The Curse of Long Horizons[*]

V. Bhaskar[†]   George J. Mailath[‡]

July 24, 2017[§]

Abstract

We study dynamic moral hazard with symmetric ex ante uncertainty about the difficulty of the job, when the principal can only commit to spot contracts. The principal and agent update their beliefs about job difficulty on observing output. Since the agent's effort is private, he has an additional incentive to shirk when the principal induces effort: shirking results in the principal having incorrect beliefs, giving rise to future informational rents. We show that the effort inducing contract must provide increasingly high powered incentives as the length of the relationship increases. Thus it is never optimal to always induce effort in very long relationships.

Keywords: principal-agency, moral hazard, differences in beliefs, high-powered incentives.

JEL Classification Codes: D01, D23, D86, J30.

Contents

1 Introduction

We analyze the long-run implications of the ratchet effect, arising from the introduction of new technology, in a context where both firm and worker are learning about its efficacy. Milgrom and Roberts (1990) provide a lucid statement of the problem: when a firm installs new equipment, firms and workers must learn the appropriate work standard. It is efficient to use future information to adjust the standard, but this reduces work incentives today.[1] The ratchet effect arises from the combination of learning, moral hazard and lack of long term commitment by the employer.

Earlier work on the ratchet effect usually assumes ex ante differential information. The agent has private information on the nature of the job, and the principal is unable to make long term commitments. The problem is formulated as one of dynamic mechanism design without commitment in which the principal aims to induce the agent to reveal her private information.

We differ from this literature in formulating the ratchet effect as arising from learning problem under symmetric incomplete information and moral hazard (since worker effort is not observed by the principal). The principal and the agent are symmetrically uncertain about the difficulty of the worker's job. We assume that the principal cannot commit to long term contracts, and has all the bargaining power when choosing optimal spot (short-term) contracts. We also assume there is no limited liability, so the agent is indifferent between accepting the principal's optimal spot contract and taking her outside option. Furthermore, since uncertainty pertains to the nature of the job, the outside option does not depend upon what is learned. Finally, we assume signals do not allow for the principal's learning about job difficulty and about agent behavior to be disentangled (in the sense that a signal that the state is good is also a signal of high effort, and conversely). Our assumption on the structure of the signals is natural, being satisfied by Holmström (1982) and most other parametric models.

The ratchet effect arises from the agent's possible manipulation of the principal's beliefs by shirking. In a pure strategy equilibrium in which high effort is chosen, the principal correctly anticipates the agent's effort choices, and the beliefs of the two parties about the nature of the job agree. However, when the agent deviates and shirks, the beliefs of the two parties differ, at least temporarily. Our analysis begins with a simple observation: In a two period world, such a deviation increases the expected continuation value of

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[1]In the sociological literature, Mathewson (1931), Roy (1952), and Edwards (1979) are workplace studies that document the importance of output restriction in order to influence the firm's beliefs.

the agent. In consequence, any incentive compatible contract inducing high effort must be sufficiently high powered to offset this deviation gain. Thus, the ratchet effect gives rise to a dynamic incentive cost (which we term the *future information rent from shirking*, or *FIRS*), since the agent must be exposed to additional risk in order to overcome the incentive problem (Proposition 1). Since the principal must compensate the agent for increased risk, his wage costs increase. This finding generalizes Milgrom and Roberts (1990), who show this in a model with a linear technology and normal model signals, since we find that this applies under a general information structure and general agent preferences.

The bulk of our analysis concerns the behavior of the dynamic incentive cost as the time horizon T increases. Our focus is on sequentially incentive efficient contracts, where the principal induces high effort in every period. While it is intuitive that the future information rents from shirking in any period should increase with the time horizon, there is a subtlety. The dynamic incentive cost is essentially the opportunity cost of not shirking, and little is known about the comparative statics of the optimal effort contract with respect to costs of shirking. Nonetheless, it turns out that the intuitive increase with the time horizon does occur if either the agent has a specific form of CRRA preferences (Proposition A.1) or if the signal distribution satisfies one additional collinearity restriction (the collinearity assumption is always satisfied with binary signals).

However, a plausible conjecture is that this effect, when present, tapers off: since the both principal and agent learn the state of the world, there is very little uncertainty remaining towards the end of the game. Our main result is that this conjecture is false. Under the collinearity restriction on the signal distribution, the cost of inducing effort in any period is at least linear in the remaining duration of the relationship (Proposition 3). The key insight is the following. Consider the cost of inducing effort in the initial period in a three period setting relative to that in a two period setting. In the three period setting, the initial period future information rents from shirking reflects the increased value of different beliefs in period 2 arising from a period 2 contract that is more high powered than the period 2 contract in the two period setting (which is just the statically optimal contract). We also provide an example showing that in the absence of this positive feedback from one period's future information rents from shirking to earlier periods, the value from having different beliefs in all future periods is bounded (Proposition 4). We also show a similar phenomenon arises under an infinite horizon with discounting (Proposition C.1). Finally, our results on the cost of inducing effort imply that it is never optimal to always induce

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effort when the time horizon is long enough, if the agent's utility function is log, or if we have a sequence of short lived principals. While characterizing the optimal optimal pattern of elicited effort is beyond the scope of the current analysis, we do report some suggestive numerical calculations in the last Section.

1.1 Related Literature

This paper is related to a growing literature on dynamic moral hazard with learning/experimentation. Holmström's (1982) career concerns model is a pioneering example.[2] Like us, Holmström (1982) assumes there is symmetric incomplete information, but critically, in that paper, learning relates to the talent of the agent (not the nature of the job), and so affects the agent's outside option.

As we mentioned earlier, much of the work on the ratchet effect focuses on the asymmetric information case, where the principal wishes to elicit the private information of the agent. Lazear (1986) argues that high powered incentives are able to overcome the ratchet effect, without any efficiency loss, assuming that the worker is risk neutral. Gibbons (1987) shows that Lazear's result depends upon an implicit assumption of long term commitment; in its absence, one cannot induce efficient effort provision by the more productive type.[3] Laffont and Tirole (1988) prove that in general one cannot induce full separation given a continuum of types.[4] Laffont and Tirole (1993) have a comprehensive discussion, and consider both the case of binary types and of a continuum of types. Gerardi and Maestri (2015) analyze an infinite horizon model with binary types.

More recently, there has been increased interest in agency models with learning, where the uncertainty also pertains to the nature of the project. Bergemann and Hege (1998, 2005), Manso (2011), Hörner and Samuelson (2015), and Kwon (2011) and analyze agency models with binary effort, binary signals and limited liability. There is also recent work on learning in agency models with private actions in continuous time and continuum action spaces including DeMarzo and Sannikov (2011), Cisternas (2014), and Pratt and Jovanovic (2014), that examines the agent's incentives for belief

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[2]Extensions of the career concerns model include Gibbons and Murphy (1992) and Dewatripont, Jewitt, and Tirole (1999).

[3]See also Freixas, Guesnerie, and Tirole (1985) and Carmichael and MacLeod (2000).

[4]Malcomson (2016) shows that the no full-separation result also obtains in a relational contracting setting, where the principal need not have all the bargaining power, as long as continuation play following full separation is efficient.

manipulation. Bhaskar (2014) studies a two-period model that makes the same informational and contracting assumptions as in the present paper, but allows for continuum effort choices (rather than binary). The main finding is that the principal cannot implement interior effort choices in the first period. Since the agent can increase his continuation value by shirking, this must be dissuaded by high powered incentives. However, this implies that the agent can deviate upwards, and increase his current payoff, without any loss in continuation value since he can always quit the job tomorrow.

2 The model

We study a risk neutral principal (whom we treat as female) who repeatedly hires a risk averse agent (whom we treat as male) to undertake some task. In each period, the principal offers a spot contract to the agent, who decides whether to accept or reject it. If the agent rejects the contract, the relationship is dissolved and the game ends. If the agent accepts the contract, the agent then decides whether to exert effort e (incurring a disutility of $c > 0$) or shirk s (which is costless). As usual, there is moral hazard, with this choice not observed by the principal. Moreover, there is uncertainty about the "difficulty" of the task. Specifically, there are two states of the world $\omega \in \{B, G\}$, with the task being easy in G, and hard in B. The uncertainty concerns how difficult it is to succeed on *this* job. Importantly, it does *not* affect outside option of the agent, which we normalize to 0.

The choice $a \in \{e, s\}$ by the agent determines, with the state of the world, the probability distribution over signals $y \in Y$, where $Y := \{y^1, y^2, \ldots, y^K\}$ is a finite set of signals. The spot contract specifies the wage payment as a function of the realized signal.

The agent updates his beliefs about the state knowing his own effort choice and the realized public signal. The principal updates her beliefs knowing only the signal, since the agent's effort is not public (i.e., it is not observed by the principal).

The agent's flow utility from a wage payment $w \in \mathbb{R}$ is $u(w)$, where u is strictly increasing and concave. To guarantee individual rationality binds, we assume unlimited liability, so that there are no constraints on the size and sign of utility payments.

We find it more convenient to work with utility schedules, so we write a spot contract as a utility schedule $u := (u^1, \ldots, u^K)$, where u^k is the utility the agent will receive after signal y^k. The wage cost of providing utility level u^k is written $w(u^k) := u^{-1}(u^k)$.

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We do not specify how output signals translate into revenues for the principal. While solving for the equilibrium of the game does require specifying the principal's trade-off between revenues and wage costs, that is not our focus. Our focus, rather, is on the important preliminary step of characterizing the expected cost minimizing sequence of spot contracts that induce effort in every period. This step is independent of the revenue consequences of effort.[5]

There are a finite number of signals with the probability of signal y^k at action $a \in \{s, e\}$ and state $\omega \in \{B, G\}$ denoted by $p_{a\omega}^k$. Our interest is in settings where a signal that the state is good is also a signal of high effort (and conversely), so that it is impossible to disentangle the two. Most parametric models in the learning/experimentation literature satisfy this assumption. For example, it is satisfied if the signal is the number of Poisson distributed successes, with an arrival rate increasing in both the ease of the job and effort. We capture this by the following assumption.

Assumption 1.

1. *There exists an* informative *signal, i.e., there exists $y^k \in Y$ such that either $p_{eG}^k \neq p_{sG}^k$ or $p_{eB}^k \neq p_{sB}^k$.*

2. *For any informative signal $y^k \in Y$,*

$$\min \left\{ p_{sB}^k, p_{eG}^k \right\} < p_{sG}^k, p_{eB}^k < \max \left\{ p_{sB}^k, p_{eG}^k \right\}.$$

3. *Signals have full support: $p_{a\omega}^k > 0$ for all k, a, ω.*

We partition the set of signals into a set of "high" signals Y^H, "low" signals Y^L, and neutral $Y \setminus (Y^H \cup Y^L)$ by setting

$$y^k \in Y^H \text{ if } p_{eG}^k > p_{sG}^k$$

and

$$y^k \in Y^L \text{ if } p_{eG}^k < p_{sG}^k.$$

A signal is high if it is informative of effort in state G. Assumption 1 implies that a high signal is informative of effort also in state B. Also, it is informative of the state being G under either effort or shirking.

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[5]This analysis also does not depend upon the principal's time preference. Also, it is possible to generalize the results to the case where the principal is risk-averse, as long as the agent's incentive constraint binds in the static contract.

A player with belief μ that the task is easy ($\omega = G$) assigns a probability to signal y^k of $p_{a\mu}^k := \mu p_{aG}^k + (1-\mu)p_{aB}^k$. Assumption 1 immediately implies

$$y^k \in Y^H \iff p_{eG}^k > p_{eB}^k, p_{sG}^k > p_{sB}^k \iff p_{e\mu}^k > p_{s\mu}^k$$

and

$$y^k \in Y^L \iff p_{eG}^k < p_{eB}^k, p_{sG}^k < p_{sB}^k \iff p_{e\mu}^k < p_{s\mu}^k.$$

In other words, high signals arise with higher probability when either the agent exerts effort *or* the state is good.

We now show two important implications of Assumption 1 that are key for the rest of the paper. First, if the principal believes that the agent is exerting effort, but the agent is in fact shirking, then on average, the principal will become more pessimistic than the agent after observing the signal realization. Second, if the agent is more optimistic than the principal, and the one-period contract incentivizes effort, the agent earns an informational rent.

Let $\psi_a^k(\mu)$ denote the Bayesian update on the initial belief μ on observing signal y^k, given that action a has been taken. Suppose that the principal believes that the agent has taken action a, while the agent in fact takes action $\tilde{a}$. The average belief of the principal, under the distribution over signals induced by $\tilde{a}$ equals

$$\mathbf{E}(\psi_a^k(\mu))|\tilde{a}) = \sum_k \psi_a^k(\mu)p_{\mu\tilde{a}}^k.$$

The martingale property of beliefs ensures that $\mathbf{E}(\psi_a^k(\mu))|a) = \mu$. The following lemma shows that under Assumption 1, the average belief of the principal is always lower than that of the agent, when the agent shirks and the principal believes that she has exerted effort.

Lemma 1. *If the signals satisfy Assumption 1, then*

$$\boldsymbol{E}(\psi_e^k(\mu))|s) < \boldsymbol{E}(\psi_s^k(\mu))|s) = \mu.$$

Proof. Assumption 1 implies

$$y^k \in Y^H \iff p_{e\mu}^k > p_{s\mu}^k \iff p_{eG}^k > p_{e\mu}^k$$

and

$$y^k \in Y^L \iff p_{e\mu}^k < p_{s\mu}^k \iff p_{eG}^k < p_{e\mu}^k.$$

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Thus,

$$\mu - \sum_k p_{s\mu}^k \frac{\mu p_{eG}^k}{p_{e\mu}^k} = \mu \sum_k (p_{e\mu}^k - p_{s\mu}^k) \frac{p_{eG}^k}{p_{e\mu}^k}$$

$$> \mu \sum_k (p_{e\mu}^k - p_{s\mu}^k) = 0. \qquad \square$$

Suppose the principal and agent both assign probability μ to the task being easy. The *statically optimal spot contract* offered by the principal is a contract $u \in \mathbb{R}^K$ minimizing its expected cost of provision

$$p_{e\mu} \cdot w(u),$$

where $p_{a\mu} := (p_{a\mu}^1, \ldots, p_{a\mu}^K)$, subject to *incentive compatibility*

$$p_{e\mu} \cdot u - c \geq p_{s\mu} \cdot u \qquad \text{(IC)}$$

and *individual rationality*

$$p_{e\mu} \cdot u - c \geq 0. \qquad \text{(IR)}$$

Since the principal is risk neutral and the agent is risk averse, both (IC) and (IR) bind at the statically optimal contract, which is unique and denoted $\hat{u}_\mu$.

Suppose the principal assigns probability μ to G and offers a static contract $\hat{u}_\mu$ at which the (IR) binds (given $p_{e\mu}$). If the agent has belief π and exerts effort, the agent's payoff from exerting effort is

$$\begin{aligned}
V^*(\pi, \mu) :&= p_{e\pi} \cdot \hat{u}_\mu - c \\
&= p_{e\mu} \cdot \hat{u}_\mu - c + (\pi - \mu)(p_{eG} - p_{eB}) \cdot \hat{u}_\mu \\
&= (\pi - \mu)(p_{eG} - p_{eB}) \cdot \hat{u}_\mu. \qquad (1)
\end{aligned}$$

We now show that the expression in (1) is strictly positive. From (IC), we have

$$(p_{e\mu} - p_{s\mu}) \cdot \hat{u}_\mu > 0.$$

Observe that $\hat{u}_\mu^k \geq \hat{u}_\mu^{k'}$ if $p_{e\mu}^k > p_{s\mu}^k$ and $p_{e\mu}^{k'} \leq p_{s\mu}^{k'}$. [If not, there exists k and k' such that $\hat{u}_\mu^k < \hat{u}_\mu^{k'}$ with $p_{e\mu}^k > p_{s\mu}^k$ and $p_{e\mu}^{k'} \leq p_{s\mu}^{k'}$. The contract that equals the old contract except at signals y^k and $y^{k'}$, where the utility promises are replaced by the constant value $(p_{e\mu}^k \hat{u}_\mu^k + p_{e\mu}^{k'} \hat{u}_\mu^{k'})/(p_{e\mu}^k + p_{e\mu}^{k'})$, satisfies (IC) and (IR), at lower cost.]

Assumption 1 then implies that $\hat{u}_\mu^k \geq \hat{u}_\mu^{k'}$ for $y^k \in Y^H$ and $y^{k'} \in Y^L$. We have therefore proved the following lemma.

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Lemma 2. *Suppose the signals satisfy Assumption 1. If the principal has belief μ and offers the statically optimal contract $\hat{u}_\mu$, then the informational rent of the agent when her belief π exceeds μ is no less than*

$$(\pi - \mu)(p_{eG} - p_{eB}) \cdot \hat{u}_\mu > 0.$$

To summarize: Assumption 1 ensures that the agent always has a dynamic incentive to shirk, since by doing so, she induces the principal to be more pessimistic than the agent, and this divergence in beliefs translates to an informational rent in a one-period contract. While this phenomenon has been noted before in the literature for specific informational assumptions, we find that Assumption 1 suffices.

3 Two time periods

We begin with the first two period case. The principal minimizes the total wage costs. The agent maximizes total expected payoff. To minimize notation, we assume the agent does not discount in the finite horizon setting. Our results hold under discounting, with obvious modifications; we discuss discounting in more detail when we analyze the infinite horizon setting in Appendix C.

Neither the principal nor the agent can commit in period 1 to wages or effort in period 2, so each period's spot contract satisfies incentive compatibility (IC) and individual rationality (IR) in that period.

We are interested in the most efficient sequence of spot contracts inducing e in every period. Since there is incomplete information, we require that both the principal and the agent's behavior be sequentially rational after every history, and that both actors update using Bayes' rule whenever possible. The common prior probability on G of the principal and agent is denoted $\mu^\dagger$. Let $\mu_a^k := \psi_a^k(\mu^\dagger)$ be the posterior probability on G after y^k under action a. While the principal does not observe effort, under the sequence of incentive efficient contracts, she assigns probability one to the agent choosing e.

Denote the first period spot contract by $u(1) := (u^1(1), \ldots, u^K(1))$, and the second period spot contract offered by the principal after signal y^k by $u(y^k) := (u^1(y^k), \ldots, u^K(y^k))$.

Definition 1. *A two period sequence of contracts $(u(1), (u(y^k))_{y^k \in Y})$ is sequentially effort incentive efficient if*

1. for every first period signal realization $y^k \in Y$, $u(y^k)$ minimizes

$$p_{e\mu_e^k} \cdot w(u) = \sum_{k'} p_{e\mu_e^k}^{k'} \, w(u^{k'})$$

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subject to the agent finding it optimal to participate and exert effort in the second period after exerting effort in the first period, and

2. *$u(1)$ minimizes $\sum p_{e\mu^\dagger}^k w(u^k)$ subject to the agent finding it optimal to participate and exert effort in the first period.*

Under a sequentially effort incentive efficient sequence of contracts, the agent exerts effort in every period, and the second period beliefs of the agent and principal agree. In particular, after y^k, the second period effort incentive efficient contract solves the static problem with public beliefs μ_e^k.

The first period is more complicated, since the agent's deviation to shirking in the first period results in the principal and agent having different beliefs. After signal y^k, the agent has update μ_s^k, which differs from the principal's update of μ_e^k. In addition, the principal is mistaken in her conviction that the agent also has the belief μ_e^k.

We saw at the end of the previous section that if $\mu_s^k > \mu_e^k$, then the agent receives a strictly positive payoff from the contract $\hat{u}_{\mu_e^k}$. As a consequence, the agent's second period expected payoff *strictly increases* from shirking in the first period:

1. Lemma 1 implies there is a signal y^k such that $\mu_s^k > \mu_e^k$, with a resulting second period gain from deviation.

2. For any signal y^k satisfying $\mu_s^k < \mu_e^k$, the IR constraint is violated, and the agent walks away, obtaining his reservation utility.

Thus, the first period spot contract must satisfy the constraint

$$p_{e\mu^\dagger} \cdot u(1) - c \geq p_{s\mu^\dagger} \cdot u(1) + W(\mu^\dagger), \qquad (2)$$

where $W(\mu^\dagger)$ is the expected payoff in the second period from shirking rather than exerting effort in the first period. This is the one period *future information rent from shirking*. We have just seen that

$$W(\mu^\dagger) \geq \sum\nolimits_{y^k} p_{s\mu^\dagger}^k \max\{V^*(\mu_s^k, \mu_e^k), 0\} > 0,$$

and so the statically optimal contract $\hat{u}_{\mu^\dagger}$ does not satisfy (2). The first period spot contract must be more high powered than the statically optimally contract in order to deter shirking.

We summarize this discussion in the following proposition.

Proposition 1. *Suppose the two period sequence of contracts $(u(1), (u(y^k))_{y^k \in Y})$ is sequentially effort incentive efficient. Then, the first period contract $u(1)$ is more high powered than the statically optimal contract $\hat{u}_{\mu^\dagger}$:*

$$(p_{e\mu^\dagger} - p_{s\mu^\dagger}) \cdot u(1) > c = (p_{e\mu^\dagger} - p_{s\mu^\dagger}) \cdot \hat{u}_{\mu^\dagger}$$

and

$$p_{e\mu^\dagger} \cdot u(1) = p_{e\mu^\dagger} \cdot \hat{u}_{\mu^\dagger} = c.$$

4 Finite Horizon

We consider next the finite horizon setting, with T periods in the relationship. We index periods backwards, so in period t, there are $t - 1$ periods remaining after the current one. In period $\tau = T, \ldots, 1$, the principal has observed the history $h^\tau \in Y^{T-\tau}$, and offers a spot contract $u(h^\tau)$. In the following definition, $\hat{h}^t$ is the common $T - t$ initial segment of each h^τ.

Definition 2. *A sequence of contracts $((u(h^\tau))_{h^\tau \in Y^{T-\tau}})_{\tau=1,\ldots,T}$ is sequentially effort incentive efficient (SEIE) if for every $t \in \{T, \ldots, 2, 1\}$ and every $\hat{h}^t \in Y^{T-t}$, the sequence minimizes*

$$\sum_{\tau=1}^{t} E_{h^\tau, y^k} \{ w(u^k(h^\tau, y^k)) \mid \hat{h}^t, a^\tau = e, a^T = \cdots = a^{\tau-1} = e]$$

subject to the agent finding it optimal to participate and exert effort in period t and in every subsequent period after every public history, conditional on the agent having exerted effort in every previous period.

Since the behavior of the principal in any period is completely determined by her beliefs about the state updated from the public history, we can solve for SEIE contracts recursively, beginning in the last period (period 1; recall we index periods backwards).

We need to consider situations in which the agent and principal have different beliefs. Let $V(\pi, \mu, t)$ denote the agent's value function in period t when his belief is π and the principal's belief is μ (for our purposes, these beliefs are the result of updating using $h^t \in Y^{T-t}$, the period t public history). Denote the effort incentive efficient contract offered by the principal in period t by $u_\mu(t)$.

In the last period, period 1, the principal, given his updated beliefs μ, offers the contract $u_\mu(1) := \hat{u}_\mu$. The agent's value from this contract is

$$V(\pi, \mu, 1) = \max \{ p_{e\pi} \cdot u_\mu(1) - c, \ p_{s\pi} \cdot u_\mu(1), \ 0 \}.$$

If beliefs agree the value is zero, i.e., $V(\mu, \mu, 1) = 0$.

Proceeding recursively, in period t,

$$V(\pi, \mu, t) = \max \Big\{ p_{e\pi} \cdot u_\mu(t) - c + \sum_k p_{e\pi}^k V(\psi_e^k(\pi), \psi_e^k(\mu), t - 1),$$
$$p_{s\pi} \cdot u_\mu(t) + \sum_k p_{s\pi}^k V(\psi_s^k(\pi), \psi_e^k(\mu), t - 1), \ 0 \Big\},$$

where $\psi_a^k(\beta)$ is the posterior probability on G after y^k under action a, given a prior β.

On the equilibrium path, the agent always exerts effort, so that in period t, at belief μ, the contract $u_\mu(t)$ satisfies the incentive constraint

$$p_{e\mu} \cdot u_\mu(t) - c + \sum_k p_{e\mu}^k V(\psi_e^k(\mu), \psi_e^k(\mu), t - 1)$$
$$\geq p_{s\mu} \cdot u_\mu(t) + \sum_k p_{s\mu}^k V(\psi_s^k(\mu), \psi_e^k(\mu), t - 1)$$

and the participation constraint

$$p_{e\mu} \cdot u_\mu(t) - c + \sum_k p_{e\mu}^k V(\psi_e^k(\mu), \psi_e^k(\mu), t - 1) = 0.$$

Since $V(\mu', \mu', 1) = 0$ for all μ', induction immediately implies $V(\mu', \mu', t) = 0$ for all μ'.

Defining
$$W(\mu, t) := \sum_k p_{s\mu}^k V(\psi_s^k(\mu), \psi_e^k(\mu), t - 1), \tag{3}$$

as the *future information rent from shirking* (FIRS) in period t, the period-t incentive constraint can then be written as

$$p_{e\mu} \cdot u_\mu(t) - c \geq p_{s\mu} \cdot u_\mu(t) + W(\mu, t).$$

Summarizing this discussion, we have:

Proposition 2. *A sequence of contracts* $((u(h^\tau))_{h^\tau \in Y^{T-\tau}})_{\tau=1,\ldots,T}$ *is sequentially effort incentive efficient (SEIE) if and only if* $u = u(h^\tau)$ *minimizes*

$$p_{e\mu(h^\tau)} \cdot w(u)$$

subject to

1. $\mu(h^\tau) = \Pr[G \mid h^\tau, a^T = \cdots a^{\tau-1} = e]$,

2. $p_{e\mu(h^\tau)} \cdot u - c \geq p_{s\mu(h^\tau)} \cdot u + W(\mu(h^\tau), t)$, *and*

3. $p_{e\mu(h^\tau)} \cdot u - c \geq 0$.

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Furthermore, the two inequalities hold as equalities in every SEIE contract.

From Section 3, we know $W(\mu, 2) > 0$. Is $W(\mu, t)$ increasing in t, and if it is increasing, does it increase without bound?

Intuitively, $W(\mu, 3)$ should be larger than $W(\mu, 2)$, because the latter reflects the value of different beliefs induced by shirking under a statically optimal contract for a less demanding incentive compatibility constraint. This is essentially a question of comparative statics on static contracts with respect to the opportunity cost of shirking, which turns out to be a lot harder than comparative statics with respect to the disutility of effort. The next section outlines the problem and a resolution.

5 Comparative Statics of Optimal Contracts

The contract $u_\mu(t)$ described in Proposition 2 solves the following static incentive problem $(w(u^k) = u^{-1}(u^k)$ denotes the wage that delivers utility u^k):

$$\min_{\{u^k\}} \sum_k p_{e\mu}^k w(u^k)$$

subject to

$$\sum_k p_{e\mu}^k u^k - c \geq \sum_k p_{s\mu}^k u^k + W \qquad\qquad (\text{IC}^*)$$

$$\text{and} \quad \sum_k p_{e\mu}^k u^k - c \geq 0. \qquad\qquad (\text{IR}^*)$$

Suppose W and $\widetilde{W}$ are two distinct future informational rents from shirking in period t, with $W > \widetilde{W}$. We would like to show that in the preceding period, $t + 1$, the informational rent from shirking is greater under W than under $\widetilde{W}$. Let u and $\tilde{u}$ denote the vectors of utilities in the corresponding optimal period t contracts. Since IC holds with equality we have

$$(p_{e\mu} - p_{s\mu}) \cdot u = c + W$$

and

$$(p_{e\mu} - p_{s\mu}) \cdot \tilde{u} = c + \widetilde{W}.$$

The informational rent from shirking in $t + 1$ is directly related to the properties of the vector $\tilde{u} - u$. In particular, recalling (1), if we show that

$$W > \widetilde{W} \implies (p_{eG} - p_{eB}) \cdot (u - \tilde{u}) > 0, \qquad\qquad (4)$$

then the informational rent is larger under W than under $\widetilde{W}$.

While we know

$$(p_{e\mu} - p_{s\mu}) \cdot (u - \tilde{u}) = W - \widetilde{W}, \tag{5}$$

without further assumptions, this does not imply (4).

If the agent has specific CRRA preferences, then it turns out that, even with general probabilities, the implication in (4) holds, and so $W(\mu, t)$ is monotonic in t (see Appendix A). But we do not know if there is a useful lower bound on $W(\mu, t))$ in this case.

We now pursue a direct path to link (4) and (5) by assuming the vectors $(p_{eG} - p_{eB})$ and $(p_{e\mu} - p_{s\mu})$ are collinear.

Assumption 2. *There exists a vector $\gamma \in \mathbb{R}^K$, $\gamma \cdot \mathbf{1} = 0$, and constants $\alpha > 0$ and β satisfying $\beta > \max\{\alpha, 1\} > 0$ such that*

$$p_{sG} = p_{sB} + \alpha\gamma,$$
$$p_{eB} = p_{sB} + \gamma,$$
$$and \quad p_{eG} = p_{sB} + \beta\gamma.$$

Remark 1. With binary signals, the collinearity assumption is automatically satisfied, since the space of probabilities is one-dimensional.

Our goal is to bound $W(\mu, t)$ as a function of t, since larger information rents require more high powered incentives. We bound $W(\mu, t)$ from below by bounding $V(\pi, \mu, t)$. Obtaining tight bounds for the value function is in general difficult. However, under the Collinearity Assumption 2, we are able to obtain useful bounds by considering a particular specification of continuation play of the agent, namely always exert effort. Denote by $V^*(\pi, \mu, t)$ the agent's value function in period t when his belief is π and the principal's belief is μ, and the agent always chooses effort. Since

$$V(\pi, \mu, t) \geq V^*(\pi, \mu, t), \tag{6}$$

it is enough to bound $V^*(\pi, \mu, t)$. The value recursion for V^* is

$$V^*(\pi, \mu, t) = p_{e\pi} \cdot u_\mu(t) - c + \sum_k p_{e\pi}^k V^*(\psi_e^k(\pi), \psi_e^k(\mu), t-1). \tag{7}$$

As we saw from (1), if $\pi > \mu$, the first flow term is positive, with subsequent flows reflecting additional rents from updated differences in beliefs. However, beliefs *merge* (Blackwell and Dubins, 1962): the difference between

the agent's and the principal's posteriors vanishes. Consequently, in a long relationship, the impact of a difference in beliefs after a deviation in the initial period on the expected information rent in the last period is small.

Nonetheless, in the last period, any small information rent leads to an increase (albeit small) in the power of the required incentives in the penultimate period. This implies that the information rents in period 2 generated from a difference in beliefs are greater than they would have been in the last period. This in turn requires more high powered incentives in period 3, and so on. This cascading effect implies that the effect of an additional period upon period 1 incentives are non-negligible, no matter how long the time horizon T is.

Proposition 3. *Suppose that Assumption 2 is satisfied. Let*

$$K := \min_{\mu} \frac{(\beta - 1)}{[\mu(\beta - \alpha) + (1 - \mu)]} > 0.$$

For any integer t,

$$V(\pi, \mu, t) \geq V^*(\pi, \mu, t) \geq (\pi - \mu)Kct. \tag{8}$$

We now prove the proposition. Assumption 1 holds without further restrictions on the parameters, with $y^k \in Y^H$ if $\gamma^k > 0$ and $y^k \in Y^L$ if $\gamma^k < 0$. Note that $p_{eG} - p_{eB} = (\beta - 1)\gamma$ and $p_{e\mu} - p_{s\mu} = [\mu(\beta - \alpha) + (1 - \mu)]\gamma$.

We first state two implications of the assumed structure on signals. The optimal spot contract in period t satisfies

$$c + W(\mu, t) = (p_{e\mu} - p_{s\mu}) \cdot u_\mu(t) = [\mu(\beta - \alpha) + (1 - \mu)]\gamma \cdot u_\mu(t) \tag{9}$$

(where $W(\mu, 1) = 0$), and so (since (IR) binds on $u_\mu(t)$ at belief μ, recalling (1))

$$p_{e\pi} \cdot u_\mu(t) - c = (\pi - \mu)(\beta - 1)\gamma \cdot u_\mu(t),$$

$$= (\pi - \mu)\frac{(\beta - 1)}{[\mu(\beta - \alpha) + (1 - \mu)]}(c + W(\mu, t)). \tag{10}$$

The first inequality in (8) is simply (6).

From the value recursion for V^* given in (7), we have

$$V^*(\pi, \mu, t) = p_{e\pi} \cdot u_\mu(t) - c + \sum_k p_{e\pi}^k V^*(\psi_e^k(\pi), \psi_e^k(\mu), t - 1)$$

$$= (\pi - \mu)\frac{(\beta - 1)}{[\mu(\beta - \alpha) + (1 - \mu)]}(c + W(\mu, t))$$

$$+ \sum_k p_{e\pi}^k V^*(\psi_e^k(\pi), \psi_e^k(\mu), t - 1). \tag{11}$$

A natural way to proceed is by induction. Suppose $t = 1$. Then,

$$\begin{aligned}
V(\pi, \mu, 1) &\geq V^*(\pi, \mu, 1) \\
&= (\pi - \mu)(p_{eG} - p_{eB}) \cdot u_\mu(1) \\
&= (\pi - \mu)(\beta - 1)\gamma \cdot u_\mu(1) \\
&\geq (\pi - \mu)Kc.
\end{aligned}$$

The inductive hypothesis is

$$V^*(\pi, \mu, t - 1) \geq (\pi - \mu)Kc(t - 1).$$

If this implied

$$\sum_k p_{e\pi}^k V^*(\psi_e^k(\pi), \psi_e^k(\mu), t - 1) \geq (\pi - \mu)Kc(t - 1), \qquad (12)$$

then we would be done, since $W(\mu, t) \geq 0$ and so

$$(\pi - \mu)K(c + W(\mu, t)) \geq (\pi - \mu)Kc.$$

However, (12) fails because beliefs merge. From the inductive hypothesis we have

$$\sum_k p_{e\pi}^k V^*(\psi_e^k(\pi), \psi_e^k(\mu), t - 1) \geq Kc(t - 1) \sum_k p_{e\pi}^k(\psi_e^k(\pi) - \psi_e^k(\mu)).$$

Using the equality $p_{e\pi}^k = p_{e\mu}^k + (\pi - \mu)(p_{eG}^k - p_{eB}^k)$, we have

$$\begin{aligned}
\sum_k p_{e\pi}^k(\psi_e^k(\pi) - \psi_e^k(\mu)) &= \pi - \sum_k p_{e\pi}^k \frac{\mu p_{eG}^k}{p_{e\mu}^k} \\
&= \pi - \mu - (\pi - \mu)\mu \sum_k (p_{eG}^k - p_{eB}^k)\frac{p_{eG}^k}{p_{e\mu}^k} \\
&= (\pi - \mu)(1 - \xi(\mu)), \qquad (13)
\end{aligned}$$

where

$$\xi(\mu) := \mu \sum_k (p_{eG}^k - p_{eB}^k)\frac{p_{eG}^k}{p_{e\mu}^k} > 0$$

is the *merging deficit*.[6] Therefore, all we can conclude from the inductive hypothesis with respect to the second term of (11) is

$$\sum_k p_{e\pi}^k V^*(\psi_e^k(\pi), \psi_e^k(\mu), t - 1) \geq (\pi - \mu)Kc(t - 1)(1 - \xi(\mu)). \qquad (14)$$

---

[6]The strict positivity of $\xi(\mu)$ for $\mu \notin \{0, 1\}$ is an immediate implication of Assumption 1. Consistent with the interpretation of ξ as a measure of the merging deficit, $\xi(\mu) \to 0$ as $\mu \to 0$ or 1 (the summation equals 0 when $\mu = 1$, i.e., when $p_{eG}^k = p_{e\mu}^k = 1$).

For future reference, a straightforward calculation shows that under the collinear parameterization,

$$\xi(\mu) = \mu(\beta - 1) \sum_k \gamma^k \frac{p_{eG}^k}{p_{e\mu}^k}. \tag{15}$$

But the inductive hypothesis also bounds the future information rents from shirking,

$$W(\mu, t) = \sum_k p_{s\mu}^k V(\psi_s^k(\mu), \psi_e^k(\mu), t - 1)$$
$$\geq K(t - 1)c \sum_k p_{s\mu}^k (\psi_s^k(\mu) - \psi_e^k(\mu)).$$

Now,

$$\sum_k p_{s\mu}^k (\psi_s^k(\mu) - \psi_e^k(\mu)) = \mu - \sum_k p_{s\mu}^k \frac{\mu p_{eG}^k}{p_{e\mu}^k}$$
$$= \mu \sum_k \left\{ p_{e\mu}^k - p_{s\mu}^k \right\} \frac{p_{eG}^k}{p_{e\mu}^k}$$
$$= \mu \sum_k [\mu(\beta - \alpha) + (1 - \mu)] \gamma^k \frac{p_{eG}^k}{p_{e\mu}^k}$$
$$= \frac{[\mu(\beta - \alpha) + (1 - \mu)]}{(\beta - 1)} \xi(\mu). \tag{16}$$

Hence,

$$\frac{(\beta - 1)}{[\mu(\beta - \alpha) + (1 - \mu)]} W(\mu, t) \geq K(t - 1)c\xi(\mu). \tag{17}$$

Substituting (14) and (17) into (11) yields

$$V^*(\pi, \mu, t) \geq (\pi - \mu)Kc[1 + (t - 1)\xi(\mu) + (t - 1)(1 - \xi(\mu))] = (\pi - \mu)Kct,$$

completing the proof.

These calculations also give via (3), a lower bound on W.

Corollary 1. *The future information rent from shirking is bounded below by a linear function of time:*

$$W(\mu, t) \geq Kc \frac{[\mu(\beta - \alpha) + 1 - \mu]}{(\beta - 1)} \xi(\mu)(t - 1).$$

The assumption on the structure of signals plays two roles in the analysis. The first is to provide a relationship between $p_{e\pi} \cdot u_\mu(t) - c$ and $W(\mu, t)$. The second is connect the merging deficit with the bound on $W(\mu, t)$. While it is possible to provide a relationship between $p_{e\pi} \cdot u_\mu(t) - c$ and $W(\mu, t)$ under weaker assumptions, the connection of the merging deficit with the bound on $W(\mu, t)$ is more subtle, and we have not found a more general condition.

$p_{a\omega}^{H}$	$a = e$	$a = s$
$\omega = G$	r	$q + (2r - 1)$
$\omega = B$	$1 - r$	q

Figure 1: The probability of the high signal y^H as a function of the state ω and action a, with $0 < q < r < 1$ and $2r - 1 > 0$ and $q + r < 1$.

6   Merging with Binary Signals

We have already seen in the two period case that the initial period contract must be more high powered than the one period contract in order to compensate for the one period FIRS. But this means that in the three period contract, the FIRS reflects the increased value of different beliefs in period 2 from the more high powered period 2 contract, in addition to the value of different beliefs in period 1.

How much of the lower bound on future information rents from shirking is due to the value from having different beliefs in all future periods, and how much is due to the positive feedback from one period's increase in the required power of the incentives to the previous period?

To shed light on this issue, we consider a symmetric environment with two signals y^H and y^L, the probability of y^H given in Figure 1.

By construction, beginning from a common prior, if the principal expects effort, but the agent shirks, then the agents is more optimistic than the principal after both y^H and y^L. We are interested in the value to the agent of shirking in the initial period (and so being more optimistic in every future period), when there are *no* expected information rents after the initial period.

Suppose that in each period (after the initial period), the principal offers the *statically optimal* contract $\hat{u}_\mu(t)$, where μ is the posterior update assuming the agent has exerted effort previously. The principal has belief $\psi_e(\mu, h^\tau) =: \mu^\tau$. This contract solves

$$u^H - u^L = \frac{c}{p_{e\mu^\tau}^H - p_{s\mu^\tau}^H}$$

and

$$p_{e\mu^\tau}^H u^H + p_{e\mu^\tau}^L u^L = 0.$$

The flow benefit to the agent from exerting effort is then, from (1),

$$[\psi_e(\pi, h^\tau) - \mu^\tau](p_{eG}^H - p_{eB}^H)(u^H - u^L) = [\psi_e(\pi, h^\tau) - \mu^\tau]\frac{(2r - 1)c}{p_{e\mu^\tau}^H - p_{s\mu^\tau}^H}.$$

19

The value to the agent of having belief $\pi > \mu$ at the end of the initial period with t periods remaining is

$$V^\dagger(\pi, \mu, t) = E_{e\pi} \sum_{\tau=1}^{t} [\psi_e(\pi, h^\tau) - \mu^\tau] \frac{(2r-1)c}{p^H_{e\mu^\tau} - p^H_{s\mu^\tau}},$$

where, as before, $h^\tau \in Y^{t-\tau}$. At the risk of emphasizing the obvious, observe that because $\pi > \mu$, for all h^τ we have that $\psi_e(\pi, h^\tau) - \mu^\tau = \psi_e(\pi, h^\tau) - \psi_e(\mu, h^\tau) > 0$.

We have the following proposition, proved in Appendix B.

Proposition 4. *Suppose there are two signals with distributions given in Figure 1 and $16r^3(1-r) < 1$. There exists $\bar{V} \in \mathbb{R}$ such that for all t, and $\pi > \mu$,*

$$V^\dagger(\pi, \mu, t) < \bar{V}.$$

While we have not been able to bound $V^\dagger$ for other parameterizations (in particular, r must be close to one and q close to zero[7]), we conjecture the result holds more generally. This result support our intuition that the incentive costs are unbounded in t due to the positive feedback from the power of the incentives.

7 The Cost of Inducing Effort

We have shown that the agent's opportunity cost of effort increases at least linearly in the length of the relationship under Assumption 2. We now examine how this translates to the principal's expected wage cost in any period, as a function of the length of the remaining relationship. The analysis of this section is based on the comparative statics of the optimal static contract when the agent's opportunity cost of shirking changes. As the time horizon t increases, $W(\mu, t)$ increases at least linearly. Let $\mathbf{w}(\mu, t)$ denote the expected wage cost of inducing effort in period t and belief μ, given that the principal induces effort in every period $s < t$. Our focus is on the behavior of $\mathbf{w}(\mu, t)$, as a function of t for a fixed μ.

Note first from Proposition A.1, that if the utility of the agent is CRRA with coefficient of relative risk aversion 2, then the wage cost is quadratic in $W(\mu, t)$, and therefore quadratic in t. We make three additional observations. First, if the agent's utility function is log, then the wage cost is

---

[7]More precisely, the bound $16r^3(1-r) < 1$ requires approximately $r > 0.92$ and so $q < .08$.

exponential in the length of the relationship. Second, if the second derivative of $w(u)$ (the inverse of the worker's utility function) is bounded away from zero, then the wage cost is at least quadratic in the length of the relationship. Finally, for any strictly concave utility function the expected wage cost is at least linear in the length of the relationship.

Consider first log utility. The expected wage cost from the spot contract $u_\mu(t)$, $\mathbf{w}(\mu, t)$, can be bounded below as

$$
\begin{aligned}
\mathbf{w}(\mu, t) &= p_{e\mu} \cdot w(u_\mu(t)) \\
&= \sum_k (p_{sB}^k + \mu(\beta - 1)\gamma^k) \exp(u_\mu^k(t)) \\
&\geq \mu(\beta - 1) \sum_k \gamma^k \exp(u_\mu^k(t)) \\
&\geq \mu(\beta - 1) \exp(\gamma \cdot u_\mu(t)) \\
&\geq \mu(\beta - 1) \exp(c + W(\mu, t)).
\end{aligned}
$$

Before we consider more general agent utility functions, we argue that we can restrict attention to binary signals without loss of generality.

Lemma 3. *Fix a general signal structure (Y, p). There exists a binary signal structure $(\{y^L, y^H\}, p_a^H)$, such that the expected wage cost of inducing effort under (Y, p) is at least the expected cost of inducing effort under the binary signal structure.*

Proof. Let y^K denote the signal with maximum likelihood ratio (p_e^k / p_s^k) and y^1 the signal with minimum likelihood ratio. Construct a new information structure from the original information structure by replacing each signal y^k, $k \neq 1, K$, with two signals $\bar{y}^k$ and $\underline{y}^k$ having probabilities $\theta_a^k p_{a\mu}^k$ and $(1 - \theta_a^k) p_{a\mu}^k$, respectively, under the action $a \in \{s, e\}$. The numbers θ_a^k are chosen to satisfy

$$
\frac{\theta_e^k p_{e\mu}^k}{\theta_s^k p_{s\mu}^k} = \frac{p_{e\mu}^K}{p_{s\mu}^K}
$$

and

$$
\frac{(1 - \theta_e^k) p_{e\mu}^k}{(1 - \theta_s^k) p_{s\mu}^k} = \frac{p_{e\mu}^1}{p_{s\mu}^1},
$$

so that the likelihood ratio of $\bar{y}^k$ equals that of y^K, while the likelihood ratio of $\underline{y}^k$ equals that of y^1.

Since the optimal spot contract under the original information structure is feasible under the new information structure (by treating the pooled $\{\underline{y}^k, \bar{y}^k\}$ as y^k), the expected wage cost of inducing effort under the original

information structure is at least as large as that from the optimal contract inducing effort under the new structure.

But since there are only two likelihood ratios under the new information structure, there are only two wages offered in the optimal spot contract. That is, the optimal spot contract partitions the signal space into two, $\{y^1\} \cup \{\underline{y}^k : k = 2, \ldots, K - 1\}$ and $\{y^K\} \cup \{\bar{y}^k : k = 2, \ldots, K - 1\}$. The desired binary signal structure treats each element of the partition as a signal. $\quad\square$

Consider now an arbitrary strictly concave and twice differentiable u. Under binary signals, the optimal contract in period t, $u_\mu(t) = (u_\mu^L(t), u_\mu^H(t))$, satisfies

$$\Delta u_\mu(t) := u_\mu^H(t) - \tilde{u}_\mu^L(t) = \frac{c + W(\mu, t)}{p_{e\mu}^H - p_{s\mu}^H}.$$

The optimal contract is the pair $(u_\mu^L(t), u_\mu^H(t))$ solving

$$u_\mu^H(t) = u_\mu^L(t) + \Delta u_\mu(t) \qquad \text{and}$$
$$c = p_{e\mu}^H u_\mu^H(t) + (1 - p_{e\mu}^H) u_\mu^L(t),$$

so that

$$u_\mu^L(t) = c - p_{e\mu}^H \Delta u_\mu(t) \text{ and } u_\mu^L(t) = c + (1 - p_{e\mu}^H)\Delta u_\mu(t). \qquad (18)$$

Expected wages under the optimal contract are

$$\mathbf{w}(\mu, t) = (1 - p_{e\mu}^H) w(u_\mu^L(t)) + p_{e\mu}^H w(u_\mu^H(t)).$$

A second order Taylor-expansion of $w(u)$ around $u = 0$ yields

$$\mathbf{w}(\mu, t) = w(0) + \frac{1}{2}(1 - p_{e\mu}^H)w''(\hat{u}^L)(u_\mu^L(t))^2 + \frac{1}{2}p_{e\mu}^H w''(\hat{u}^H)(u_\mu^H(t))^2,$$

where $\hat{u}^L \in (u_\mu^L(t)), 0)$ and $\hat{u}^H \in (0, (u_\mu^H(t))$. If w'' is bounded below by $\rho > 0$, then the expected wage cost is a quadratic function of $u_\mu^L(t)$ and $u_\mu^H(t)$, and thus of W and so t.

Finally, we show that the expected wage cost increases at least linearly in t even if $w''(.)$ is not bounded away from zero. Let $u_\mu^H(1)$ and $u_\mu^L(1)$ denote the optimal contract in the static case, i.e., when $t = 1$, and let $w_\mu^H(1)$ and $w_\mu^L(1)$ denote the corresponding wages. Since $u_\mu^H(1) - u_\mu^L(1) = \frac{c}{(p_{e\mu}^H - p_{s\mu}^H)}$ (due to the incentive constraint), and since u is strictly concave $u'(w_\mu^H(1)) <$

$u'(w_\mu^L(1))$. Let $a := u'(w_\mu^H(1))$, and $b := u'(w_\mu^L(1))$. We approximate the function u by the piece-wise linear function $\tilde{u}$,

$$\tilde{u}(w) = \begin{cases} \tilde{u}_0 + a(w - \tilde{w}), & w \geq \tilde{w}, \\ \tilde{u}_0 - b(\tilde{w} - w), & w < \tilde{w}, \end{cases}$$

where $(\tilde{w}, \tilde{u}_0)$ is defined so that $\tilde{u}$ is a continuous function, by the condition

$$\tilde{u}_0 := u_\mu^L(1) + b\left(\tilde{w} - w_\mu^L(1)\right) = u_\mu^H(1) - a\left(w_\mu^H(1) - \tilde{w}\right)$$

as depicted in Figure 2.

As before, the optimal contract satisfies (18). It is straightforward to verify that the expected cost of this contract is of the same order as $W(\mu, t)$, and so linear in t – see Figure 2. It is also straightforward to verify that the expected wage cost under the true utility function u is strictly greater than that under $\tilde{u}$. Thus the expected wage costs under the true utility function are bounded below by a linear function.[8]

Summarizing the preceeding discussion, we have the following proposition.

Proposition 5. *Under the collinearity Assumption 2, $\mathbf{w}(\mu, t)$ is bounded below by a strictly increasing linear function of t. If additionally, w'' is bounded below by $\rho > 0$, then the bound on $\mathbf{w}(\mu, t)$ is quadratic in t. If agent utility is logarithmic, the bound is exponential in t.*

8   Endogenous Effort

We conclude with a few comments on endogenous effort. We maintain the assumptions of Proposition 3, and consider the case of finite horizons only.

Suppose the signal is the level of revenue accruing to the principal, so that the expected revenue under action a and belief μ is $E_{a\mu}y$. Let $w_0 := w^{-1}(0)$ denote the constant wage that meets the agent's IR constraint when the agent shirks. We assume $E_{a\mu}y - w_0 > 0$ for all a and μ so that employing the agent is always optimal in the one-period problem. This implies that employment is also efficient in the dynamic case, since the principal can always hire the worker for a constant wage of w_0 (inducing shirking in every period). Let

$$R(\mu) := E_{e\mu}y - E_{s\mu}y$$

---

[8]Since we can find a sequence of strictly concave utility functions that converge to $\tilde{u}$, one cannot in general do better than a linear bound, if we do not assume a bound on the second derivative.
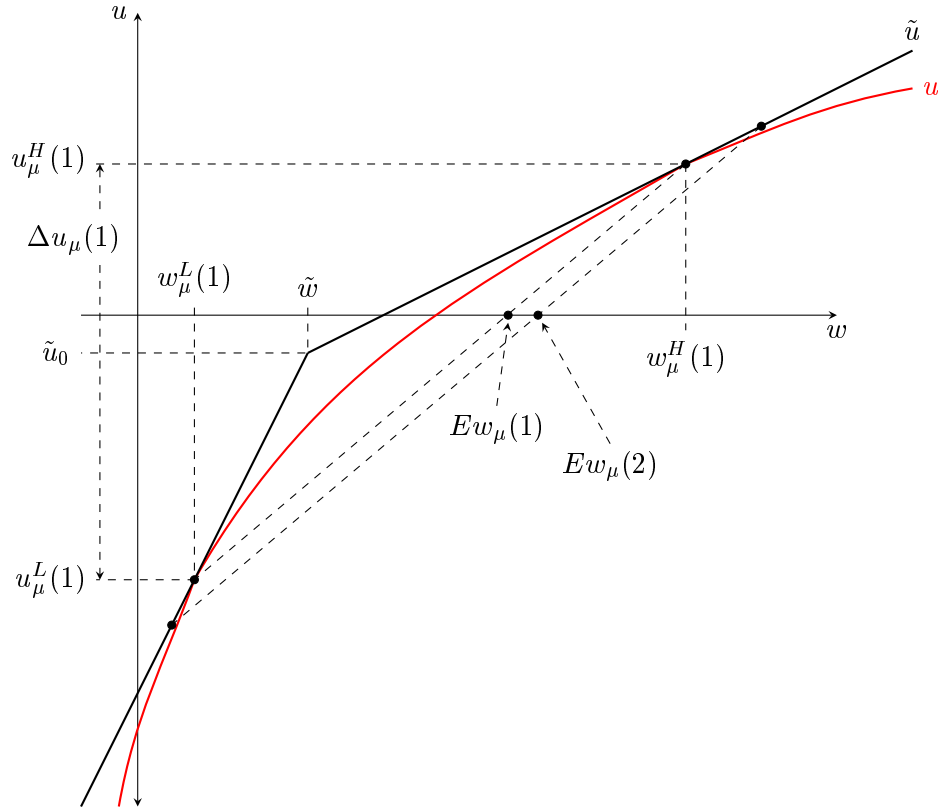
Figure 2: The original utility function u, the approximating piecewise linear utility function $\tilde{u}$, and the statically optimal contract $u_\mu(1)$ for binary signals. The contract $u_\mu(1)$ is determined by $\Delta u_\mu(1)$ and the requirement that expected utility (under $p_{e\mu}$) is zero. The expected cost of the contract is then the corresponding value on the w-axis.

denote the principal's incremental revenue from effort over shirk at belief μ. Denote the expected wage cost of the contract $u_\mu(t)$ by $w_\mu(t)$.

In the one-period problem, the principal's optimal policy is to induce effort if

$$R(\mu) > w_\mu(1) - w_0,$$

and to induce shirking otherwise.[9] We assume inducing effort is optimal in the static setting for all beliefs μ.

Suppose first that, like the agent, the principal does not discount future payoffs. In addition to increasing expected revenues, effort may generate more informative signals than shirking. If effort is more informative, then the principal may find it profitable to induce even if the flow returns are negative. Nonetheless, we do have two straightforward observations about the profitability of always inducing effort. Since the value of information is less than linear in the horizon (because of merging), for a fixed prior, for a sufficiently long horizon, it cannot be optimal for the principal to induce effort in every period. On the other hand, for a fixed horizon, if the prior is sufficiently close the boundary, then always inducing effort can be optimal.

Proposition 6. *Suppose for all $\mu \in [0,1]$,*

$$R(\mu) > w_\mu(1) - w_0.$$

1. *For all $\mu_0 \in (0,1)$, there exists T_0 such that if $T \geq T_0$, then it is not optimal for the principal to induce effort in every period.*

2. *For all T_0, there exists $\varepsilon > 0$ such that if $\mu_0 \in (0,\varepsilon) \cup (1 - \varepsilon, 1)$, then it is optimal for the principal to induce effort in every period of a T-period relationship if $T \leq T_0$.*

Proof.   1. It is immediate from Proposition 5 and the observation that the value of information is less than linear in the horizon that not inducing effort in the initial period (and then inducing effort in every period) is for sufficiently long horizons more profitable than inducing effort in the initial period as well.

2. Fix T_0. Define

$$B(\mu, \mu') := \max_{t \leq T_0} |w_\mu(t) - w_{\mu'}(t)|.$$

---

[9]We assume that the principal induces shirking when she is indifferent, thereby focusing attention on the principal optimal equilibrium, since such a policy minimizes the deviation gain of the agent. Such an indifference does not arise generically.

Observe that $B(\mu, \mu')$ is continuous in μ and μ', and equals 0 when $\mu = \mu'$.

By assumption, inducing effort is statically strictly optimal, so that $\eta := \max_\mu R(\mu) - (w_\mu(1) - w_0) > 0$. Since the domain of B is compact, B is uniformly continuous, and so there exists $\varepsilon' > 0$ such that if the posterior belief μ after any history of signals and agent action choices is within ε' of the prior μ_0, then $B(\mu_0 . \mu) < \eta$.

The proof is completed by obsreving that if the prior μ_0 is sufficiently close to the boundary (either 0 or 1) then every potential posterior must be within ε' of μ_0.

$\square$

In general, determining the principal's optimal sequence of induced efforts is complicated, not least because it will also involve elements of active learning (experimentation) and possibly randomization over effort.

The principal will not induce randomized effort if the information structure such that $\psi_s^k(\mu) \geq \psi_e^k(\mu)$ for every k, so that the agent never quits after shirking. Under this informational assumption, it is always optimal for the principal to induce a deterministic level of effort, i.e. it is strictly dominated for the principal to induce the agent to randomize between effort and shirking. If the principal induces random effort at date t, then at $t - 1$, then after any signal realization y^k, the principal faces a screening problem, where the types of the agent correspond to the beliefs associated with the two different effort choices. If $\psi_s^k(\mu) \geq \psi_e^k(\mu)$ for every k, the agent never gets any informational rent after working, while he gets a rent from shirking, just as in the case the principal induces working for sure. This implies that the incentive constraint and participation constraint at date t for inducing random effort are identical to the constraints for inducing work for sure, thus ensuring that inducing random effort is dominated by one of the two pure effort levels. If the informational structure is such that there is a signal realization for which the agent is more pessimistic after shirking than after working, then inducing random effort allows the principal to commit to pay future rents to the agent who exerted effort. This relaxes both incentive and participation constraints, and in this case, inducing random effort can be optimal.

To illustrate how complicated the optimal sequence could be, we now focus on the case of a sequence of short-lived principals contracting with the long-lived agent. A short-lived principal will not induce effort for purposes of experimentation/learning. She will only induce effort if the expected wage

cost of doing so is less than $R(\mu)$.

As for a long-lived principal and log utility agent, it is obviously never optimal for a short-lived principal to induce effort when all future principals induce effort and the horizon is long. A short run principal may induce random effort. As discussed above, a sufficient condition for not inducing random effort is that $\psi_s^k(\mu) \geq \psi_e^k(\mu)$ for every k.

A plausible conjecture is that for a sufficiently long lived agent, initially the short-lived principals induce shirking, followed by a second and final phase where they induce work. Intuitively, the initial phase reduces the time horizon and uncertainty regarding the state, both of which reduce the future informational rents from shirking, thereby permitting the principal to induce effort in the second phase. This conjecture is incorrect. As the time horizon T increases, the initial phase of shirking must also increase. But once the shirking phase becomes sufficiently long, and T increases further, it becomes profitable to induce effort in the first period of the relationship, since the future information rent from shirking is this period becomes very small, given that it is followed by a long phase of shirking.

The above considerations suggest that even for the case of short-lived principals, the optimal sequence of induced agent behavior is complicated. This is illustrated in Figures 3 and 4, which report the expected wage cost of inducing effort given *optimal* future effort inducement (that is, effort in a period is only induced if that period's principal finds it optimal to do so). For example, for $T = 2$, the initial period principal does not induce effort for $\mu \in (.1, .2)$. Consequently, for $T = 3$, there is a discontinuity in the expected wage cost of inducing effort in the initial period at $\mu = \frac{4}{7}$: For $\mu > \frac{4}{7}$, effort is induced in the second periods after all signals, and so the FIRS is high. For μ just below $\frac{4}{7}$, the low signal leads to a posterior in $(.1, .2)$ and so a constant wage (no effort induced) in the second period, with a consequently lower FIRS. Moreover, if slightly larger $y^H - y^L$, we see it is possible for some priors to induce effort in the initial period for $T = 3$ but not for $T = 2$. Finally, we see that is there is no monotonicity of the expected wage cost in either T or μ. At $\mu = .65$, for example, this results in effort being induced in the initial period for $T = 1, 2$, and 4 but not for $T = 3$ and 5.
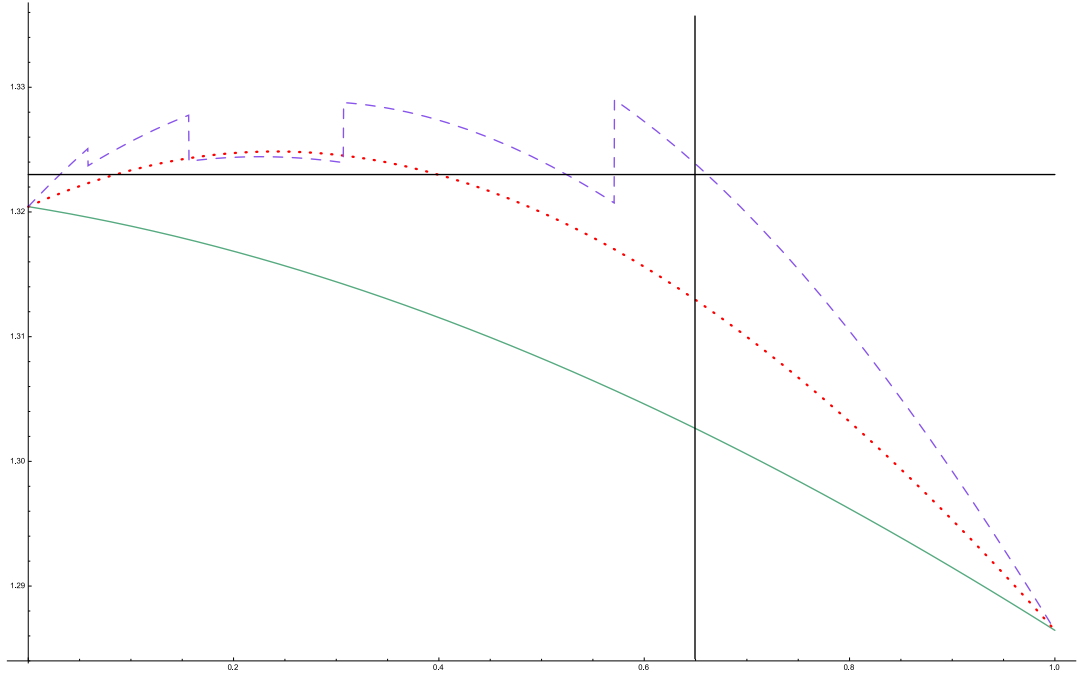
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Figure 3: The expected wage cost of inducing effort in the initial period, as a function of the prior μ and length of the horizon. There are two signals, with $p_{eG}^H = \frac{3}{4}$, $p_{eB}^H = p_{sG}^H = \frac{1}{2}$, and $p_{sB}^H = \frac{1}{4}$. The cost of effort is $c = 2$ and the agent's utility functions if $u(w) = 10 \log w$. Finally, $y^H - y^L = 1.292$, so that $R(\mu) + w_0 = 1.323$. The horizontal line is $R(\mu) + w_0$, the vertical line is $\mu = 0.65$. The cyan solid line is for $T = 1$, the red dotted line is for $T = 2$, and the purple dashed line is for $T = 3$.

Appendix

A CRRA Preferences

We assume the agent's utility function is unbounded below, but only to ensure individual rationality is always binding. While CRRA utility functions are not unbounded below, individual rationality will still be binding if $w(u^k) > -A$ for all y^k. In particular, it will be for many periods for the utility function in Proposition A.1 for sufficiently large A.
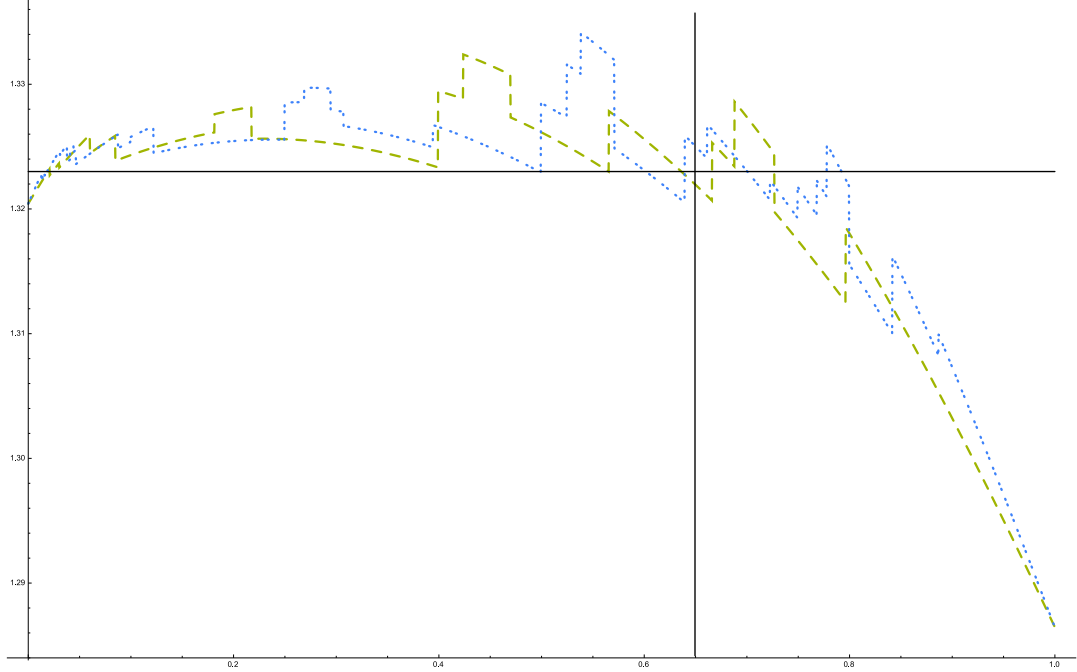
Figure 4: The expected wage cost of inducing effort in the initial period, as a function of the prior μ for $T = 4$ and $T = 5$. The green dashed line is for $T = 4$, and the blue dotted line is for $T = 5$. The parameter values are the same as in Figure 3.

Proposition A.1. *Suppose the agent's utility function is given by*

$$u(w) = \sqrt{A + w},$$

where $A > 0$. If $w(u^k) > -A$ for all y^k under W and $\widetilde{W}$, then the implication (4) holds. There exists $\tilde{T}$ such that for all time horizons $T < \tilde{T}$, the agent's future informational rent from shirking, $W(T)$, is strictly increasing in T, and the principal's wage cost in the intial period is quadratic in $W(T)$.

Proof. Since $w'(u^k) = 2u^k$, the first order conditions for the principal's problem can be written as

$$2u^k = \lambda + \zeta \left(1 - \frac{p_{s\mu}^k}{p_{e\mu}^k} \right), \qquad k = 1, \ldots, K, \tag{A.1}$$

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where λ is the multiplier on the IR constraint and ζ is the multiplier on the IC constraint. The incentive constraint $(p_{e\mu} - p_{s\mu}) \cdot u = c + W$ can then be rewritten as

$$
\begin{aligned}
c + W &= \sum_k (p_{e\mu}^k - p_{s\mu}^k) \left[\frac{\lambda}{2} + \frac{\zeta}{2} \left(1 - \frac{p_{s\mu}^k}{p_{e\mu}^k} \right) \right] \\
&= \sum_k \frac{\zeta}{2} (p_{e\mu}^k - p_{s\mu}^k) \left(1 - \frac{p_{s\mu}^k}{p_{e\mu}^k} \right) \\
&=: \frac{\zeta X(\mu)}{2},
\end{aligned}
$$

where $X(\mu) > 0$ since there exists an informative signal such that $p_{e\mu}^k \neq p_{s\mu}^k$. This implies

$$
\zeta = \frac{2(c + W)}{X(\mu)},
$$

and so

$$
\begin{aligned}
(p_{eG} - p_{eB}) \cdot (u - \tilde{u}) &= \sum_k (p_{eG}^k - p_{eB}^k) \frac{(W - \widetilde{W})}{X(\mu)} \left(1 - \frac{p_{s\mu}^k}{p_{e\mu}^k} \right) \\
&=: X^*(\mu)(W - \widetilde{W}),
\end{aligned}
$$

where $X^*(\mu) > 0$ from Assumption 1.

Further, from (A.1),

$$
2\mathbf{E}(u) = \lambda + \zeta \sum_k (p_{e\mu}^k - p_{s\mu}^k) = \lambda,
$$

and since $\mathbf{E}(u) - c = 0$ from the binding IR constraint, λ is independent of W, the opportunity cost of shirking.

Turning to expected wage costs, from (A.1),

$$
\begin{aligned}
4\mathbf{E}(w) &= \lambda^2 + \zeta^2 \sum_k \left(p_{e\mu}^k - p_{s\mu}^k \right) \left(1 - \frac{p_{s\mu}^k}{p_{e\mu}^k} \right) \\
&= \lambda^2 + \frac{4(c + W)^2}{X(\mu)}.
\end{aligned}
$$

Since λ does not depend upon W, $\mathbf{E}(w)$ is quadratic in $c + W$. $\qquad\square$

B   Proofs for Section 6

Lemma B.1. *Suppose $\mu,\,\pi > \frac{1}{2}$. Then, there exists $\sigma \in (0,1)$ such that for all $\mu,\pi \geq \frac{1}{2}$ and for all $y^k \in Y^H$,*

$$\left| \psi_e^k(\pi) - \psi_e^k(\mu) \right| \leq \sigma \left| \pi - \mu \right|.$$

Proof. From some straightforward calculations, we have

$$\psi_e^k(\pi) - \psi_e^k(\mu) = \frac{\pi p_{eG}^k}{p_{e\pi}^k} - \frac{\mu p_{eG}^k}{p_{e\mu}^k}$$
$$= \frac{(\pi - \mu)p_{eG}^k p_{eB}^k}{p_{e\pi}^k p_{e\mu}^k},$$

and so it remains to bound the ratio of probabilities.

Now, consider

$$f^k(\pi,\mu) := p_{e\pi}^k p_{e\mu}^k - p_{eG}^k p_{eB}^k$$
$$= \pi\mu(p_{eG}^k)^2 + (1-\pi)(1-\mu)(p_{eB}^k)^2$$
$$\quad - [\pi\mu + (1-\pi)(1-\mu)]p_{eG}^k p_{eB}^k.$$

This function is increasing in π and μ (since $y^k \in Y^H$), and so is minimized at $\pi = \mu = \frac{1}{2}$ over $\pi,\mu \geq \frac{1}{2}$. That is,

$$f^k(\pi,\mu) \geq \frac{1}{4}(p_{eG}^k - p_{eB}^k)^2 \quad \forall \pi,\mu \geq \frac{1}{2}.$$

Define

$$X := \min_{y^k \in Y^H} \frac{(p_{eG}^k - p_{eB}^k)^2}{4 p_{eG}^k p_{eB}^k}$$

and set

$$\sigma = \frac{1}{1+X} \in (0,1). \tag{B.1}$$

Then,

$$p_{e\pi}^k p_{e\mu}^k - p_{eG}^k p_{eB}^k = f^k(\pi,\mu)$$
$$\geq X p_{eG}^k p_{eB}^k$$
$$= \left(\frac{1}{\sigma} - 1\right) p_{eG}^k p_{eB}^k,$$

and so

$$\frac{p_{eG}^k p_{eB}^k}{p_{e\pi}^k p_{e\mu}^k} \leq \sigma. \qquad \square$$

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Proof of Proposition 4. For the purposes of this proof, it is more convenient to index periods forward rather than backward, so that h^τ is the τ length history leading to period τ, with $T - \tau$ periods remaining.

Given h^τ, let $n(h^\tau)$ denote the difference between the number of y^H and y^L realizations in h^τ. Then, since $p_{eB}^H = p_{eG}^L$, histories of different lengths lead to the same posterior as long as they agree in $n(h^\tau)$, i.e., for all h^τ and $h^{\tau'}$, with τ possibly different from τ',

$$n(h^\tau) = n(\hat{h}^{\tau'}) \Rightarrow \psi_e(\mu, h^\tau) = \psi_e(\mu, \hat{h}^{\tau'}).$$

We proceed by conditioning on G (the unconditional expectation is then the average of the conditioning on G and the symmetric term from B). Moreover, for large t, conditional on G, the probability that $n(h^t)$ is negative goes to zero sufficiently fast, that it is enough to show that

$$\Pr\{n(h^\tau) \geq 0 \text{ for } \tau = 0, \dots, t-1\} \times$$

$$E\left\{\left.\sum_{\tau=0}^{t-1}[\psi_e(\pi, h^\tau) - \psi_e(\mu, h^\tau)]\right| G, a^\tau = e, n(h^\tau) \geq 0\right\} \quad \text{(B.2)}$$

is bounded. Moreover, we can also assume $\mu > 1/2$, since conditional on G, the probability that $n(h^\tau)$ is small becomes arbitrarily small as t becomes large.

From Lemma B.1 (using the value of σ from (B.1)), we have that for $\sigma := 4r(1-r) \in (0,1)$, if $\pi, \mu > \frac{1}{2}$, then

$$\psi_e(\pi, n(h^\tau)) - \psi_e(\mu, n(h^\tau)) < \sigma^{n(h^\tau)}(\pi - \mu).$$

Then the expression in (B.2) is bounded above by

$$\sum_{\tau=0}^{t-1}\sum_{n=0}^{\tau}\Pr(n(h^\tau) = n)\sigma^n(\pi - \mu)$$

$$= (\pi - \mu)\sum_{n=0}^{t-1}\sigma^n\sum_{\tau=n}^{t-1}\Pr(n(h^\tau) = n)$$

$$\leq (\pi - \mu)\sum_{n=0}^{\infty}\sigma^n\sum_{\tau=n}^{\infty}\Pr(n(h^\tau) = n). \quad \text{(B.3)}$$

We first bound

$$\Pr(n(h^\tau) = n) = b((\tau + n)/2; \tau, p) = \binom{\tau}{(\tau + n)/2} r^{(\tau+n)/2}(1 - r)^{(\tau-n)/2}.$$

Using Stirling's formula[10]

$$\sqrt{2\pi}\ m^{m+1/2}e^{-m} \le m! \le e\ m^{m+1/2}e^{-m} \text{ for all positive integers } m,$$

we bound the binomial coefficients as follows

$$\binom{\tau}{(\tau+n)/2} = \frac{\tau!}{\frac{(\tau+n)}{2}!\frac{(\tau-n)}{2}!}$$

$$\le \frac{e\ \tau^{\tau+\frac{1}{2}}e^{-\tau}}{2\pi\left[\frac{(\tau+n)}{2}\right]^{\frac{(\tau+n)}{2}+\frac{1}{2}}e^{-\frac{(\tau+n)}{2}}\left[\frac{(\tau-n)}{2}\right]^{\frac{(\tau-n)}{2}+\frac{1}{2}}e^{-\frac{(\tau-n)}{2}}}$$

$$\le \frac{\tau^{\tau+\frac{1}{2}}}{\sqrt{2}\left[\frac{(\tau+n)}{2}\right]^{\frac{(\tau+n)}{2}+\frac{1}{2}}\left[\frac{(\tau-n)}{2}\right]^{\frac{(\tau-n)}{2}+\frac{1}{2}}}$$

$$= \frac{(2\tau)^{\tau+\frac{1}{2}}}{(\tau^2-n^2)^{\frac{\tau}{2}+\frac{1}{2}}} \times \left(\frac{\tau-n}{\tau+n}\right)^{n/2}$$

$$\le \frac{(2\tau)^{\tau+\frac{1}{2}}}{(\tau^2-n^2)^{\frac{\tau}{2}+\frac{1}{2}}}$$

$$\le 2^{\tau+\frac{1}{2}}\left(\frac{\tau^2}{\tau^2-n^2}\right)^{\frac{\tau}{2}+\frac{1}{4}}.$$

We also need the following calculation. Setting $k := \sqrt{(1+\sigma)/(1-\sigma)}$, gives for all $\tau > kn$,

$$\frac{\tau^2}{\tau^2-n^2}\sigma < \frac{k^2n^2}{k^2n^2-n^2}\sigma$$

$$= \frac{k^2}{k^2-1}\sigma$$

$$= \frac{1+\sigma}{2\sigma}\sigma = \frac{1+\sigma}{2} =: y < 1,$$

where the final inequality holds because $\sigma < 1$.

---

[10]See, for example, Abramowitz and Stegun (1972, 6.1.38).

We are now in a position to bound (B.3), since

$$\sum_{n=0}^{\infty} \sigma^n \sum_{\tau=n}^{\infty} \Pr(n(h^\tau) = n) = \sum_{n=0}^{\infty} \sigma^n \sum_{\tau=n}^{kn} \Pr(n(h^\tau) = n)$$

$$+ \sum_{n=0}^{\infty} \left(\sigma^2 \frac{r}{1-r} \right)^{n/2} \sum_{\tau=kn+1}^{\infty} \binom{\tau}{(\tau+n)/2} [r(1-r)]^{\tau/2}$$

$$\leq \sum_{n=0}^{\infty} \sigma^n (k-1)n$$

$$+ \sqrt{2} \sum_{n=0}^{\infty} \left(\sigma^2 \frac{r}{1-r} \right)^{n/2} \sum_{\tau=kn+1}^{\infty} \left(\frac{\tau^2}{\tau^2 - n^2} \right)^{\frac{\tau}{2} + \frac{1}{4}} [4r(1-r)]^{\tau/2}$$

$$\leq \sum_{n=0}^{\infty} \sigma^n (k-1)n$$

$$+ \sqrt{2} \sum_{n=0}^{\infty} \left(\sigma^2 \frac{r}{1-r} \right)^{n/2} \sum_{\tau=kn+1}^{\infty} \left(\frac{\tau^2}{\tau^2 - n^2} \right)^{\frac{1}{4}} y^{\tau/2}.$$

Since $\sigma < 1$ and $y < 1$, this expression is bounded if

$$1 > \sigma^2 \frac{r}{1-r} = 16r^2(1-r)^2 \frac{r}{1-r} = 16r^3(1-r).$$

C   Infinite Horizon

In this section, we maintain the hypotheses on the probability distributions of Proposition 3 and show that a similar phenomenon arises with an infinite horizon. We assume both the principal and agent discount with possibly different discount factors δ_A and $\delta_P < 1$. We focus on stationary high effort incentive efficient contracts.

Proposition C.1. *Suppose the probability distributions satisfy the conditions in Proposition 3, and K is the constant defined in that proposition. Suppose a stationary high effort incentive efficient contract exists and $V(\pi, \mu)$ is the agent's value when his belief is π and the principal's belief is μ. Then,*

$$V(\pi, \mu) \geq \frac{Kc(\pi - \mu)}{1 - \delta_A}. \tag{C.1}$$

The denominator $1 - \delta_A$ replaces the horizon, and analogously to the finite horizon, as the agent becomes patient, future information rents from shirking become arbitrarily large.

Let $\mathcal{Y}$ be the set of all functions mapping $[0,1]^2$ to $\mathbb{R}$ equalling zero on the diagonal (i.e., $V(\mu, \mu) = 0$ for all $\mu \in [0,1]$ and all $V \in \mathcal{Y}$),[11] and let $\Psi : \mathcal{Y} \to \mathcal{Y}$ be the mapping defined by $V' = \Psi(V)$ given by

$$V'(\pi, \mu) := \max \left\{ p_{e\pi} \cdot u_\mu^V - c + \delta_A \sum_k p_{e\pi}^k V(\psi_e^k(\pi), \psi_e^k(\mu)), \right.$$
$$\left. p_{s\pi} \cdot u_\mu^V + \delta_A \sum_k p_{s\pi}^k V(\psi_s^k(\pi), \psi_e^k(\mu)), 0 \right\}, \quad \text{(C.2)}$$

where u_μ^V is the unique cost minimizing vector of utilities satisfying

$$p_{e\mu} \cdot u_\mu^V - c \geq p_{s\mu} \cdot u_\mu^V + \delta_A \sum_k p_{s\pi}^k V(\psi_s^k(\pi), \psi_e^k(\mu)) \qquad \text{(C.3)}$$
$$\text{and} \qquad p_{e\mu} \cdot u_\mu^V - c \geq 0. \qquad \text{(C.4)}$$

For any stationary high effort incentive efficient contract, the value function V describing the agent's value when his belief is π and the principal's belief is μ is a fixed point of Ψ.

We proceed as in the finite horizon case, bounding V by the value function when the agent exerts effort. Consequently, as for the finite horizon case, we do not need to know the precise details of the spot contracts, here u_μ^V. It is enough to know that

$$p_{e\pi} \cdot u_\mu^V - c = (\pi - \mu) \frac{(\beta - 1)}{[\mu(\beta - \alpha) + (1 - \mu)]} \left(c + \delta_A \sum_k p_{s\pi}^k V(\psi_s^k(\pi), \psi_e^k(\mu)) \right),$$

which follows from familiar arguments (see (9) and (10)).

Lemma C.1. *Denote by $\mathcal{V}$ the subset of $\mathcal{Y}$ satisfying the inequality in (C.1). The mapping $\Psi^e : \mathcal{Y} \to \mathcal{Y}$ defined by $V^* = \Psi^e(V)$, where*

$$V^*(\pi, \mu) := p_{e\pi} \cdot u_\mu^V - c + \delta_A \sum_k p_{e\pi}^k V(\psi_e^k(\pi), \psi_e^k(\mu)) \qquad \text{(C.5)}$$

is a self-map on $\mathcal{V}$, i.e.,
$$\Psi^e : \mathcal{V} \to \mathcal{V}.$$

Proof. For $V \in \mathcal{V}$,

$$\sum_k p_{s\mu}^k V(\psi_s^k(\mu), \psi_e^k(\mu)) \geq \sum_k p_{s\mu}^k \frac{Kc(\psi_s^k(\mu) - \psi_e^k(\mu))}{1 - \delta_A}$$
$$\geq \frac{[\mu(\beta - \alpha) + (1 - \mu)]}{(\beta - 1)} \frac{Kc\xi(\mu)}{(1 - \delta_A)}$$

---

[11] We have already seen in the finite horizon setting that this property holds, and it could be deduced here as well. Assuming it directly is without loss of generality and simplifies our analysis.

(where the last inequality follows from (16)). This gives

$$V^*(\pi, \mu) \geq (\pi - \mu)K \left\{ c + \delta_A \frac{c\xi(\mu)}{(1 - \delta_A)} \right\} + \delta_A \sum_k p_{e\pi}^k V(\psi_e^k(\pi), \psi_e^k(\mu)).$$

Turning to the second term and applying (13) to obtain the equality gives

$$\sum_k p_{e\pi}^k V(\psi_e^k(\pi), \psi_e^k(\mu)) \geq \frac{Kc \sum_k p_{e\pi}^k (\psi_e^k(\pi) - \psi_e^k(\mu))}{1 - \delta_A}$$

$$= (\pi - \mu) \frac{Kc}{(1 - \delta_A)} (1 - \xi(\mu)),$$

so that

$$V^*(\pi, \mu) \geq (\pi - \mu) \frac{Kc}{(1 - \delta_A)} \{1 - \delta_A + \delta_A \xi(\mu) + \delta_A(1 - \xi(\mu))\}$$

$$= (\pi - \mu) \frac{Kc}{(1 - \delta_A)},$$

and so $V^* \in \mathcal{V}$. $\qquad\qquad\square$

Since

$$\Psi(V) \geq \Psi^e(V)$$

pointwise (i.e., for all (π, μ), $\Psi(V)(\pi, \mu) \geq \Psi^e(V)(\pi, \mu)$) and $\Psi^e : \mathcal{V} \to \mathcal{V}$, we have $\Psi : \mathcal{V} \to \mathcal{V}$.

We now argue that any fixed point of Ψ must lie in $\mathcal{V}$, which proves Proposition C.1. Since Ψ need not be a contraction, we argue indirectly.

Let $\mathcal{Y}^0 := \{V \in \mathcal{Y} \mid V(\pi, \mu) \geq 0 \ \forall(\pi, \mu)\}$. Clearly, $\Psi : \mathcal{Y} \to \mathcal{Y}^0$. For all $V \in \mathcal{Y}^0$,

$$\Psi(V)(\pi, \mu) \geq \Psi^e(V)(\pi, \mu) \geq (\pi - \mu)Kc.$$

Lemma C.2. *Defining*

$$\mathcal{Y}^\kappa := \{V \in \mathcal{Y}^{\kappa-1} \mid V(\pi, \mu) \geq (\pi - \mu)Kc(1 - \delta_A^\kappa)/(1 - \delta_A), \ \forall(\pi, \mu)\},$$

we have

$$\Psi : \mathcal{Y}^\kappa \to \mathcal{Y}^{\kappa+1}, \qquad \forall \kappa \geq 0.$$

Proof. For $V \in \mathcal{Y}^\kappa$, applying (16),

$$\sum_k p_{s\mu}^k V(\psi_s^k(\mu), \psi_e^k(\mu)) \geq \sum_k p_{s\mu}^k \frac{Kc(1 - \delta_A^\kappa)(\psi_s^k(\mu) - \psi_e^k(\mu))}{(1 - \delta_A)}$$

$$\geq \frac{c(1 - \delta_A^\kappa)\xi(\mu)}{(1 - \delta_A)} (p_{e\mu}^H - p_{s\mu}^H).$$

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Then, as in the beginning of the proof of Lemma C.1,

$$\Psi^e(V)(\pi,\mu) \geq (\pi-\mu)K\left\{c + \delta_A \frac{(1-\delta_A^\kappa)c\xi(\mu)}{(1-\delta_A)}\right\}$$
$$+ \delta_A \sum_k p_{e\pi}^k V(\psi_e^k(\pi), \psi_e^k(\mu)).$$

But

$$\sum_k p_{e\pi}^k V(\psi_e^k(\pi), \psi_e^k(\mu)) \geq \frac{Kc(1-\delta_A^\kappa)\sum_k p_{e\pi}^k(\psi_e^k(\pi) - \psi_e^k(\mu))}{1-\delta_A}$$
$$= (\pi-\mu)\frac{K(1-\delta_A^\kappa)c}{(1-\delta_A)}(1-\xi(\mu)),$$

so that

$$\Psi(V)(\pi,\mu) \geq \Psi^e(V)(\pi,\mu)$$
$$\geq (\pi-\mu)\frac{Kc}{(1-\delta_A)}\{1 - \delta_A + \delta_A(1-\delta_A^\kappa)\xi(\mu) + \delta_A(1-\delta_A^\kappa)(1-\xi(\mu))\}$$
$$= (\pi-\mu)\frac{Kc(1-\delta_A^{\kappa+1})}{(1-\delta_A)},$$

and so $V^* \in \mathcal{Y}^{\kappa+1}$. □

Since $\mathcal{V} = \bigcap \mathcal{Y}^\kappa$, we have the desired result.

Lemma C.3. *Every fixed point of Ψ is in $\mathcal{V}$.*

Proof. Each fixed point of Ψ must be in every $\mathcal{Y}^\kappa$, so that

$$V(\pi,\mu) \geq \frac{(\pi-\mu)Kc(1-\delta_A^\kappa)}{(1-\delta_A)}, \qquad \forall(\pi,\mu),$$

for all κ, implying $V \in \mathcal{V}$. □

Proposition C.1 implies a similar lower bound on the future information rent from shirking to that in Corollary 1.

Corollary C.1. *The future information rent from shirking becomes unbounded as the agent becomes arbitrarily patient:*

$$W(\mu) \geq \frac{Kc}{(1-\delta_A)}\frac{[\mu(\beta-\alpha)+1-\mu]}{(\beta-1)}\xi(\mu).$$

C.1 Existence of stationary high effort incentive efficient contracts

A natural approach to obtaining existence of a well-defined value function is to find conditions under which Ψ is a contraction. Since Ψ is the point-wise maximum of Ψ^e (defined in (C.5)), Ψ^s (the analogous operator in which the agent shirks in the current period, corresponding to the second term in (C.2)), and the zero function, Ψ will be a contraction (under the sup norm) if Ψ^e and Ψ^s are (again, under the sup norm).

Suppose $V, \hat{V} \in \mathcal{V}$. Then,

$$|\Psi^e(V) - \Psi^e(\hat{V})|$$
$$\leq \sup_{\pi,\mu} \left| \frac{(\pi - \mu)(\beta - 1)}{\mu(\beta - \alpha) + (1 - \mu)} \right|$$
$$\times \delta_A \left| \sum_k p_{s\pi}^k [V(\psi_s^k(\pi), \psi_e^k(\mu)) - \hat{V}(\psi_s^k(\pi), \psi_e^k(\mu))] \right|$$
$$+ \delta_A \left| \sum_k p_{e\pi}^k [V(\psi_e^k(\pi), \psi_e^k(\mu)) - \hat{V}(\psi_e^k(\pi), \psi_e^k(\mu))] \right|$$
$$\leq |V - \hat{V}| \times \left\{ \sup_{\pi,\mu} \left| \frac{(\pi - \mu)(\beta - 1)}{\mu(\beta - \alpha) + (1 - \mu)} \right| + 1 \right\} \delta_A.$$

This simple calculation shows that when Ψ^e is not a contraction, the failure arises from the future information rent from shirking (which contributes the sup term in the last expression. We also see that Ψ^e is a contraction if that sup term is sufficiently small (relative to $(1 - \delta_A)/\delta_A$). A similar calculation shows that Ψ^s is also a contraction if a similar sup term is sufficiently small (also relative to $(1 - \delta_A)/\delta_A$).[12]

A second approach to obtaining existence is to impose a parameter restriction that allows us to obtain a closed form expression for the value functions.

Lemma C.4. *Suppose $\beta = \alpha + 1$. The mapping Ψ has as a fixed point the function*

$$V(\pi, \mu) = \frac{\alpha c(\pi - \mu)}{1 - \delta_A}, \tag{C.6}$$

and the associated stationary high effort incentive efficient contract is the unique cost minimizing vector of utilities satisfying (C.3) and (C.4).

---

[12] The sup in Ψ^e is being taken over $|(p_{e\pi}^H - p_{e\mu}^H)/(p_{e\pi}^H - p_{s\mu}^H)|$, while the sup in Ψ^s is being taken over $|(p_{s\pi}^H - p_{e\mu}^H)/(p_{e\pi}^H - p_{s\mu}^H)|$.

Proof. We need only show that the function specified in (C.6) is a fixed point of Ψ. It is straightforward to verify that (C.6)) is a fixed point of Ψ^e. Some algebra verifies that $\Psi^e(V) - \Psi^s(V) = 0$ for V given by (C.6), and so (C.6) does indeed describe a fixed point of Ψ.[13] $\qquad\square$

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[13] Analogous calculations for the finite horizon can be found in the proof of Proposition 3 in an earlier version of this paper, Bhaskar and Mailath (2016).

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