_k\} \). Define the \( k \)-plane scalar curvature of \( L \), \( \kappa : L \rightarrow \mathbb{R} \), by

\[
\kappa(L) = \sum_{j>i=1}^{k} \kappa(f_i, f_j).
\]

\( k \)-Plane Constant Sectional Curvature

A model space \( \mathcal{M} \) has \( k \)-plane constant sectional curvature \( \varepsilon \), denoted \( k\text{-csc}(\varepsilon) \), if \( \kappa(L) = \varepsilon \) for all non-degenerate \( k \)-planes \( L \).

\( k \)-Plane Constant Vector Curvature

A model space \( \mathcal{M} \) has \( k \)-plane constant vector curvature \( \varepsilon \), denoted \( k\text{-csc}(\varepsilon) \), if for all \( v \in V \) there is some non-degenerate \( k \)-plane \( L \) containing \( v \) such that \( \kappa(L) = \varepsilon \).
1. If a model space $\mathcal{M}$ has $k\text{-csc}(\varepsilon)$ then it has $k\text{-cvc}(\varepsilon)$.

2. Let $M_1 = (V, \langle \ , \rangle, R_1)$ have $k\text{-csc}(\varepsilon)$ and $M_2 = (V, \langle \ , \rangle, R_2)$ have $k\text{-cvc}(\delta)$. Then $M = (V, \langle \ , \rangle, R = R_1 + R_2)$ has $k\text{-cvc}(\varepsilon + \delta)$.

3. Suppose $M = (V, \langle \ , \rangle, R)$ has $k\text{-cvc}(\varepsilon)$. Let $c \in \mathbb{R}$. Then $M = (V, \langle \ , \rangle, cR)$ has $k\text{-cvc}(c\varepsilon)$.

4. Let $M = (V, \langle \ , \rangle, R)$ where $\dim(\ker(R)) \geq k - 1$. Then $\mathcal{M}$ has $k\text{-cvc}(0)$. 
Recall the Definition:
A model space $\mathcal{M}$ has \textit{k-plane constant sectional curvature} $\varepsilon$, denoted $k\text{-csc}(\varepsilon)$, if $\mathcal{K}(L) = \varepsilon$ for all non-degenerate $k$-planes $L$. 
Theorem

Set $2 \leq k \leq n - 2$. Let $M = (V, \langle \cdot, \cdot \rangle, R)$ be a model space. Suppose $\mathcal{K}(L) = 0$ for all $k$-planes $L$. Then $R = 0$. 
$k$-csc: Main Result and Some Nice Corollaries

**Theorem**

Set $2 \leq k \leq n - 2$. Let $M = (V, \langle \ , \ , \rangle, R)$ be a model space. Suppose $K(L) = 0$ for all $k$-planes $L$. Then $R = 0$.

**Corollaries**

1. Suppose $K_{R_1}(L) = K_{R_2}(L)$ for all $k$-planes $L$. Then $R_1 = R_2$.

2. There is a unique $R$ giving $k$-csc$(\varepsilon)$ where $R = \frac{2\varepsilon}{k(k-1)}R_*$.

3. $\mathcal{M}$ has $k$-csc$(\varepsilon)$ if and only if it has $j$-csc$(\delta)$, where $\delta = \varepsilon \frac{j(j-1)}{k(k-1)}$. 
Theorem

Set \( 2 \leq k \leq n - 2 \). Let \( M = (V, \langle \cdot, \cdot \rangle, R) \) be a model space. Suppose \( \mathcal{K}(L) = 0 \) for all \( k \)-planes \( L \). Then \( R = 0 \).

Corollaries

1. Suppose \( \mathcal{K}_{R_1}(L) = \mathcal{K}_{R_2}(L) \) for all \( k \)-planes \( L \). Then \( R_1 = R_2 \).
2. There is a unique \( R \) giving \( k\text{-csc}(\varepsilon) \) where \( R = \frac{2\varepsilon}{k(k-1)} R_* \).
3. \( M \) has \( k\text{-csc}(\varepsilon) \) if and only if it has \( j\text{-csc}(\delta) \), where \( \delta = \varepsilon \frac{j(j-1)}{k(k-1)} \).

”What’s the deal with \( (n - 1)\text{-csc}(0) \)?”

- Weird things happen!
- Conjecture: \( R \neq 0 \).
Recall the Definition:
A model space $\mathcal{M}$ has $k$-plane constant vector curvature $\varepsilon$, denoted $k\text{-csc}(\varepsilon)$, if for all $v \in V$ there is some non-degenerate $k$-plane $L$ containing $v$ such that $\mathcal{K}(L) = \varepsilon$. 
**$k$-cvc: Calculating $k$-cvc Values**

**Methods:**

- Work in model spaces with canonical tensors,
- Use the eigenspaces!

For a model space with $k$-cvc($\varepsilon$), we can...

- Calculate multiple values for $\varepsilon$ for given $k$,
- Set loose bounds for $\varepsilon$,
- Rotate $(k - 1)$-planes in $v^\perp$ to get a connected set of curvature values.
**k-cvc: Example**

Let $M = (V, \langle \cdot, \cdot \rangle, R)$ be a model space such that $V \subseteq \mathbb{R}^n$, the inner product is positive definite, and $R = R_\phi$ where $\phi$ is represented by

$$
\begin{bmatrix}
  I_2 & 0_2 & \ldots & 0_2 \\
  0_2 & 0 & \ldots & 0 \\
  \vdots & \vdots & \ddots & \vdots \\
  0_2 & 0 & \ldots & 0
\end{bmatrix}
$$

where $I_2$ is the $2 \times 2$ identity matrix and $0_2$ is the $2 \times 2$ matrix whose entries are all 0. Note that $\lambda_1 = 1$ where $\dim(E_1) = 2$ and $\lambda_2 = 0$ where $\dim(E_2) = n - 2$.

- For $k \geq 3$, $M$ has $k$-cvc(0) and $k$-cvc(1).
- If $M$ has 3-cvc($\varepsilon$), then $\varepsilon \in [0, 1]$.
- For $k \geq 4$, if $M$ has $k$-cvc($\varepsilon$), then $\varepsilon \in [0, k - 1)$.
**k-cvc: Connected Sets of Curvature Values**

For any $v \in V$, rotate $(k-1)$-planes in $v^\perp$ to get a connected set of curvature values. Let the linear transformation $A_\theta : [0, \frac{\pi}{2}] \rightarrow SO(n)$ be represented by

$$A_\theta = \begin{bmatrix}
I_{k-1} & 0 & 0 \\
0 & R & 0 \\
0 & 0 & I_{n-k-1}
\end{bmatrix}$$

where

$$R = \begin{bmatrix}
\cos \theta & \sin \theta \\
-\sin \theta & \cos \theta
\end{bmatrix}$$

and $I_m$ is the $m \times m$ identity matrix.

- Let $v \in V$. For all $\varepsilon \in [0, 1]$, there is a $k$-plane $L$ containing $v$ and some $\theta$ such that $\mathcal{K}(A_\theta L) = \varepsilon$.
- Audience Participation: give me an $\varepsilon$, any $\varepsilon$!\(^1\)

\(^1\)*Any $\varepsilon \in [0, 1]$. 

Maxine Calle

**k-Plane Constant Curvature Conditions**

**Splashing in the Shallow End**

Preliminaries

**Jumping in the Deep End**

$k$-Plane Curvature

**Main Result / Corollaries**

$k$-Plane Constant Sectional Curvature

**General Approach**

**Example**
In Conclusion,

Goals:

- To study $k$-csc and $k$-cvc in Riemannian model spaces.
- To generalize some previously known results from 2-plane constant curvature conditions.

Results:

- For $k$-csc:
  - $k$-csc$(\varepsilon)$ uniquely determines $R$ for $2 \leq k \leq n - 2$.
  - $k$-csc$(\varepsilon) \Leftrightarrow j$-csc$(\delta)$, where $\delta = \varepsilon \frac{j(j-1)}{k(k-1)}$.
  - $(n - 1)$-csc$(0)$ is strange and interesting.

- For $k$-cvc:
  - $\varepsilon$ can be found in terms of products of eigenvalues.
  - $\varepsilon$ can be bounded based on sectional curvatures.
  - For any $[a, b] \subset \mathbb{R}$, there exists $\mathcal{M}$ with $k$-cvc of at least that interval.
References

B.Y. Chen (1999)
Relations between Ricci curvature and shape operator for submanifolds with arbitrary codimension

P. Gilkey (2001)
Geometric Properties of Natural Operators Defined by the Riemann Curvature Tensor
World Scientific Pub.

R. Klinger (1991)
A Basis that Reduces to Zero as many Curvature Components as Possible

M. Beveridge (2017)
Constant Vector Curvature for Skew-Adjoint and Self-Adjoint Canonical Algebraic Curvature Tensors
CSUSB REU.
Thank You!
(The End.)