

Who Wants a Good Reputation? Two Corrections

George J. Mailath and Larry Samuelson

February 17, 2012

1 Introduction

In Mailath and Samuelson (2001), Proposition 1.2 is incorrect as written and the proof of Proposition 3 contains an error. We thank Lucas Maestri for bringing the first and Eduardo Faingold for bringing the second to our attention. We refer the reader to Mailath and Samuelson (2001) for the setting and notation.

2 Proposition 1.2

The correct statement of Proposition 1.2 is:

Proposition 1.2 *If $\theta = 0$ and*

$$\lambda < \frac{1 - 2\rho}{1 - \rho}$$

then for any

$$\phi' \in \left(0, \frac{\rho}{\rho + \lambda(1 - \rho)}\right),$$

there exists $\bar{c} > 0$ such that for all $0 \leq c < \bar{c}$, the pure strategy profile in which the competent firm chooses high effort in period t if and only if $\phi_t \geq \phi'$ is a Markov perfect equilibrium.

The first displayed equation is a new requirement, while the second displayed equation tightens the previous condition that $\phi' \in (0, 1 - \lambda)$ (note that $\rho/[\rho + \lambda(1 - \rho)] < 1 - \lambda$). Both conditions hold if λ and ϕ' are not too large, which is the case we have in mind. The first addition ensures that the slope of $\varphi(\phi|g)$ with respect to ϕ , evaluated at zero, is positive. This ensures that there are values of ϕ at which a good outcome is sufficiently good news as to push the posterior probability of a competent firm upward, i.e., that there exist values of ϕ at which $\varphi(\phi|g) > \phi$. The second requirement ensures that $\varphi(\phi|g) > \phi$ for values of ϕ above ϕ' . The competent firm then chooses high effort in order to reduce the probability that the posterior falls below the cutoff ϕ' at which expectations of high effort vanish.

With these changes, the proof of the proposition is correct as written.

3 The Proof of Proposition 3

The corrected proof has a structure analogous to that of the published proof, replacing arguments based on bounded derivatives with those based on Lipschitz bounds.

We are concerned here with the endogenous-replacements model examined in Section 5. With the exception of the new lemma 0, numbered items correspond to their counterparts in Mailath and Samuelson (2001).

We first collect from Mailath and Samuelson (2001) some ideas that will play prominent roles in the proposition and proof. In an equilibrium in which competent firms always exert high effort, posterior beliefs of the consumers are given by

$$\varphi(\phi | g) = (1 - \lambda) \frac{(1 - \rho)\phi}{(1 - \rho)\phi + \rho(1 - \phi)} + \lambda\nu + \lambda\kappa D(V_C(\varphi(\phi | g)) - V_I(\varphi(\phi | g))), \quad (9)$$

and

$$\varphi(\phi | b) = (1 - \lambda) \frac{\rho\phi}{\rho\phi + (1 - \rho)(1 - \phi)} + \lambda\nu + \lambda\kappa D(V_C(\varphi(\phi | b)) - V_I(\varphi(\phi | b))). \quad (10)$$

Given that competent firms exert high effort, the equilibrium belief function for consumers is given by

$$p(\phi) = (1 - 2\rho)\phi + \rho. \quad (11)$$

The value function of the inept firm is then

$$V_I(\phi) = (1 - 2\rho)\phi + \rho + \delta(1 - \lambda) \{ \rho V_I(\varphi(\phi | g)) + (1 - \rho) V_I(\varphi(\phi | b)) \}, \quad (12)$$

and the value function of the competent firm is

$$V_C(\phi) = (1 - 2\rho)\phi + \rho - c + \delta(1 - \lambda) \{ (1 - \rho) V_C(\varphi(\phi | g)) + \rho V_C(\varphi(\phi | b)) \}. \quad (13)$$

From (13), a necessary condition for always exerting high effort to be consistent with equilibrium is that for all possible posteriors ϕ , a one period deviation to low effort not be profitable, i.e.,

$$\delta(1 - \lambda)(1 - 2\rho) \{ V_C(\varphi(\phi | g)) - V_C(\varphi(\phi | b)) \} \geq c. \quad (14)$$

An argument analogous to that of the one-shot deviation principle for repeated games shows that (14) is also sufficient.

Definition 2 *The triple (τ, p, φ) is a reputation equilibrium if the competent firm chooses high effort in every state ($\tau(\phi) = 1$ for all ϕ), the expectation updating rules and the value functions of the firms satisfy (9)–(13), and the competent firm is maximizing at every ϕ .*

Proposition 3 *Suppose $\nu > 0$, $\lambda > 0$, $\delta(1 - \lambda) < \rho(1 - \rho)/(1 - 3\rho + 3\rho^2)$, and D' is bounded. Then there exists $\kappa^* > 0$ and $c^* > 0$ such that a reputation equilibrium exists for all $\kappa \in [0, \kappa^*]$ and $c \in [0, c^*]$.*

4 The Proof

Proof. The proof requires some lemmas that we collect at the end of the proof.

[**Step 1**] We first show that any admissible posterior updating rule implies a unique pair of value functions. Fix $\eta \in (\delta(1 - \lambda)(1 - 3\rho + 3\rho^2)/(\rho - \rho^2), 1)$, and let¹

$$\begin{aligned} \mathfrak{X} \equiv \left\{ f \in \mathcal{C}([0, 1], [0, 1]^2) : \right. \\ \delta |\rho f_1(\phi) + (1 - \rho)f_2(\phi) - (\rho f_1(\phi') + (1 - \rho)f_2(\phi'))| \leq \eta |\phi - \phi'|, \\ \delta |(1 - \rho)f_1(\phi) + \rho f_2(\phi) - ((1 - \rho)f_1(\phi') + \rho f_2(\phi'))| \leq \eta |\phi - \phi'|, \\ |f_1(\phi) - f_1(\phi')| \leq \frac{2}{\rho} |\phi - \phi'|, \quad |f_2(\phi) - f_2(\phi')| \leq \frac{2}{\rho} |\phi - \phi'| \\ \left. \text{for all distinct } \phi, \phi' \in [0, 1] \right\}. \end{aligned}$$

Any function $\varphi(\phi) \equiv (\varphi(\phi | g), \varphi(\phi | b)) \in \mathfrak{X}$ is a potential posterior updating rule, giving, for any prior probability ϕ , the posteriors $(\varphi(\phi | g), \varphi(\phi | b))$ that follow a good and bad utility realization. The set \mathfrak{X} is a convex, compact subset of the normed space $\mathcal{C}([0, 1], [0, 1]^2)$, with the norm

$$\|f\| = \max \left\{ \sup_{\phi} |f_1(\phi)|, \sup_{\phi, \phi'} \left| \frac{f_1(\phi) - f_1(\phi')}{\phi - \phi'} \right|, \sup_{\phi} |f_2(\phi)|, \sup_{\phi, \phi'} \left| \frac{f_2(\phi) - f_2(\phi')}{\phi - \phi'} \right| \right\}. \quad (*)$$

The set \mathfrak{X} is nonempty. In particular, let φ_0 denote the exogenous updating rule, i.e., κ is set equal to zero in (9) and (10). Noting that this updating rule is differentiable, the proof of lemma 0 verifies that, for $x \neq y \in \{g, b\}$, $0 \leq \rho \varphi'_0(\phi | x) + (1 - \rho)\varphi'_0(\phi | y) \leq (1 - \lambda)(1 - 3\rho + 3\rho^2)/(\rho - \rho^2) < \eta/\delta$ and $\varphi'_0(\phi | x) \leq 2/\rho$, implying that $\varphi_0 \equiv (\varphi_0(\cdot | g), \varphi_0(\cdot | b)) \in \mathfrak{X}$.

Let $Y \equiv (1 - \rho)/(1 - \delta)$ and

$$\begin{aligned} \mathfrak{Y} \equiv \left\{ f \in \mathcal{C}([0, 1], [-Y, Y]^2) : \right. \\ \sup_{x, y} |(f_1(x) - f_1(y))/(x - y)| \leq (1 - 2\rho)/(1 - \eta), \\ \left. \sup_{x, y} |(f_2(x) - f_2(y))/(x - y)| \leq (1 - 2\rho)/(1 - \eta) \right\}. \end{aligned}$$

Interpret an element of \mathfrak{Y} as a pair of possible value functions (V_I, V_C) , one for the inept firm and one for the competent firm. Fix an updating rule $\varphi \equiv$

¹We correct here the difficulty in the proof of this result in Mailath and Samuelson (2001), where we defined the set \mathfrak{X} as the set of continuously differentiable functions with bounds on derivatives, instead of continuous functions satisfying Lipschitz bounds. The former definition does not allow us to draw the needed conclusion that \mathfrak{X} is compact.

$(\varphi(\cdot | g), \varphi(\cdot | b)) \in \mathfrak{X}$, and let $\Psi^\varphi : \mathfrak{Y} \rightarrow \mathcal{C}([0, 1], \mathfrak{R}^2)$ denote the mapping whose coordinates are the functions:

$$\Psi_1^\varphi(V_I, V_C)(\phi) = (1 - 2\rho)\phi + \rho + \delta(1 - \lambda) \{\rho V_I(\varphi(\phi | g)) + (1 - \rho)V_I(\varphi(\phi | b))\},$$

and

$$\Psi_2^\varphi(V_I, V_C)(\phi) = (1 - 2\rho)\phi + \rho - c + \delta(1 - \lambda) \{(1 - \rho)V_C(\varphi(\phi | g)) + \rho V_C(\varphi(\phi | b))\}.$$

The mapping Ψ^φ is a contraction on \mathfrak{Y} (Lemmas A and B), and so has a unique fixed point. For any updating rule φ , this fixed point identifies the unique value functions that are consistent with φ , in the sense of satisfying (12) and (13). Let $\Phi : \mathfrak{X} \rightarrow \mathfrak{Y}$ denote the mapping that associates, for any updating rule φ in \mathfrak{X} , the fixed point of Ψ^φ . The mapping Φ is continuous (Lemma C).

[Step 2] We now show that there exist updating rules and value functions that are consistent, in the sense that using Φ to obtain value functions from an updating rule and then applying (9)–(10) returns the original updating rules. Let $\hat{\varphi}$ denote the updating rule obtained from φ and $\Phi(\varphi) = (V_I, V_C)$ by using (9)–(10):

$$\hat{\varphi}(\phi | g) = \varphi_0(\phi | g) + \lambda\kappa D[V_C(\varphi(\phi | g)) - V_I(\varphi(\phi | g))],$$

and

$$\hat{\varphi}(\phi | b) = \varphi_0(\phi | b) + \lambda\kappa D[V_C(\varphi(\phi | b)) - V_I(\varphi(\phi | b))].$$

We have, for $x \neq y \in \{g, b\}$.

$$\begin{aligned} & |\rho\hat{\varphi}(\phi | x) + (1 - \rho)\hat{\varphi}(\phi | y) - (\rho\hat{\varphi}(\phi' | x) + (1 - \rho)\hat{\varphi}(\phi' | y))| \\ & \leq |\rho\varphi_0(\phi | x) + (1 - \rho)\varphi_0(\phi | y) - (\rho\varphi_0(\phi' | x) + (1 - \rho)\varphi_0(\phi' | y))| \\ & \quad + \lambda\kappa \left| \rho D[V_C(\varphi(\phi | x)) - V_I(\varphi(\phi | x))] + (1 - \rho)D[V_C(\varphi(\phi | y)) - V_I(\varphi(\phi | y))] \right. \\ & \quad \left. - \rho D[V_C(\varphi(\phi' | x)) - V_I(\varphi(\phi' | x))] - (1 - \rho)D[V_C(\varphi(\phi' | y)) - V_I(\varphi(\phi' | y))] \right| \\ & \leq |\rho\varphi_0(\phi | x) + (1 - \rho)\varphi_0(\phi | y) - (\rho\varphi_0(\phi' | x) + (1 - \rho)\varphi_0(\phi' | y))| \\ & \quad + \lambda\kappa \sup_{d_C, d'_C} \left| \frac{D(d_C) - D(d'_C)}{d_C - d'_C} \right| \sup_{\phi, \phi'} \left| \frac{V_C(\phi) - V_I(\phi) - (V_C(\phi') - V_I(\phi'))}{\phi - \phi'} \right| \\ & \quad \times |\rho\varphi(\phi | x) + (1 - \rho)\varphi(\phi | y) - (\rho\varphi(\phi' | x) + (1 - \rho)\varphi(\phi' | y))|. \end{aligned}$$

The first and third line are each smaller than $\frac{\eta}{\delta}|\phi - \phi'|$, since φ_0 and φ are both contained in \mathfrak{X} . The two absolute values in the middle line have bounded suprema, since $(V_I(\cdot), V_C(\cdot)) \in \mathfrak{Y}$ and the distribution function D has bounded derivative and hence is Lipschitz continuous. There is then a sufficiently small (but positive) value of κ so that

$$|\rho\hat{\varphi}(\phi | x) + (1 - \rho)\hat{\varphi}(\phi | y) - (\rho\hat{\varphi}(\phi' | x) + (1 - \rho)\hat{\varphi}(\phi' | y))| \leq \frac{\eta}{\delta}|\phi - \phi'|.$$

Hence, for sufficiently small values of κ (say $\kappa \leq \kappa^{**}$), the mapping $\Upsilon^\kappa(\varphi) = \hat{\varphi}$ is a mapping from \mathfrak{X} into \mathfrak{X} . The space $\mathcal{C}([0, 1], [0, 1]^2)$ with the norm $\|\cdot\|$ defined in (*) is a locally convex, linear topological space. The set \mathfrak{X} is convex and compact. Moreover, the mapping $\Upsilon^\kappa(\varphi) = \hat{\varphi}$ is clearly continuous (because Φ is continuous), and hence, by the Schauder-Tychonoff theorem (Dunford and Schwartz, 1988, p. 456), has a fixed point. For each value of κ , we denote a posterior updating function which is a fixed point of the mapping Υ^κ by φ_κ , and let $(V_I^\kappa, V_C^\kappa) = \Phi(\varphi_\kappa)$ denote the corresponding value functions. Together, φ_κ and (V_I^κ, V_C^κ) satisfy (9)–(10) and (12)–(13).

[Step 3] We now verify (14). There is a unique triple $(\varphi_0, V_I^0, V_C^0)$ satisfying (9), (10), (12), and (13) when $\kappa = 0$. Since $\nu > 0$ and $\lambda > 0$, there exist $\underline{\phi}$, $\underline{\phi}_0$, $\bar{\phi}_0$, and $\bar{\phi}$, $0 < \underline{\phi} < \underline{\phi}_0 < \bar{\phi}_0 < \bar{\phi} < 1$, such that $\varphi_0(\phi | x) \in [\underline{\phi}_0, \bar{\phi}_0]$ for all $\phi \in [0, 1]$ and $x \in \{g, b\}$. Moreover, there exists c_0^* such that $\delta(1 - \lambda)(1 - 2\rho) \{V_C^0(\varphi_0(\phi | g)) - V_C^0(\varphi_0(\phi | b))\} > c$ for all $\phi \in [\underline{\phi}, \bar{\phi}]$ and $c < c_0^*$ (Lemma D).

Fix $c^* < c_0^*$. The sequential compactness of \mathfrak{X} and \mathfrak{Y} then implies the existence of κ^* ($\leq \kappa^{**}$) such that for all $\kappa \leq \kappa^*$, $\varphi_\kappa(\phi | x) \in [\underline{\phi}, \bar{\phi}]$ for all $\phi \in [0, 1]$ and $x \in \{g, b\}$, and $\delta(1 - \lambda)(1 - 2\rho) \{V_C^\kappa(\varphi_\kappa(\phi | g)) - V_C^\kappa(\varphi_\kappa(\phi | b))\} > c^*$ for all $\phi \in [\underline{\phi}, \bar{\phi}]$. ■

Lemma 0 $\varphi_0 \in \mathfrak{X}$.

Proof. The function φ_0 is differentiable, and hence it suffices to show that its derivatives satisfy the Lipschitz bounds in the specification (*) of the norm $\|\cdot\|$. Differentiating φ_0 gives

$$\varphi_0'(\phi | g) = \frac{(1 - \lambda)\rho(1 - \rho)}{[(1 - 2\rho)\phi + \rho]^2} \leq \frac{2}{\rho}$$

and

$$\varphi_0'(\phi | b) = \frac{(1 - \lambda)\rho(1 - \rho)}{[(1 - \rho) - (1 - 2\rho)\phi]^2} \leq \frac{2}{\rho}.$$

Thus, $\rho\varphi_0'(\phi | g) + (1 - \rho)\varphi_0'(\phi | b) =$

$$\frac{(1 - \lambda)\rho(1 - \rho)}{[(1 - 2\rho)\phi + \rho]^2 [(1 - \rho) - (1 - 2\rho)\phi]^2} \left\{ \rho [(1 - \rho) - (1 - 2\rho)\phi]^2 + (1 - \rho) [(1 - 2\rho)\phi + \rho]^2 \right\}. \quad (**)$$

We want to bound this expression from above by $\frac{2}{\delta}$. Differentiating the term in braces with respect to ϕ yields

$$\begin{aligned} & -2(1 - 2\rho)\rho[(1 - \rho) - (1 - 2\rho)\phi] + 2(1 - 2\rho)(1 - \rho)[(1 - 2\rho)\phi + \rho] \\ &= 2(1 - 2\rho)[- \rho(1 - \rho) + \rho(1 - 2\rho)\phi + (1 - \rho)(1 - 2\rho)\phi + (1 - \rho)\rho] \\ &= 2(1 - 2\rho)^2\phi \geq 0, \end{aligned}$$

and so an upper bound for the term in braces is obtained by evaluating it at $\phi = 1$, i.e., $\{\rho^3 + (1 - \rho)^3\}$.

Differentiating the denominator with respect to ϕ in order to find its minimum yields a first order condition

$$\begin{aligned} & 2(1-2\rho)[(1-2\rho)\phi + \rho][(1-\rho) - (1-2\rho)\phi]^2 \\ & - 2(1-2\rho)[(1-2\rho)\phi + \rho]^2[(1-\rho) - (1-2\rho)\phi] = 0, \end{aligned}$$

i.e.,

$$(1-\rho) - (1-2\rho)\phi = (1-2\rho)\phi + \rho$$

or

$$1-2\rho = 2(1-2\rho)\phi \implies \phi = \frac{1}{2}.$$

This yields three possible minimizers for the denominator, $\phi = 0$, $\frac{1}{2}$, and 1. Evaluating the denominator at $\phi = 0$ and 1 yields $\rho^2(1-\rho)^2$, while evaluating it at $\phi = 1/2$ yields

$$\left[(1-2\rho)\frac{1}{2} + \rho\right]^2 \left[(1-\rho) - (1-2\rho)\frac{1}{2}\right]^2 = \frac{1}{16}.$$

Since $\rho^2(1-\rho)^2 \leq 1/16$, we combine these three cases to conclude that (***) is no larger than

$$\frac{(1-\lambda)\rho(1-\rho)}{\rho^2(1-\rho)^2} \left\{ \rho^3 + (1-\rho)^3 \right\} = \frac{(1-\lambda)(1-3\rho+3\rho^2)}{\rho(1-\rho)} < \frac{\eta}{\delta}.$$

Turning now to the other case, $\rho\varphi'_0(\phi | b) + (1-\rho)\varphi'_0(\phi | g) =$

$$\frac{(1-\lambda)\rho(1-\rho)}{[(1-2\rho)\phi + \rho]^2 [(1-\rho) - (1-2\rho)\phi]^2} \left\{ \rho[(1-2\rho)\phi + \rho]^2 + (1-\rho)[(1-\rho) - (1-2\rho)\phi]^2 \right\}.$$

The denominator is bounded below, as before, by $\rho^2(1-\rho)^2$. The term in braces has derivative

$$\begin{aligned} & 2(1-2\rho)\rho[(1-2\rho)\phi + \rho] - 2(1-2\rho)(1-\rho)[(1-\rho) - (1-2\rho)\phi] \\ = & 2(1-2\rho) \left[\rho(1-2\rho)\phi + \rho^2 - (1-\rho)^2 + (1-\rho)(1-2\rho)\phi \right] \\ = & 2(1-2\rho) \left[(1-2\rho)\phi + \rho^2 - (1-\rho)^2 \right] = 2(1-2\rho)^2\phi + 2(1-2\rho)(2\rho-1) \\ = & 2(1-2\rho)^2(\phi-1) \leq 0, \end{aligned}$$

and so an upper bound for the term in parentheses is obtained by evaluating it at $\phi = 0$, i.e., $\left\{ \rho^3 + (1-\rho)^3 \right\}$. We thus have the same bound as in the previous case, allowing us to conclude that $\varphi_0 \in \mathfrak{X}$. ■

Lemma A $\Psi^\varphi(\mathfrak{Y}) \subset \mathfrak{Y}$.

Proof. Denote the image of (V_I, V_C) under Ψ^φ by (\hat{V}_I, \hat{V}_C) . We verify that $(\hat{V}_I, \hat{V}_C) \in \mathfrak{Y}$ for all $(V_I, V_C) \in \mathfrak{Y}$. Clearly, both \hat{V}_I and \hat{V}_C are continuous, and it is straightforward that $|\hat{V}_I(\phi)|, |\hat{V}_C(\phi)| \leq Y$. Now,

$$\begin{aligned} |\hat{V}_I(\phi) - \hat{V}_I(\phi')| &\leq (1 - 2\rho) |\phi - \phi'| \\ &\quad + \delta(1 - \lambda) \left| [\rho V_I(\varphi(\phi | g)) + (1 - \rho)V_I(\varphi(\phi | b))] \right. \\ &\quad \quad \left. - [\rho V_I(\varphi(\phi' | g)) + (1 - \rho)V_I(\varphi(\phi' | b))] \right| \\ &\leq (1 - 2\rho) |\phi - \phi'| + \delta(1 - \lambda) \frac{1 - 2\rho}{1 - \eta} \\ &\quad \times |(\rho\varphi(\phi | g) + (1 - \rho)\varphi(\phi | b)) - (\rho\varphi(\phi' | g) + (1 - \rho)\varphi(\phi' | b))| \end{aligned}$$

(since $(V_I, V_C) \in \mathfrak{Y}$)

$$\leq (1 - 2\rho) |\phi - \phi'| + \eta \frac{1 - 2\rho}{1 - \eta} |\phi - \phi'|$$

(since $\varphi \in \mathfrak{X}$)

$$= \frac{(1 - 2\rho)}{1 - \eta} |\phi - \phi'|.$$

A similar calculation holds for $|\hat{V}_C(\phi)|$, and so Ψ^φ maps \mathfrak{Y} into \mathfrak{Y} . ■

Lemma B Ψ^φ is a contraction under the norm $\|\cdot\|$ from (*).

Proof. First note that

$$\begin{aligned} &\sup_\phi \left| \Psi_1^\varphi(V_I, V_C)(\phi) - \Psi_1^\varphi(\hat{V}_I, \hat{V}_C)(\phi) \right| \\ &\leq \delta(1 - \lambda) \left\{ \sup_\phi \rho \left| V_I(\varphi(\phi | g)) - \hat{V}_I(\varphi(\phi | g)) \right| + \sup_\phi (1 - \rho) \left| V_I(\varphi(\phi | b)) - \hat{V}_I(\varphi(\phi | b)) \right| \right\} \\ &\leq \delta(1 - \lambda) \sup_\phi \left| V_I(\phi) - \hat{V}_I(\phi) \right| \end{aligned}$$

and similarly that $\sup_\phi \left| \Psi_2^\varphi(V_I, V_C)(\phi) - \Psi_2^\varphi(\hat{V}_I, \hat{V}_C)(\phi) \right| \leq \delta(1 - \lambda) \sup_\phi \left| V_C(\phi) - \hat{V}_C(\phi) \right|$.

For the next component of the norm, we note that

$$\begin{aligned}
& \sup_{\phi, \phi'} \left| \frac{(\Psi_1^\varphi(V_I, V_C))(\phi) - (\Psi_1^\varphi(\hat{V}_I, \hat{V}_C))(\phi) - (\Psi_1^\varphi(V_I, V_C))(\phi') + (\Psi_1^\varphi(\hat{V}_I, \hat{V}_C))(\phi')}{\phi - \phi'} \right| \\
& \leq \delta(1 - \lambda) \sup_{\phi, \phi'} \left| \frac{1}{\phi - \phi'} \left(\rho(V_I(\varphi(\phi | g)) - \hat{V}_I(\varphi(\phi | g))) + (1 - \rho)(V_I(\varphi(\phi | b)) - \hat{V}_I(\varphi(\phi | b))) \right) \right. \\
& \quad \left. - \rho(V_I(\varphi(\phi' | g)) - \hat{V}_I(\varphi(\phi' | g))) + (1 - \rho)(V_I(\varphi(\phi' | b)) - \hat{V}_I(\varphi(\phi' | b))) \right| \\
& \leq \delta(1 - \lambda) \sup_{\phi, \phi'} \left| \rho \frac{\varphi(\phi | g) - \varphi(\phi' | g)}{\phi - \phi'} \frac{V_I(\varphi(\phi | g)) - \hat{V}_I(\varphi(\phi | g)) - V_I(\varphi(\phi' | g)) + \hat{V}_I(\varphi(\phi' | g))}{\varphi(\phi | g) - \varphi(\phi' | g)} \right. \\
& \quad \left. + (1 - \rho) \frac{\varphi(\phi | b) - \varphi(\phi' | b)}{\phi - \phi'} \frac{V_I(\varphi(\phi | b)) - \hat{V}_I(\varphi(\phi | b)) - V_I(\varphi(\phi' | b)) + \hat{V}_I(\varphi(\phi' | b))}{\varphi(\phi | b) - \varphi(\phi' | b)} \right| \\
& \leq \eta(1 - \lambda) \sup_{\phi, \phi'} \left| \frac{V_I(\phi) - \hat{V}_I(\phi) - (V_I(\phi') - \hat{V}_I(\phi'))}{\phi - \phi'} \right|
\end{aligned}$$

while a similar calculation shows that

$$\begin{aligned}
& \sup_{\phi, \phi'} \left| \frac{(\Psi_2^\varphi(V_I, V_C))(\phi) - (\Psi_2^\varphi(\hat{V}_I, \hat{V}_C))(\phi) - (\Psi_2^\varphi(V_I, V_C))(\phi') + (\Psi_2^\varphi(\hat{V}_I, \hat{V}_C))(\phi')}{\phi - \phi'} \right| \\
& \leq \eta(1 - \lambda) \sup_{\phi, \phi'} \left| \frac{V_C(\phi) - \hat{V}_C(\phi) - (V_C(\phi') - \hat{V}_C(\phi'))}{\phi - \phi'} \right|
\end{aligned}$$

Thus,

$$\left\| \Psi^\varphi(V_I, V_C) - \Psi^\varphi(\hat{V}_I, \hat{V}_C) \right\| \leq \max\{\delta(1 - \lambda), \eta(1 - \lambda)\} \left\| (V_I, V_C) - (\hat{V}_I, \hat{V}_C) \right\|$$

and, as claimed, Ψ^φ is a contraction. \blacksquare

Lemma C Φ is continuous.

Proof. Suppose $\varphi_n \rightarrow \varphi_\infty$. Since \mathfrak{Y} is sequentially compact (it is an equicontinuous collection of uniformly bounded functions on a compact space), there is a subsequence, denoted $\{\varphi_m\}$, with $(V_I^m, V_C^m) \equiv \Phi(\varphi_m)$ uniformly converging

to some $(V_I, V_C) \in \mathfrak{V}$. To see that V_I satisfies (12), note that

$$\begin{aligned}
& |V_I(\phi) - (1 - 2\rho)\phi - \rho - \delta(1 - \lambda) \{ \rho V_I(\varphi_\infty(\phi | g)) + (1 - \rho)V_I(\varphi_\infty(\phi | b)) \}| \\
& \leq |V_I(\phi) - V_I^m(\phi)| \\
& \quad + |V_I^m(\phi) - (1 - 2\rho)\phi - \rho - \delta(1 - \lambda) \{ \rho V_I^m(\varphi_m(\phi | g)) + (1 - \rho)V_I^m(\varphi_m(\phi | b)) \}| \\
& \quad + \delta(1 - \lambda)\rho |V_I(\varphi_\infty(\phi | g)) - V_I^m(\varphi_m(\phi | g))| \\
& \quad + \delta(1 - \lambda)(1 - \rho) |V_I(\varphi_\infty(\phi | b)) - V_I^m(\varphi_m(\phi | b))| \\
& = |V_I(\phi) - V_I^m(\phi)| + \delta(1 - \lambda)\rho |V_I(\varphi_\infty(\phi | g)) - V_I^m(\varphi_m(\phi | g))| \\
& \quad + \delta(1 - \lambda)(1 - \rho) |V_I(\varphi_\infty(\phi | b)) - V_I^m(\varphi_m(\phi | b))|, \tag{A.2}
\end{aligned}$$

where the equality holds because $(V_I^m, V_C^m) \equiv \Phi(\varphi_m)$. Now, fix $\epsilon > 0$. There exists m_ϵ such that for all $m \geq m_\epsilon$ and all ϕ , $|V_I(\phi) - V_I^m(\phi)| < \epsilon/3$. Moreover, since V_I is uniformly continuous and $\Phi(\varphi_m)$ converges uniformly to φ_∞ , m_ϵ can be chosen such that $|V_I(\varphi_\infty(\phi | x)) - V_I^m(\varphi_m(\phi | x))| \leq |V_I(\varphi_\infty(\phi | x)) - V_I^m(\varphi_\infty(\phi | x))| + |V_I^m(\varphi_\infty(\phi | x)) - V_I^m(\varphi_m(\phi | x))| \leq \epsilon/3$ for $x \in \{g, b\}$. Thus, (A.2) is less than or equal to ϵ , for all $\epsilon > 0$, and so V_I satisfies (12) for the updating rule φ_∞ . Because there is a unique solution to (12) given φ_∞ , it must then be that V_I^m converges to V_I . A similar argument shows that $V_C^m(\varphi_m)$ converges to $V_C(\varphi_\infty)$, giving the result. \blacksquare

Lemma D *There exists a cost c_0^* such that $V_C^0(\varphi_0(\phi | g)) - V_C^0(\varphi_0(\phi | b)) > c / [\delta(1 - \lambda)(1 - 2\rho)]$ for all $\phi \in [\underline{\phi}, \bar{\phi}]$ and all $c < c_0^*$.*

Proof. There exists $\zeta > 0$ such that for all $\phi \in [\underline{\phi}, \bar{\phi}]$,

$$\varphi_0(\phi | g) - \varphi_0(\phi | b) > \zeta. \tag{A.3}$$

Given $h^t \in \{g, b\}^t$, denote the consumers' posterior using φ_0 after observing the sequence $h^t = (x_1, \dots, x_t)$ by $\varphi_0(\phi | h^t) \equiv \varphi_0(\dots \varphi_0(\varphi_0(\phi | x_1) | x_2) \dots | x_t)$. The value function V_C^0 can be written as, by recursively substituting,

$$V_C^0(\phi) = \frac{\rho - c}{1 - (1 - \lambda)\delta} + (1 - 2\rho)\phi + (1 - 2\rho) \sum_{t=1}^{\infty} \delta^t (1 - \lambda)^t \sum_{h^t \in \{g, b\}^t} \varphi_0(\phi | h^t) \Pr(h^t | H), \tag{A.4}$$

where $\Pr(h^t | H)$ is the probability of realizing the sequence of outcomes h^t given that the firm chooses high effort in every period.

Then, $V_C^0(\varphi_0(\phi | g)) - V_C^0(\varphi_0(\phi | b)) =$

$$\begin{aligned}
& (1 - 2\rho) (\varphi_0(\phi | g) - \varphi_0(\phi | b)) \\
& + (1 - 2\rho) \sum_{t=1}^{\infty} \delta^t (1 - \lambda)^t \sum_{h^t \in \{g, b\}^t} \{ \varphi_0(\phi | gh^t) - \varphi_0(\phi | bh^t) \} \Pr(h^t | H) \\
\geq & (1 - 2\rho) (\varphi_0(\phi | g) - \varphi_0(\phi | b)),
\end{aligned}$$

since $\varphi_0(\phi | gh^t) - \varphi_0(\phi | bh^t) \geq 0$ for all ϕ and all h^t . Thus, using (A.3),

$$V_C^0(\varphi_0(\phi | g)) - V_C^0(\varphi_0(\phi | b)) > (1 - 2\rho) \zeta$$

and so an appropriate upper bound on c is

$$c_0^* \equiv \delta (1 - \lambda) (1 - 2\rho)^2 \zeta.$$

Note that this not a tight bound, since we used only the inequalities that pertain to the first period of the value-function calculations. ■

References

- DUNFORD, N., AND J. T. SCHWARTZ (1988): *Linear Operators Part I: General Theory*. John Wiley and Sons, New York, Wiley Classics Library Edition.
- MAILATH, G. J., AND L. SAMUELSON (2001): “Who Wants a Good Reputation?,” *Review of Economic Studies*, 68(1), 425–442.