Recitation 09/15

Today: Relations + equivalence classes

Warm up: 1) Draw a picture of the reln on \( R \) given by \( R = \{(a,b) \mid a, b \in \mathbb{R} \text{ and } a^2 < b^2\} \)

2) Let \( R \) be the reln on \( \mathbb{R} \) with \( a \sim b \) if \( a - b \in \mathbb{Q} \).
   (a) Convince yourself \( R \) is an equivalence reln.
   (b) Find 3 disjoint equivalence classes. \([a] = \{b \mid a - b \in \mathbb{Q}\}\)

1) \[ y = x^2 \]
   \[(a, b) \text{ s.t. } a^2 < b \implies R \]

2) \[ a - a = 0 \in \mathbb{Q} \checkmark \]
   (a) \[ a - b \in \mathbb{Q} \]
      \[ b - a = -(a - b) \in \mathbb{Q} \checkmark \]
      \[ a - b, b - c \in \mathbb{Q} \]
      \[ a - c = (a - b) + (b - c) \in \mathbb{Q} \checkmark \]

(b) \[ [0] = \{b \mid b \sim 0 \} \]

[\( \pi \)] Pretty sure \( \pi - \pi \notin \mathbb{Q} \)

[\( \sqrt{2} \)]
Relations between sets $A$ and $B$ is a subset of $A \times B$ usually $B = A$.

**Reflexive:** $\forall a \in A, aRa$

- $A = \{0,1,2\}$
- $R = \{(0,0), (1,1), (2,2), (1,2), (2,1)\}$

- Not Symm: $0 \not\sim 1$
- Not Trans: $0 \sim 1, 1 \sim 2, 0 \not\sim 2$

**Symmetric:** $\forall a, b \in A, aRb \iff bRa$

- "being a cousin of", people

**Equivalence Reln**

$\leq, R$ ($\equiv, A$)

**Transitive:** $\forall a, b, c \in A, aRb \land bRc \implies aRc$

Recall: The equivalence class of $a \in A$ is $[a] = \{b \in A : aRb\}$. The projection map $A \to A/\equiv$ sends $a \to [a]$.

Think: Let $R$ be an equivalence reln. Then $\forall a, b \in A$, either $[a] = [b]$ or $[a] \cap [b] = \emptyset$.

Pf: If $[a] \cap [b] = \emptyset$, then done, so suppose $[a] \cap [b] \neq \emptyset$. (WTS $[a] = [b]$)

This means $\exists c \in A$ s.t. $c \in [a] \cap [b]$, i.e. $c \in [a]$ and $c \in [b]$.

$\leq$: Let $a' \in [a]$, so $a \sim a'$. Then:

- $a' \sim a$ by symmetry,
- $a' \sim c$ by transitivity, since $a \sim c$,
- and $c \sim b$ by symmetry,
- so $a' \sim b$ by transitivity,
- $b \sim a'$ by symmetry,

Hence $a' \in [b]$. Thus $[a] \subseteq [b]$.

$\geq$: Similar. (Exc or see notes). $\square$

**Cor.** If $R$ is equiv. reln, get partition of $A = \bigcup_{i \in I} A_i$ for $A_i = [a_i]$ and $A_i \cap A_j = \emptyset$ for $i \neq j$. 

Thm. \( \{ \text{Equivalence relns on } A^2 \} \leftrightarrow \{ \text{partitions of } A \} \).

Pf/ (\( \Rightarrow \)) Suppose \( A = \bigcup_{i \in I} A_i \), with \( A_i \cap A_j = \emptyset \) for \( i \neq j \). Define \( R \) s.t. \( aRb \)
if \( a, b \in A_i \cdot \) (WTS: \( R \) is (i) ref (ii) symm (iii) trans)

(i) For any \( a \in A_i \), \( aRa \) since \( a \in A_i \).
(ii) Suppose \( aRb \), so \( a, b \in A_i \), which means \( bRa \).
(iii) Suppose \( aRb \), \( bRc \), so \( a, b \in A_i \) and \( b, c \in A_j \) for some \( i, j \in I \).
Since \( b \in A_i \), \( A_i \cap A_j = \emptyset \Rightarrow A_i = A_j \) and so \( a, c \in A_i \), i.e. \( aRc \).

\( \Box \)

Important Example:
Let \( n \in \mathbb{Z} \) and consider the reln on \( \mathbb{Z} \) given by \( a \sim b \) if \( a - b \) is a multiple of \( n \).

Claim. This is an equivalence relation.

Pf/ (reflexive) For any \( a \in \mathbb{Z} \), \( a \sim a \) since \( a - a = 0 = 0 \cdot n \).

(Symmetry) Suppose \( a \sim b \), so \( a - b = kn \) for some \( k \in \mathbb{Z} \). Then \( b - a = -(a - b) = -(kn) = (-k)n \)
so \( bRa \).

(Transitive) If \( a \sim b \) and \( b \sim c \), then \( a - b = kn \) and \( b - c = k'n \) for some \( k, k' \in \mathbb{Z} \). Then \( a - c = a + (-b + b - c) = (a - b) + (b - c) = kn + k'n = (k + k')n \), so \( a \sim c \).

\( \Box \)

Defn. The equivalence classes are \( \mathbb{Z}/n =: \mathbb{Z}/n\mathbb{Z} \). "\( \mathbb{Z} \) mod \( n \mathbb{Z} \)"

Claim \( |\mathbb{Z}/n\mathbb{Z}| = n \)

Pf/ We will show there is a bijection \( f: \mathbb{Z}/n\mathbb{Z} \to \{0, 1, \ldots, n-1\} \), which implies \( |\mathbb{Z}/n\mathbb{Z}| = |\{0, 1, \ldots, n-1\}| = n \).

Define \( f[a] := \text{remainder of } a/n \), i.e. \( a = qn + r \) \( \quad \bigcirc \)

Well-defined: Suppose \( [a] = [b] \). (WTS: \( f[a] = f[b] \))
This means \( a \sim b \), so \( a - b = kn \) for some \( k \in \mathbb{Z} \). If \( b = q'n + r' \), then \( a = kn + b \).

\( a = kn + b = kn + q'n + r' = (k + q) + r' \) so \( f[a] = f[b] \).

Injective: Suppose \( f[a] = f[b] \). (WTS \( [a] = [b] \)) Write \( a = qn + r \) and \( b = q'n + k' \), so \( b/c \) \( f[a] = f[b] \), this implies \( r = r' \). Then \( a - b = (qn + r) - (q'n + r') = qn + r - q'n - r' = qn - q'n = (q - q')n \). Hence \( a \sim b \), i.e. \( [a] = [b] \), so \( f \) is injective.
Surjective: Let $r \in \{0,1,\ldots,n-1\}$. Then $f[r] = r$ since $r < n$, i.e. $r = 0 \cdot n + r$. So this shows $f$ is surjective.

Therefore $f$ is a bijection hence $|\mathbb{Z}/n\mathbb{Z}| = n$. □

**Example of not well-defined**

$n = 2$ : $\mathbb{Z}/2\mathbb{Z} \rightarrow \{1,2,3\}$

$\begin{align*}
0 & \mapsto 1 \\
1 & \mapsto 1 \\
2 & \mapsto 2 \\
3 & \mapsto 3 \\
4 & \mapsto 1 \\
5 & \mapsto 2 \\
\vdots & \quad \vdots
\end{align*}$

- $f[0] = 1$
- $f[2] = 2$
- $[0] = [2]$

\[\begin{align*}
x &= y^2 \\
y &= \sqrt[3]{x}
\end{align*}\]
**Practice**

**Part I.**
1. Find examples of \( f: \mathbb{R} \rightarrow \mathbb{R} \) s.t. \( f \) is
   - (a) bijective
   - (b) not injective nor surjective
   - (c) injective but not surjective
   - (d) surjective but not injective

2. (from Lecture) \( f: \mathbb{R} \rightarrow \mathbb{R} \) is bijective \( \iff \) it has an inverse \( g: \mathbb{R} \rightarrow \mathbb{R} \)

**Part II.**
1. Let \( |A| = n \). How many distinct relations are there on \( A \)?
   
   **Bonus:** How many reflexive, symmetric, transitive, etc?

2. (from HW2) Let \( \sim \) be the relation on \( \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\} \) given by \( x \sim y \) if \( xy > 0 \). Describe the corresponding partition.

**Part III.**
1. Let \( \mathcal{A} = \{ \text{differentiable functions } \mathbb{R} \rightarrow \mathbb{R} \} \). Define \( R \) by \( f \sim g \) if \( f(0) = g(0) \).
   - (a) Prove \( R \) is an equivalence relation
   - (b) Let \( S = \mathcal{A}/_R \) and define \( F: S \rightarrow \mathbb{R} \) by \( F[f] = f(0) \).
     Prove \( F \) is well-defined + bijective.

2. Let \( \mathcal{A} = \{ \text{lines in the plane } \mathbb{R}^2 \} \). Prove:
   - (a) "is parallel to" is an equivalence relation
   - (b) "is perpendicular to" is not.
   - (c) Show slope: \( \mathcal{A}/_{\text{parallel}} \rightarrow \mathbb{R} \) is well-defined + bijective.

**Bonus**
Can you construct \( \mathbb{Z} \) from a relation on \( \mathbb{N} \times \mathbb{N} \)?

How about \( \mathbb{Q} \) from a relation on \( \mathcal{Z} \times \mathcal{Z} \setminus \{0\} \)?

Define: \((a,b)R(c,d)\) if \(a+d=b+c\). 

\( R \) is an equivalence relation

\( a-b = c-d \)

1. (reflexive) \( wts: (ab)R(ab) \)
   
   We have \( a+b = b+a \) since addition is commutative, hence \( R \) is reflexive.

2. (symm) Suppose \((a,b)R(c,d)\); so \(a+d=b+c\). \( wts: (c,d)R(a,b) \iff c+b=d+a \)
   
   We know \( b+c = a+d \) by symmetry of \( = \), 
   \( c+b = d+a \) by \( \text{comm.} \) of \( + \).

3. (trans) If \((a,b)R(c,d)\) and \((c,d)R(e,f)\), then \(a+d=b+c\) and \(c+f=d+e\). Then \( (a+d)+(c+f) = (b+c)+(d+e) \)
   
   \( (d+c)+(a+f) = (d+c)+(b+e) \) by \( \text{comm. \ assoc. of } + \)

4. \( a+f=b+e \) by cancellation.
(2) **Note**: if \( a > b \), then \((a,b)R(a-b,0)\)
if \( a < b \), then \((a,b)R(0, b-a)\)
So \([(a,b)] \) looks like \([(0,n)] \) or \([(n,0)] \). Define

\[
\begin{align*}
N \times N/R & \to \{ -\infty, -1, 0, 1, \ldots \} \\
[(n,0)] & \mapsto n \\
[0,n)] & \mapsto -n. \quad \square
\end{align*}
\]

If \(|A|=n\), show \# relations on \( A^2 \) = ?

\[
\#\text{Subset of } A \times A^2 = |\mathcal{P}(A \times A)| = 2^{|A \times A|} = 2^{n^2} = 2^n^2.
\]

Recall: If \(|X|=m\), then \(|\mathcal{P}(X)|=2^m\)

- Reflexive: \((a,a) \in \text{subset}\)
- Symmetric: \[
\begin{bmatrix}
\frac{n^2 + n}{2} \\
2
\end{bmatrix}
\]
- Transitive:

\[\begin{array}{c}
b \cap a \cup b
\end{array}\]