Introduction to
Games of Incomplete Information

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Games of Complete Information

All the previous examples are games of complete information:

- All players know the nature of other players’ information and
- beliefs are known (being determined by the probability distribution of nature’s moves).
An Example without Complete Information

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Suppose I does not know which of the payoff matrices is true; II knows matrix, so both players know their own payoff.

II plays L in left matrix, and R in right matrix.

How will I choose in the presence of uncertainty?
Decision Making with Randomness

- How does a decision maker choose when faced with randomness?

  - **Risk**: choices have random consequences, but probability distribution of these consequences is known (lottery tickets, roulette wheel).

  - **Uncertainty**: choices have random consequences, and the probability distribution of these consequences is unknown (horse race). (Is probability even defined in these circumstances?)
Decision Making Under Risk

- Von Neumann and Morgenstern (1944): Primitive is a preference order over lotteries (i.e., a specified objective probability distribution) over outcomes $X$. 
Von Neumann and Morgenstern (1944): Primitive is a preference order \( \succeq \) over lotteries (i.e., a specified objective probability distribution) over outcomes \( X \).

If the preference order \( \succeq \) satisfies some “reasonable rationality” axioms, then there is a utility function \( u : X \rightarrow \mathbb{R} \) such that

\[
p \succeq q \iff \sum_x p(x)u(x) \geq \sum_x q(x)u(x).
\]
Decision Making Under Risk

- Von Neumann and Morgenstern (1944): Primitive is a preference order $\succeq$ over lotteries (i.e., a specified objective probability distribution) over outcomes $X$.

- If the preference order $\succeq$ satisfies some “reasonable rationality” axioms, then there is a utility function $u : X \rightarrow \mathbb{R}$ such that

  $$ p \succeq q \iff \sum_x p(x)u(x) \geq \sum_x q(x)u(x). $$

- Moreover, the utility function is unique up to positive affine transformations: If $v$ is another utility function also representing the preference order, then there exists two constants, $a > 0$ and $b$ such that for all $x \in X$,

  $$ v(x) = au(x) + b. $$
Savage (1972): Primitives are

- a state space $\Omega$, where a state resolves all uncertainty,
- acts (bets), $f : \Omega \to X$, where $X$ is (as in vNM) the space of outcomes, and
- a preference order $\succeq$ over the space of acts.
Decision Making Under Uncertainty

Savage (1972): Primitives are
- a state space $\Omega$, where a state resolves all uncertainty,
- acts (bets), $f : \Omega \to X$, where $X$ is (as in vNM) the space of outcomes, and
- a preference order $\succeq$ over the space of acts.

If the preference order $\succeq$ satisfies some “reasonable rationality” axioms, then there is a utility function $u : X \to \mathbb{R}$ and a subjective finitely-additive probability measure $\mu$ such that

$$f \succeq g \iff \int u(f(\omega))d\mu(\omega) \geq \int u(g(\omega))d\mu(\omega).$$
Savage (1972): Primitives are
- a state space $\Omega$, where a state resolves all uncertainty,
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$$f \succeq g \iff \int u(f(\omega))d\mu(\omega) \geq \int u(g(\omega))d\mu(\omega).$$

Moreover, the utility function is unique up to positive affine transformations and $\mu$ is unique.
Savage (with the extensions to accommodate countably additive beliefs and objective probabilities as well) is the standard (classical) approach to dealing with uncertainty.

Objective beliefs are determined by nature. Subjective beliefs are not and reflect the agent’s speculations or theories.
**An Example without Complete Information**

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- Suppose $I$ does not which of the payoff matrices is true; $II$ knows matrix, so both players know their own payoff.
- $II$ plays $L$ in left matrix, and $R$ in right matrix.
- $I$ assigns probability $\alpha$ to left matrix. Then, $I$ plays $T$ if $\alpha > \frac{1}{2}$, and plays $B$ if $\alpha < \frac{1}{2}$ (and is indifferent if $\alpha = \frac{1}{2}$).
An Example without Complete Information

Should II still feel comfortable playing $R$ if the payoffs are the right matrix?

Optimality of II’s action choice of $R$ depends on believing that I will play $B$, which only occurs if $\alpha \leq \frac{1}{2}$.

But suppose II does not know I’s beliefs $\alpha$. Then, II has beliefs over I’s beliefs and so II finds $R$ optimal if he assigns probability at least $\frac{1}{2}$ to I assigning probability at least $\frac{1}{2}$ to the right matrix.

But, how confident is I that II will play $R$ in the right matrix?....
A game of incomplete information or Bayesian game is the collection \( \{(A_i, T_i, p_i, u_i)_{i=1}^n\} \), where

- \( A_i \) is \( i \)'s action space,
- \( T_i \) is \( i \)'s type space,
- \( p_i : T_i \rightarrow \Delta\left(\prod_{j \neq i} T_j\right) \) is \( i \)'s subjective beliefs about the other players’ types, given \( i \)'s type and
- \( u_i : \prod_j A_j \times \prod_j T_j \rightarrow \mathbb{R} \) is \( i \)'s payoff function.

A player’s type \( t_i \) describes everything that \( i \) knows that is not common knowledge (including player \( i \)'s beliefs).
Bayes-Nash Equilibrium I

A strategy for $i$ is

$$s_i : T_i \rightarrow A_i.$$

Let $s(t) := (s_1(t_1), \ldots, s_n(t_n))$, etc.

**Definition**

The profile $(\hat{s}_1, \ldots, \hat{s}_n)$ is a Bayes-Nash (or Bayesian-Nash) equilibrium if, for all $i$ and all $t_i \in T_i$,

$$E_{t_{-i}}[u_i(\hat{s}(t), t)] \geq E_{t_{-i}}[u_i(a_i, \hat{s}_{-i}(t_{-i}), t)], \quad \forall a_i \in A_i,$$

where the expectation over $t_{-i}$ is taken with respect to $p_i(t_i)$. 
Bayes-Nash Equilibrium II

If the type spaces are finite, then the probability \( i \) assigns to the vector \( t_{-i} \in \prod_{j \neq i} T_j =: T_{-i} \) when his type is \( t_i \) can be denoted \( p_i(t_{-i}; t_i) \), and the profile \((\hat{s}_1, \ldots, \hat{s}_n)\) is a Bayes-Nash (or Bayesian-Nash) equilibrium if, for all \( i \) and all \( t_i \in T_i \),

\[
\sum_{t_{-i}} u_i(\hat{s}(t), t)p_i(t_{-i}; t_i) \geq \sum_{t_{-i}} u_i(a_i, \hat{s}_{-i}(t_{-i}))p_i(t_{-i}; t_i), \quad \forall a_i \in A_i.
\]
Return to Cournot duopoly example

- Firm 1’s costs are private information, while firm 2’s are public.
- Nature determines the costs of firm 1 at the beginning of the game, with $\Pr(c_1 = c_L) = \theta \in (0, 1)$.
- $A_i = \mathbb{R}_+$, firm 1’s type space is $T_1 = \{t^L_1, t^H_1\}$, firm 2’s is $T_2 = \{t_2\}$.
- Belief mapping $p_1$ for firm 1 is trivial: both types assign prob. 1 to $t_2$.
- The belief mapping for firm 2 is $p_2(t_2) = \theta \circ t^L_1 + (1 - \theta) \circ t^H_1 \in \Delta(T_1)$.
- Finally, payoffs are

$$u_1(q_1, q_2, t_1, t_2) = \begin{cases} 
[(a - q_1 - q_2) - c_L]q_1, & \text{if } t_1 = t^L_1, \\
[(a - q_1 - q_2) - c_H]q_1, & \text{if } t_1 = t^H_1,
\end{cases}$$

$$u_2(q_1, q_2, t_1, t_2) = [(a - q_1 - q_2) - c_2]q_2.$$
Return to Cournot Duopoly Example with a twist

Firm 2 may know that firm 1 has low costs, $c_L$

- $T_1 = \{t_1^L, t_1^H\} = \{c_L, c_H\}$, $T_2 = \{t_2^I, t_2^U\} = \{t_I, t_U\}$. The prior distribution is

$$
\Pr(t_1, t_2) = \begin{cases} 
1 - p' - p'', & \text{if } (t_1, t_2) = (t_1^L, t_2^I), \\
p', & \text{if } (t_1, t_2) = (t_1^L, t_2^U), \\
p'', & \text{if } (t_1, t_2) = (t_1^H, t_2^U).
\end{cases}
$$

- The belief mappings are

$$
p_1(t_1) = \begin{cases} 
(1 - \alpha) \circ t_2^I + \alpha \circ t_2^U, & t_1 = t_1^L, \\
1 \circ t_2^U, & t_1 = t_1^H,
\end{cases}
$$

$$
p_2(t_2) = \begin{cases} 
1 \circ t_1^L, & t_2 = t_2^I, \\
\theta \circ t_1^L + (1 - \theta) \circ t_1^H, & t_2 = t_2^U.
\end{cases}
$$
Interim perspective and the role of priors

- The perspective of a game of incomplete information is interim: the beliefs of player $i$ are specified type by type.
Interim perspective and the role of priors

- The perspective of a game of incomplete information is interim: the beliefs of player $i$ are specified type by type.
- Suppose the type spaces are finite or countably infinite. Let $\hat{q}_i$ be an arbitrary full support distribution on $T_i$, i.e., $\hat{q}_i \in \Delta(T_i)$. Then defining
  \[ q_i(t) := \hat{q}_i(t_i)p_i(t_{-i}; t_i) \quad \forall t \]
generates a prior $q_i \in \Delta(T)$ for player $i$ with the property that $p_i(\cdot; t_i) \in \Delta(T_{-i})$ is the belief on $t_{-i}$ conditional on $t_i$.
- There are many priors consistent with the subjective beliefs (since $\hat{q}_i$ is arbitrary).
Common Prior Assumption

Definition

The subjective beliefs are consistent or satisfy the Common Prior Assumption (CPA) if there exists a single probability distribution $p \in \Delta (\prod_i T_i)$ such that, for each $i$, $p_i(t_i)$ is the probability distribution on $T_{-i}$ conditional on $t_i$ implied by $p$.

If the type spaces are finite, this is equivalent to the existence of a distribution $p$ over type profiles such that

$$p_i(t_{-i}; t_i) = p(t_{-i}|t_i) = \frac{p(t)}{\sum_{t'_{-i}} p(t'_{-i}, t_i)}.$$
If beliefs are consistent, the Bayesian game can be interpreted as having an initial move by nature, which selects \( t \in T \) according to \( p \).

Suppose type spaces are finite. Viewed as a game of complete information, a profile \( \hat{s} \) is a Nash equilibrium if, for all \( i \), for all \( s_i : T_i \to A_i \),

\[
\sum_{t} u_i(\hat{s}(t), t)p(t) \geq \sum_{t} u_i(s_i(t_i), \hat{s}_{-i}(t_{-i}), t)p(t).
\]
If beliefs are consistent, the Bayesian game can be interpreted as having an initial move by nature, which selects $t \in T$ according to $p$.

Suppose type spaces are finite. Viewed as a game of complete information, a profile $\hat{s}$ is a Nash equilibrium if, for all $i$, for all $s_i : T_i \to A_i$,

$$\sum_t u_i(\hat{s}(t), t)p(t) \geq \sum_t u_i(s_i(t_i), \hat{s}_{-i}(t_{-i}), t)p(t).$$

This inequality can be rewritten as (where $p^*_i(t_i) := \sum_{t_{-i}} p(t_{-i}, t_i)$)

$$\sum_{t_i} \left\{ \sum_{t_{-i}} u_i(\hat{s}(t), t)p_i(t_{-i}; t_i) \right\} p^*_i(t_i) \geq$$

$$\sum_{t_i} \left\{ \sum_{t_{-i}} u_i(s_i(t_i), \hat{s}_{-i}(t_{-i}), t)p_i(t_{-i}; t_i) \right\} p^*_i(t_i).$$
Global Games
Carlsson and van Damme (1993)

\[
\begin{array}{c|cc}
& A & B \\
\hline
A & \theta, \theta & \theta - 9, 5 \\
B & 5, \theta - 9 & 7, 7 \\
\end{array}
\]

- The parameter \(\theta\) is uniformly distributed on the interval \([0, 20]\).
- For \(\theta < 5\), \(B\) is strictly dominant, while for \(\theta > 16\), \(A\) is strictly dominant.
- Each player \(i\) receives a signal \(x_i\), with \(x_1\) and \(x_2\) independently and uniformly drawn from the interval \([\theta - \varepsilon, \theta + \varepsilon]\) for \(\varepsilon > 0\).
- A pure strategy for player \(i\) is a function

\[
s_i : [-\varepsilon, 20 + \varepsilon] \rightarrow \{A, B\}.
\]
• For \( x_i \in [\varepsilon, 20 - \varepsilon] \), player \( i \)'s posterior on \( \theta \) is uniform on \([x_i - \varepsilon, x_i + \varepsilon]\).

• For \( x_i \in [\varepsilon, 20 - \varepsilon] \), player \( i \)'s posterior on \( x_j \) is symmetric around \( x_i \) with support \([x_i - 2\varepsilon, x_i + 2\varepsilon]\). Hence,

\[
\Pr\{x_j > x_i \mid x_i\} = \Pr\{x_j < x_i \mid x_i\} = \frac{1}{2}.
\]
For $x_i \in [\varepsilon, 20 - \varepsilon]$, player $i$’s posterior on $\theta$ is uniform on $[x_i - \varepsilon, x_i + \varepsilon]$.

For $x_i \in [\varepsilon, 20 - \varepsilon]$, player $i$’s posterior on $x_j$ is symmetric around $x_i$ with support $[x_i - 2\varepsilon, x_i + 2\varepsilon]$. Hence,

$$\Pr\{x_j > x_i \mid x_i\} = \Pr\{x_j < x_i \mid x_i\} = \frac{1}{2}.$$ 

**Lemma**

*For $\varepsilon < \frac{5}{2}$, the game has an essentially unique Nash equilibrium $(s_1^*, s_2^*)$, given by

$$s_i^*(x_i) = \begin{cases} A, & \text{if } x_i \geq 10 \frac{1}{2}, \\ B, & \text{if } x_i < 10 \frac{1}{2}. \end{cases}$$*
Proof

Suppose $x_i < 5 - \varepsilon$, so that $\theta < 5$ (so that $B$ is strictly dominant).
Proof

- Suppose \( x_i < 5 - \varepsilon \), so that \( \theta < 5 \) (so that \( B \) is strictly dominant).
- Then, player \( i \)'s payoff from \( A \) is less than that from \( B \) irrespective of player \( j \)'s action, and so \( i \) plays \( B \) for \( x_i < 5 - \varepsilon \) (as does \( j \) for \( x_j < 5 - \varepsilon \)).
Proof

- Suppose $x_i < 5 - \varepsilon$, so that $\theta < 5$ (so that $B$ is strictly dominant).
- Then, player $i$’s payoff from $A$ is less than that from $B$ irrespective of player $j$’s action, and so $i$ plays $B$ for $x_i < 5 - \varepsilon$ (as does $j$ for $x_j < 5 - \varepsilon$).
- But then at $x_i = 5 - \varepsilon$, since $\varepsilon < 5 - \varepsilon$, player $i$ assigns at least probability $\frac{1}{2}$ to $j$ playing $B$, and so $i$ strictly prefers $B$. 
Proof

- Suppose $x_i < 5 - \varepsilon$, so that $\theta < 5$ (so that $B$ is strictly dominant).
- Then, player $i$'s payoff from $A$ is less than that from $B$ irrespective of player $j$'s action, and so $i$ plays $B$ for $x_i < 5 - \varepsilon$ (as does $j$ for $x_j < 5 - \varepsilon$).
- But then at $x_i = 5 - \varepsilon$, since $\varepsilon < 5 - \varepsilon$, player $i$ assigns at least probability $\frac{1}{2}$ to $j$ playing $B$, and so $i$ strictly prefers $B$.
- Define

$$x_i^* := \sup\{x'_i \mid B \text{ is implied by iterated strict dominance for all } x_i < x'_i\}.$$
Proof

- Suppose $x_i < 5 - \varepsilon$, so that $\theta < 5$ (so that $B$ is strictly dominant).
- Then, player $i$’s payoff from $A$ is less than that from $B$ irrespective of player $j$’s action, and so $i$ plays $B$ for $x_i < 5 - \varepsilon$ (as does $j$ for $x_j < 5 - \varepsilon$).
- But then at $x_i = 5 - \varepsilon$, since $\varepsilon < 5 - \varepsilon$, player $i$ assigns at least probability $\frac{1}{2}$ to $j$ playing $B$, and so $i$ strictly prefers $B$.
- Define

$$x_i^* := \sup\{x_i' \mid B \text{ is implied by iterated strict dominance for all } x_i < x_i'\}.$$ 

- By symmetry, $x_1^* = x_2^* = x^*$. At $x_i = x^*$, player $i$ cannot strictly prefer $B$ to $A$ (since $E[\theta \mid x^*] = x^*$ and $p \leq \frac{1}{2}$):

$$px^* + (1 - p)(x^* - 9) = p5 + (1 - p)7$$ 

and so $x^* \geq 10 \frac{1}{2}$. 

Proof

Define

\[ x_{i}^{**} := \inf \{ x_{i}'' \mid A \text{ is implied by iterated strict dominance for all } x_{i} > x_{i}'' \} . \]

Then,

\[ x^{**} \leq 10^\frac{1}{2}, \]

and so

\[ 10^\frac{1}{2} \leq x^* \leq x^{**} \leq 10^\frac{1}{2}. \]
The assumption that players can randomize is sometimes criticized on three grounds:

1. players don’t randomize;
2. there is no reason for a player to randomize with just the right probability, when the player is indifferent over all possible randomization probabilities (including 0 and 1); and
3. a randomizing player is subject to ex post regret.
Ex post regret

- A player is said to be subject to **ex post regret** if after all uncertainty is resolved, a player would like to change his/her decision (i.e., has regret).
- In a game with no moves of nature, no player has ex post regret in a pure (but not mixed) strategy equilibrium.
- Any pure strategy equilibrium of a game with moves of nature will typically also have ex post regret.
  - Ex post regret should not be viewed as a criticism of mixing, but rather a caution to modelers. If a player has ex post regret, then that player has an incentive to change his/her choice. Whether a player is able to do so depends upon the scenario being modeled. If the player cannot do so, then there is no issue. If, however, the player can do so, then that option should be included in the game description.
Purification

Player $i$’s mixed strategy $\sigma_i$ of a game $G$ is said to be purified if in an “approximating” version of $G$ with private information (with player $i$’s private information given by $T_i$), that player’s behavior can be written as a pure strategy $s_i : T_i \rightarrow A_i$ such that

$$\sigma_i(a_i) \approx \Pr\{s_i(t_i) = a_i\},$$

where $\Pr$ is given by the prior distribution over $T_i$ (and so describes player $j \neq i$ beliefs over $T_i$).
Example of Purification

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The game has two strict pure strategy Nash equilibria and one symmetric mixed strategy Nash equilibrium. Let \( p = \Pr \{A\} \), then in the mixed strategy eq

\[
9p = 5p + 7(1 - p)
\]

\[\iff\]

\[
9p = 7 - 2p
\]

\[\iff\]

\[
11p = 7 \iff p = \frac{7}{11}.
\]
Example of Purification (cont)

Trivial purification: Give player $i$ payoff-irrelevant information $t_i$, where $t_i \sim \mathcal{U}([0, 1])$, and $t_1$ and $t_2$ are independent. This is a game with private information, where player $i$ learns $t_i$ before choosing his or her action.

- The mixed strategy equilibrium is purified by many pure strategy equilibria in the game with private information, such as

\[ s_i(t_i) = \begin{cases} 
B, & \text{if } t_i \leq 4/11, \\
A, & \text{if } t_i \geq 4/11. 
\end{cases} \]
Better Purification
Harsanyi (1973)

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- Player $i$’s type $t_i \sim \mathcal{U}([0, 1])$ and $t_1$ and $t_2$ are independent.
- A pure strategy for player $i$ is $s_i : [0, 1] \rightarrow \{A, B\}$. Suppose 2 is following a cutoff strategy (with $\bar{t}_2 \in (0, 1)$),

\[
s_2(t_2) = \begin{cases} 
A, & t_2 \geq \bar{t}_2, \\
B, & t_2 < \bar{t}_2.
\end{cases}
\]
Type $t_1$ expected payoff from $A$ is

\[
U_1 (A, t_1, s_2) = (9 + \varepsilon t_1) \Pr \{ s_2(t_2) = A \} \\
= (9 + \varepsilon t_1) \Pr \{ t_2 \geq \bar{t}_2 \} \\
= (9 + \varepsilon t_1) (1 - \bar{t}_2),
\]

while from $B$ is

\[
U_1 (B, t_1, s_2) = 5 \Pr \{ t_2 \geq \bar{t}_2 \} + 7 \Pr \{ t_2 < \bar{t}_2 \} \\
= 5(1 - \bar{t}_2) + 7 \bar{t}_2 \\
= 5 + 2\bar{t}_2.
\]

Thus, $A$ is optimal if and only if

\[
(9 + \varepsilon t_1)(1 - \bar{t}_2) \geq 5 + 2\bar{t}_2,
\]

i.e.,

\[
t_1 \geq \frac{11\bar{t}_2 - 4}{\varepsilon (1 - \bar{t}_2)}.
\]
In the symmetric equilibrium: \( \bar{t}_1 = \bar{t}_2 = \bar{t} \), that is,

\[
\bar{t} = \frac{11\bar{t} - 4}{\varepsilon(1 - \bar{t})},
\]

or

\[
\varepsilon \bar{t}^2 + (11 - \varepsilon)\bar{t} - 4 = 0.
\]

Let \( t(\varepsilon) \) denote the value of \( \bar{t} \) satisfying this equality.

- Note that \( t(0) = 4/11 \).
- Write the equality as \( g(\bar{t}, \varepsilon) = 0 \).
- Apply the implicit function theorem (since \( \partial g/\partial \bar{t} \neq 0 \) at \( \varepsilon = 0 \)) to conclude that for \( \varepsilon > 0 \) but close to 0, the cutoff value of \( \bar{t} \), \( t(\varepsilon) \), is close to \( 4/11 \) (the probability of the mixed strategy equilibrium in the unperturbed game).
Auctions
First-Price Sealed Bid Private-Value Auctions

Bidder $i$’s value for the object, $v_i$, is known only to $i$.

Nature chooses $v_i$, $i = 1, 2$, with $v_i$ being independently drawn from the interval $[v_i, \bar{v}_i]$, with distribution $F_i$ and density $f_i$.

Bidders know $F_i$ (and so $f_i$).

The set of possible bids is $\mathbb{R}_+$.

Bidder $i$’s ex post payoff as a function of $b_1$ and $b_2$, and values $v_1$ and $v_2$:

$$u_i(b_1, b_2, v_1, v_2) = \begin{cases} 
0, & \text{if } b_i < b_j, \\
\frac{1}{2} (v_i - b_i), & \text{if } b_i = b_j, \\
v_i - b_i, & \text{if } b_i > b_j.
\end{cases}$$
Suppose bidder 2 uses a strategy $\sigma_2 : [v_2, \bar{v}_2] \rightarrow \mathbb{R}_+$. 
Suppose bidder 2 uses a strategy $\sigma_2 : [\bar{v}_2, \bar{v}_2] \rightarrow \mathbb{R}_+$. Then, bidder 1’s expected (or interim) payoff from bidding $b_1$ at $v_1$ is

$$U_1(b_1, v_1; \sigma_2) = \int u_1(b_1, \sigma_2(v_2), v_1, v_2) \, dF_2(v_2)$$

$$= \frac{1}{2} (v_1 - b_1) \Pr \{\sigma_2(v_2) = b_1\}$$

$$+ \int_{\{v_2 : \sigma_2(v_2) < b_1\}} (v_1 - b_1) \, f_2(v_2) \, dv_2.$$
Suppose bidder 2 uses a strategy \( \sigma_2 : [\bar{v}_2, \tilde{v}_2] \to \mathbb{R}_+ \).

Then, bidder 1’s expected (or interim) payoff from bidding \( b_1 \) at \( v_1 \) is

\[
U_1(b_1, v_1; \sigma_2) = \int u_1(b_1, \sigma_2(v_2), v_1, v_2) \, dF_2(v_2)
\]

\[
= \frac{1}{2} (v_1 - b_1) \Pr\{\sigma_2(v_2) = b_1\}
\]

\[
+ \int_{\{v_2 : \sigma_2(v_2) < b_1\}} (v_1 - b_1) f_2(v_2) \, dv_2.
\]

Player 1’s ex ante payoff from the strategy \( \sigma_1 \) is given by

\[
\int U_1(\sigma_1(v_1), v_1; \sigma_2) \, dF_1(v_1),
\]

and so for an optimal strategy \( \sigma_1 \), the bid \( b_1 = \sigma_1(v_1) \) must maximize \( U_1(b_1, v_1; \sigma_2) \) for almost all \( v_1 \).
Proceed by “guess and verify”: that is, we impose a sequence of increasingly demanding conditions on the strategy of player 2, and prove that there is a best reply for player 1 satisfying these conditions.
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Without loss of generality, restrict attention to bids \( b_1 \) in the range of \( \sigma_2 \).

Then,

\[
U_1 (b_1, v_1; \sigma_2) = \int_{\{v_2: \sigma_2(v_2) < b_1\}} (v_1 - b_1) f_2(v_2) \, dv_2
\]

\[
= E[v_1 - b_1 \mid \text{winning}] \Pr\{\text{winning}\}
\]

\[
= (v_1 - b_1) \Pr \{ \sigma_2 (v_2) < b_1 \}
\]

\[
= (v_1 - b_1) \Pr \{ v_2 < \sigma_2^{-1}(b_1) \}
\]

\[
= (v_1 - b_1) F_2(\sigma_2^{-1}(b_1)).
\]
Need to choose $b_1$ to max $U_1(b_1, v_1; \sigma_2) = (v_1 - b_1)F_2(\sigma_2^{-1}(b_1))$. 
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Suppose $\sigma_2$ is differentiable, and that the bid $b_1 = \sigma_1(v_1)$ is an interior maximum.

The first order condition is

$$0 = -F_2(\sigma_2^{-1}(b_1)) + (v_1 - b_1) f_2(\sigma_2^{-1}(b_1)) \frac{d\sigma_2^{-1}(b_1)}{db_1}. $$
Need to choose \( b_1 \) to max \( U_1(b_1, v_1; \sigma_2) = (v_1 - b_1)F_2(\sigma_2^{-1}(b_1)) \).

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\]

But

\[
\frac{d\sigma_2^{-1}(b_1)}{db_1} = \frac{1}{\sigma_2'(\sigma_2^{-1}(b_1))},
\]

so

\[
F_2(\sigma_2^{-1}(b_1)) \sigma_2'(\sigma_2^{-1}(b_1)) = (v_1 - b_1) f_2(\sigma_2^{-1}(b_1)),
\]

i.e.,

\[
\sigma_2'(\sigma_2^{-1}(b_1)) = \frac{(v_1 - b_1) f_2(\sigma_2^{-1}(b_1))}{F_2(\sigma_2^{-1}(b_1))}.
\]
Assume $F_1 = F_2$, and suppose the equilibrium is symmetric, so that $\sigma_1 = \sigma_2 = \tilde{\sigma}$, and $b_1 = \sigma_1(v)$ implies $v = \sigma_2^{-1}(b_1)$.

Then,

$$\sigma'_2 \left( \sigma_2^{-1}(b_1) \right) = \frac{(v_1 - b_1) f_2 \left( \sigma_2^{-1}(b_1) \right)}{F_2 \left( \sigma_2^{-1}(b_1) \right)}.$$

becomes (dropping subscripts),

$$\tilde{\sigma}'(v) = \frac{(v - \tilde{\sigma}(v)) f(v)}{F(v)},$$

or

$$\tilde{\sigma}'(v) F(v) + \tilde{\sigma}(v) f(v) = vf(v).$$
We have

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But

\[ \frac{d}{dv} \tilde{\sigma}(v)F(v) = \tilde{\sigma}'(v)F(v) + \tilde{\sigma}(v)f(v), \]

so

\[ \tilde{\sigma}(\hat{v})F(\hat{v}) = \int_{\hat{v}}^{\hat{v}} vf(v)dv + k, \]

where \( k \) is a constant of integration.
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so
\[ \tilde{\sigma}(\hat{v})F(\hat{v}) = \int_{v}^{\hat{v}} vf(v)dv + k, \]
where \( k \) is a constant of integration.

Moreover, evaluating both sides at \( \hat{v} = v \) shows that \( k = 0 \), and so
\[ \tilde{\sigma}(\hat{v}) = \frac{1}{F(\hat{v})} \int_{v}^{\hat{v}} vf(v)dv = E[v \mid v \leq \hat{v}]. \]
Summary

- Each bidder bids the expectation of the other bidder’s valuation, conditional on that valuation being less than his (i.e., conditional on his value being the highest). This is not an accident.

- Summarizing the calculations till this point, we have shown that if $(\tilde{\sigma}, \tilde{\sigma})$ is a Nash equilibrium in which $\tilde{\sigma}$ is a strictly increasing and differentiable function, and $\tilde{\sigma}(v)$ is interior (which here means strictly positive), then it is given by $\tilde{\sigma}(\hat{v}) = E[v \mid v \leq \hat{v}]$.
  Note that $E[v \mid v \leq \hat{v}]$ is increasing in $\hat{v}$ and lies in the interval $[v, Ev]$.

- It remains to verify the hypotheses. It is immediate that $\tilde{\sigma}$ is strictly increasing and differentiable. Moreover, for $v > \underline{v}$, $\tilde{\sigma}(v)$ is strictly positive. It remains to verify the optimality of bids.
Optimality

- It is not optimal to bid $b_1 < v = \tilde{\sigma}(v)$ or $b_1 > E v = \tilde{\sigma}(\tilde{v})$.
- Since $\tilde{\sigma}$ is strictly increasing and continuous, any bid in $[v, E v]$ is the bid of some valuation $v$.
- Bidding as if valuation $\hat{v}$ has valuation $v'$ is suboptimal:

$$U(v'; \hat{v}) = (\hat{v} - \tilde{\sigma}(v')) Pr(v_2 \leq \tilde{\sigma}^{-1}(\tilde{\sigma}(v'))) = (\hat{v} - E[v \mid v \leq v']) F(v')$$

$$= \left( \hat{v} - \frac{1}{F(v')} \int_{v}^{v'} vf(v)\,dv \right) F(v')$$

$$= \int_{v}^{v'} (\hat{v} - v) f(v)\,dv.$$
Common Value Auctions

- Each bidder receives a private signal about the value of the object, $t_i$, with $t_i \in T_i = [0, 1]$, uniformly independently distributed.
- The common (to both players) value of the object is $v = t_1 + t_2$.
- Ex post payoffs are given by

$$u_i(b_1, b_2, t_1, t_2) = \begin{cases} 
  t_1 + t_2 - b_i, & \text{if } b_i > b_j, \\
  \frac{1}{2}(t_1 + t_2 - b_i), & \text{if } b_i = b_j, \\
  0, & \text{if } b_i < b_j.
\end{cases}$$
Suppose bidder 2 uses strategy $\sigma_2 : T_2 \rightarrow \mathbb{R}_+$. Suppose $\sigma_2$ is strictly increasing. Then, $t_1$’s expected payoff from bidding $b_1$ is

$$U_1(b_1, t_1; \sigma_2) = E[t_1 + t_2 - b_1 \mid \text{winning}] \Pr\{\text{winning}\}$$

$$= E[t_1 + t_2 - b_1 \mid t_2 < \sigma_2^{-1}(b_1)] \Pr\{t_2 < \sigma_2^{-1}(b_1)\}$$

$$= (t_1 - b_1)\sigma_2^{-1}(b_1) + \int_0^{\sigma_2^{-1}(b_1)} t_2 \, dt_2$$

$$= (t_1 - b_1)\sigma_2^{-1}(b_1) + (\sigma_2^{-1}(b_1))^2/2.$$
Maximizing $U_1(b_1, t_1; \sigma_2) = (t_1 - b_1)\sigma_2^{-1}(b_1) + (\sigma_2^{-1}(b_1))^2/2$.

If $\sigma_2$ is differentiable, the first order condition is

$$0 = -\sigma_2^{-1}(b_1) + (t_1 - b_1)\frac{d\sigma_2^{-1}(b_1)}{db_1} + \sigma_2^{-1}(b_1)\frac{d\sigma_2^{-1}(b_1)}{db_1},$$

and so

$$\sigma_2^{-1}(b_1)\sigma'_2(\sigma_2^{-1}(b_1)) = (t_1 + \sigma_2^{-1}(b_1) - b_1).$$

Suppose the equilibrium is symmetric, so that $\sigma_1 = \sigma_2 = \sigma$. Then,

$$t\sigma'(t) = 2t - \sigma(t).$$

Integrating,

$$t\sigma(t) = t^2 + k,$$

where $k$ is a constant of integration. Evaluating at $t = 0$ shows that $k = 0$, and so

$$\sigma(t) = t.$$
**Winner’s Curse**

- Note that this is **not** the profile that results from the analysis of the private value auction when \( v = 1/2 \) \( (E[t_1 + t_2 | t_1] = t_1 + 1/2) \).
- In particular, letting \( v' = t + \frac{1}{2} \), we have

\[
\sigma_{\text{private value}}(t) = \tilde{\sigma}(v') = \frac{v' + 1/2}{2} = \frac{t + 1}{2} > t = \sigma_{\text{common value}}(t).
\]

- This illustrates the **winner’s curse**: \( E[v | t_1] > E[v | t_1, \text{winning}] \). In particular, in the equilibrium just calculated,

\[
E[v | t_1, \text{winning}] = E[t_1 + t_2 | t_1, t_2 < t_1] = t_1 + \frac{1}{t_1} \left[ \frac{(t_2)^2}{2} \right]_{t_1}^{t_1} = \frac{3t_1}{2},
\]

while \( E[v | t_1] = t_1 + \frac{1}{2} > \frac{3t_1}{2} \) (recall \( t_1 \in [0, 1] \)).


