Group Actions

idea: understand $G$ by "what it does" to an object

examples we've seen:

1. Elmts of $S_n$ are (by defn) permutations of \{1,...,n\}
   \[(12) : \{1,2,3\} \rightarrow \{1,2,3\}\]
   
   
   \[(12) \cdot 1 = 2\]
   \[(12) \cdot 2 = 1\]
   \[(12) \cdot 3 = 3\]

2. Elmts of $D_4$ are symmetries of a square

If we label the vertices of the square, can view $r = (1234)$ and $s = (24)$

Think of $D_4$ as acting on the vertices of the square \{1,2,3,4\}

\[
\begin{align*}
   r \cdot 1 &= 2 \\
   r \cdot 2 &= 3 \\
   r \cdot 3 &= 4 \\
   r \cdot 4 &= 1 \\
   s \cdot 1 &= 1 \\
   s \cdot 2 &= 4 \\
   s \cdot 3 &= 3 \\
   s \cdot 4 &= 2
\end{align*}
\]

Group actions abstract & make this rigorous:

**Defn.** An action of $G$ on a set $X$ is an assignment $G \rightarrow \Sigma_X = \text{Bij}(X)$ s.t.

\[
\begin{align*}
   (i) \quad &d(e) = \text{id} \\
   (ii) \quad &d(g) \circ d(h) = d(gh).
\end{align*}
\]

i.e. $d$ is a gp hom

Equivalently, an action is a map $\ast : G \times X \rightarrow X$ s.t.

\[
\begin{align*}
   (i) \quad &e \times x = x \quad \forall x \in X \\
   (ii) \quad &g \times (h \times x) = (gh) \times x \quad \forall g, h \in G, x \in X.
\end{align*}
\]

Notation $G \times X$
(1) Every group acts on itself \( (X = G) \)

Define \( * : G \times G \rightarrow G \) by group operation. Check:

(i) \( e * g = e \cdot g = g \quad \forall g \in G \)

(ii) \( g * (h * k) = g * (h \cdot k) = g \cdot (h \cdot k) = (g \cdot h) \cdot k = g \cdot (h \cdot k) \quad \forall g, h, k \in G \)

(1.5) \( (\mathbb{R}^n, +) \) acts on itself by translations (geometric interpretation)

Given a vector \( v \in \mathbb{R}^n \), define translation by \( v \) by \( T_v : \mathbb{R}^n \rightarrow \mathbb{R}^n \).

Check: (i) \( T_0(w) = 0 + w = w \quad \forall w \in \mathbb{R}^n \)

(ii) \( T_{v_1}(T_{v_2}(w)) = v_1 + T_{v_2}(w) = v_1 + (v_2 + w) = (v_1 + v_2) + w = T_{v_1+v_2}(w) \) \( \text{properties of vector addition} \)

(2) \( GL_n(\mathbb{R}) \) acts on \( \mathbb{R}^n \) by matrix/vector multiplication: \( A \cdot v = Av \)

Check: (i) \( \text{Id} \cdot v = \text{Id} \cdot v = v \)

(ii) \( A \cdot (B \cdot v) = A(B \cdot v) = A(Bv) = (AB)v \)

Properties of matrix multiplication.

(3) Handwaving: Laws of motion in physics should be the same across location, time, in every direction, traveling in fixed direction @ fixed speed

\( \mathbb{R}^4 \) can be described as \( \mathbb{R}^{10 \cdot \text{dim}} \)

(4) If \( H \leq G \) then \( \mathbb{R} \setminus G/H \) by \( g \cdot \hat{g}H = (g \hat{g})H \)

Check: well-defined - if \( \hat{g} \cdot \hat{h} \in \hat{h} \cdot H \) then \( g \hat{g} \cdot \hat{h} \cdot H = g \hat{g} \cdot H \)

\[ \hat{g}_2 \hat{g}_1 \in H \]

\[ (g \hat{g}_2 \cdot \hat{g}_1) \in H \]

(i) \( e \cdot \hat{g}H = (e \hat{g})H = \hat{g}H \)

(ii) \( g_1 \cdot (g_2 \cdot \hat{g}H) = g_1 \cdot (g_2 \hat{g})H = (g_1 g_2 \hat{g})H = (g_1 g_2) \hat{g}H = (g_1 g_2) \cdot \hat{g}H \).
(5) Suppose \( X \) and let \( Y \) be any set. Consider \( \text{Hom}(X,Y) \); can we give it a \( G \)-action? Want: \( G \times \text{Hom}(X,Y) \to \text{Hom}(X,Y) \)

\[
\begin{align*}
(g, f : X \to Y) &\mapsto (g \cdot f) : X \to Y \\
(g \cdot f)(x) &= f(g \cdot x) ?
\end{align*}
\]

Check: (i) \( (e \cdot f)(x) = f(x) \implies e \cdot f = f \checkmark \) 

(ii) \( g \cdot (h \cdot f)(x) = g \cdot f(h \cdot x) = f(g \cdot h \cdot x) \) 

\[
((gh) \cdot f)(x) = f((gh) \cdot x) \quad ?
\]

\( \triangledown \) there's a mistake here! 

\( f(h \cdot x) \in Y \) so \( g \cdot f(h \cdot x) \) doesn't make sense! \( (g \cdot f)(x) \neq g \cdot f(x) \)

\[
\begin{align*}
g \cdot (h \cdot f)(x) &= (h \cdot f)(g \cdot x) \\
&= f(h \cdot g \cdot x) \\
&= f(h \cdot g \cdot x) \neq f(g \cdot h \cdot x) = ((gh) \cdot f)(x)
\end{align*}
\]

To fix: define \( (g \cdot f)(x) = f(g \cdot x) \).

Q. If \( G \subseteq X \), what can we say about \( G \)? about \( X \)?

open-ended

More specific Qs: (1) Given \( X \), what kinds of \( G \) act on it? 

(2) Given \( G \), what kinds of \( X \) have \( G \)-action?

(1) Example Suppose \( X = \{ \varnothing, \varnothing, \varnothing \} \)

- How many actions of \( \mathbb{Z}/5 \) on \( X \)?

  Suppose \( \alpha : \mathbb{Z}/5 \to \Sigma_X \) is a group hom. Since \( \mathbb{Z}/5 = \langle 1 \rangle \), \( \alpha \) is determined by \( \alpha(1) \).

  By \( 1^\text{st} \) iso thm, \( |d(1)| \) divides \( |\mathbb{Z}/5| = 5 \) and \( |\Sigma_X| = |X|! = 6 \)

  \( G/\ker(\alpha) \cong \text{im}(\alpha) \leq \Sigma_X \)

  \( |\ker(\alpha)| = \varnothing \) so \( d(1) = \text{id} \) so \( \alpha \) is the trivial action.

- How many actions of \( \mathbb{Z}/15 \) on \( X \)?

  Similarly, must have \( |d(1)| \mid 15 \), \( \varnothing \) so \( |d(1)| \) is 1 or 3

  - if \( |d(1)| = 1 \), then \( \alpha \) is trivial action
  
  - if \( |d(1)| = 3 \), then \( d(1) = (123) \) or \( (132) \)
Thus we have 3 distinct actions.

In general: If \( \gcd(161, 1x1) = 1 \), then only 6 action on \( X \) is trivial
PFN exercise. Hint: 1^{st} iso-thm + Lagrange's Thm.

(2) Orbits + Stabilizers

**Defn** Let \( G \) \( X \). The orbit of \( x \in X \) is \( \Theta_x = \{ g \cdot x \mid g \in G \} \subseteq X \)

The stabilizer of \( x \in X \) is \( \text{Stab}_x = \{ g \in G \mid g \cdot x = x \} \subseteq G \)

**Ex** \( D_4 \) \( \square \)

\[ \Theta_1 = \{ 1, r, r^2, r^3 \} = X \]
\[ \Theta_2 = \{ 1, 2, 3, 4, 1^2 \} = X \]

in fact \( \Theta_3 = \Theta_4 = X \) also

\( \text{Stab}_1 = \{ e, s \} = \langle s \rangle = \text{Stab}_2 \)
\( \text{Stab}_2 = \{ e, r^2, r^3 : r \cdot s \cdot r = r^{-1} \cdot r \cdot s \cdot r = \} = \text{Stab}_4 \)

**Ex** \( GL_2(\mathbb{R}) \) \( \mathbb{R}^2 \) by \( A \cdot v = Av \)

\[ \Theta_{(0,0)} = \{ A(0) \mid \text{AEGL}_2(\mathbb{R}) \} \]
\[ = \{(0,0)\} \quad \text{fixed pt of the action} \]

\[ \Theta_{(1,0)} = \{ A(1) \mid \text{AEGL}_2(\mathbb{R}) \} \]
\[ = \{(a, 1) \mid A = [a_{11} \ a_{12}^\top \ a_{21} \ a_{22}] \in GL_2(\mathbb{R}) \} \]
\[ = \mathbb{R}^2 \setminus \{(0,0)\} \]

\[ \text{Stab}_{(0,0)} = \{ A \mid A(0) = (0,0) \} \]
\[ = \text{GL}_2(\mathbb{R}) \]

\[ \text{Stab}_{(1,0)} = \{ A \mid A(1) = (1,0) \} \]
\[ = \{ [a_{11} \ a_{12}] \mid a_{12} = 0 \} \]

**Important Theorems** Let \( X \subset G \)

(a) the orbits partition \( X \)

\[ \Theta_x \cap \Theta_y = \emptyset \text{ or } \Theta_x = \Theta_y \]

(b) bijection \( \Theta_x \rightarrow G/\text{Stab}_x \) and in particular

\[ |\Theta_x| = |G : \text{Stab}_x| \text{, "Orbit Stabilizer-thm"} \]
(c) Let $X = \Theta_1 \cup \Theta_2 \cup \cdots \cup \Theta_n$, then $|X| = \sum_{i=1}^{n} |\Theta_i| = \sum_{i=1}^{n} |G : \text{Stab}_G|$.

Ex. $D_4$ (a) trivial partition

(b) $\Theta_1 = \{1, 2, 3, 4\}$
    $\text{Stab}_1 = \{e, s\} = \langle s \rangle$
    $G/\langle s \rangle = \{e, s, r, sr\}$

(c) not interesting by $\langle 1 \rangle$ orbit

**Application of Group Actions**

**Sylow's Theorems** (quickly)

Let $G$ be a group and $p$ a prime.

**Defn.** $G$ is a $p$-group if $|G| = p^k$ for some $k \geq 1$.

- A subgroup $H$ (in any group) is a $p$-subgp if $|H| = p^k$.
- If $|G| = p^n$ where $p \nmid n$, then a subgp $H \leq G$.
- $W$ order $p^d$ is a Sylow $p$-subgp.

**Notation.** $\text{Syl}_p(G) := \{ \text{Sylow p-subgps of G} \}$

- $n_p(G) = |\text{Syl}_p(G)|$

**Thms.**

1. $\text{Syl}_p(G) \neq \emptyset$.
2. If $P_1, P_2 \in \text{Syl}_p(G)$ then $\exists g \in G$ st. $P_1 = gP_2g^{-1}$.
3. If $H \leq G$ is a $p$-subgp, then $\exists P \in \text{Syl}_p(G)$ st. $H \subseteq P$.
4. $n_p(G) \equiv 1 \pmod{p}$.

**Important Consequences:**

- $n_p(G) = |G : N_G(P)|$ so $n_p(G) \mid |G|/p^d$.
- $P \in \text{Syl}_p(G)$ is normal $\iff n_p(G) = 1$.
- If $G$ is Abelian, any subgp is normal, so $n_p(G) = 1$.
  use Thm of f.g. Ab gps to say more...
- If $P \in \text{Syl}_p(G)$ and $Q \in \text{Syl}_q(G)$ for $p \neq q$, then $P \cap Q = e$.

**Example Problems**

(1) Exc 3(1) pt 1 There is only one gp of order 1001.
For any $n$ we have $\mathbb{Z}/n\mathbb{Z}$, so there's at least one. Suppose $|G| = 1001$; we will show $G \cong \mathbb{Z}/1001\mathbb{Z}$.

Note $|G| = 1001 = 7 \cdot 11 \cdot 13$. Apply Sylow Thms to $n = 7$:

$$n_7 | 11 \cdot 13 \quad \text{and} \quad n_7 \equiv 1 \pmod{7}$$

$$\Rightarrow n_7 = 11 \text{ or } 13$$

$$11 \equiv 4 \pmod{7} \quad \text{or} \quad 13 \equiv 6 \pmod{7} \quad \Rightarrow \quad n_7 = 1$$

Similarly argue $n_{11} = n_{13} = 1$. Let $P_7, P_{11}, P_{13}$ be the Sylow subgps. Since $P_7 \cap P_{11} = e$, $P_7P_{11} \cong P_7 \times P_{11} \cong \mathbb{Z}/7 \times \mathbb{Z}/11 \cong \mathbb{Z}/77$.

Similarly, $P_7 \cap P_{13} = e$, so $P_7P_{13} \cong \mathbb{Z}/7 \times \mathbb{Z}/13 \cong \mathbb{Z}/91$.

But $P_7P_{11}P_{13} \leq G$ has the same size as $G$, hence $G = P_7P_{11}P_{13} \cong \mathbb{Z}/1001\mathbb{Z}$.

(2) **Exc 3(2) pt 1** Show $G$ is not simple if $|G| = 80$

**idea:** Show $n_p = 1$ for some $p$.

If $N \not\leq G$ then $N = e$ or $N = G$.

Since $161 = 80 = 2^4 \times 5$, the Sylow Thms tell us $n_5 | 2^4$ and $n_5 \equiv 1 \pmod{5}$ so

$$n_5 = 1, 2 \equiv 5^2, 4 \equiv 5^4, 8 \equiv 5^3, 16 \equiv 5^1$$

If $n_5 = 1$, done, so assume $n_5 = 16$. By 1st Sylow, each of these 16 subgps has order 5 (hence $\cong \mathbb{Z}/5\mathbb{Z}$), and are all distinct so intersect only at $e$. Thus we get $16 \cdot 4 = 64$ elmts of order 5 in $G$. This leaves $80 - 64 = 16$ elmts of $G$.

Now, $n_2 | 5$ and $n_2 \equiv 1 \pmod{2} \Rightarrow n_2 = 1$ or 5. But if $n_2 = 5$, then we have 5 Sylow subgps each of size $2^4 = 16$, which isn't possible ($G$ is too small).

So $n_2 = 1$ and we're done.

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**HW time**

**Exc 2** Consider $T = \mathbb{S} \left[ \begin{array}{cc} a & b \\ c & d \end{array} \right]$. Under mtx mult.

- Show $\left[ \begin{array}{c} x \\ y \end{array} \right] = \left[ \begin{array}{c} ax \\ dy \end{array} \right]$ defines $TR^2$. 

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**Minute sheet:** - What was most helpful today?
- What would you like to see more of?
- Less of?
(2) Find the orbits + stabilizers
(3) Show \( U = \{ [a, b] \mid k \in \mathbb{R} \} \leq T \) and identify \( T/U \)

**Exe 4)** Let \( |G| = pq \) for \( p,q \) prime

1. Show \( G \) has a unique Sylow \( p \)-group \( P \) which is normal in \( G \).
2. Find \( |\text{Aut}(P)| \).
3. Let \( H \in \text{Syl}_p(G) \). Show if \( p \not\equiv 1 \pmod{q} \) then \( H \triangleleft G \) by conjugation is trivial.
4. Conclude \( G \) is cyclic if \( p \not\equiv 1 \pmod{q} \).