

Equilibrium of Affiliated Value Second Price Auctions with Financially Constrained Bidders: The Two-Bidder Case¹

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We study affiliated value second price auctions with two financially constrained bidders. We prove the existence of a symmetric equilibrium under quite general conditions. Comparative static results are provided. *Journal of Economic Literature* Classification Number: D44. © 2002 Elsevier Science (USA)

1. INTRODUCTION

A range of empirical and anecdotal evidence demonstrates the importance of buyers' financial constraints in auctions. Financial constraints are used by Cramton (1995) to explain some bidders' exit decisions in the personal communications services (PCS) auctions and by Genesove (1993) to explain the end-of-day drop in prices at used car auctions. The importance of financial constraints is also recognized in auction design. The U.S.

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government often limits the length and size of mineral leases and sets some leases aside for sale to small firms. Recently governments in many parts of the world have aggressively sought to privatize once socially held assets. Given the magnitude of these privatization sales, it is often realistic to assume that buyers may run up against financial constraints.

There is a growing body of literature on auctions with financially constrained bidders. In a series of seminal papers, Che and Gale (1996a, 1998) study independent private value (IPV) auctions of a single object with financially constrained bidders. They show that the revenue equivalence between standard IPV auctions no longer holds once financial constraints are introduced. The IPV setting, however, precludes one from analyzing how bidding constraints interact with information revelation in auctions. Che and Gale (1996b) consider a model in which bidders have identical common valuation of the object but have independent and privately observed bidding constraints. Their focus is to show that, in the presence of budget constraints, standard auctions are not revenue equivalent and may be dominated by a lottery and an all-pay auction. Again, the setting of known common valuation prevents them from studying the interaction among the budget constraints, the information revelation, and the winner's curse. Benoît and Krishna (2001) study the auctions of multiple objects with financially constrained bidders under complete information. They focus on how the optimal orders of sale depend on the bidding constraints and on whether the multiple objects are complements or substitutes. Maskin (2000) studies the constrained efficient auction mechanism with liquidity constrained buyers in an IPV setting. Zheng (2000) provides a complete solution to the first price IPV auction with financially constrained bidders when costly outside financing and bankruptcy are allowed.

In this paper we introduce financial constraints in a two-bidder version of Milgrom and Weber's (1982) general affiliated value auction model. Each bidder privately observes a signal about the value of the object and is privately informed of his or her available budget in the auction. We establish, under quite general conditions, the existence of a symmetric equilibrium for the second price auction (SPA). It is shown that, with financially constrained bidders, the budget constraints and the signals closely interact in the equilibrium conditions of the SPA. The symmetric equilibrium we identify takes the following form: a bidder with a bidding budget w and a private signal x will bid $\min\{w, b(x)\}$ where $b(\cdot)$, called the "*unconstrained*" *bidding function*, is an increasing function in x . In words, $b(x)$ is the amount that an unconstrained bidder will bid in equilibrium, knowing that her opponent may be constrained and may think that she is constrained and so on. It is shown that the unconstrained bidders are more aggressive when his or her opponents may be financially constrained in environments with interdependent values. The intuition is simple: The likelihood that one's

opponent may be financially constrained entails the possibility that a bidder wins the object even though her opponent has a higher signal, which attenuates the winners' curse and makes a bidder more aggressive.

The remainder of the paper is structured as follows. Section 2 presents the model. Section 3 derives the ordinary differential equation that the "unconstrained" bidding function must satisfy in equilibrium. Section 4 proves, under quite general conditions, the existence of a symmetric equilibrium. Section 5 presents the comparative statics of the "unconstrained" bidding function with respect to public signals and the severity of the bidding constraints. Section 6 provides further discussions and extensions.

2. THE MODEL

The main text of this paper studies the *two* bidder case, but we will here present the model of an arbitrary $n \geq 2$ risk neutral bidders who compete for a single object in a SPA. Each bidder possesses some information concerning the value of the object for sale: let X_i be the real-valued information variable (or value estimate, or signal) observed by bidder i . Let $X = (X_1, \dots, X_n)$ be a vector of information variables observed by all bidders. Let $S = (S_1, \dots, S_m)$ be a vector of additional real-valued variables which influence the value of the object to the bidders. Some of the components of S might be observed by the seller. Let $f(s, x)$ denote the joint probability density of the signals and let F be the corresponding cumulative distribution function.

The actual value of the object to bidder i depends on all the information variables and is denoted by $V_i = u_i(S, X_i, X_{-i})$. The following assumptions from Milgrom and Weber (1982) are maintained:

Assumption 1. There is a function u on \mathbb{R}^{n+m} such that for all i , $u_i(S, X) = u(S, X_i, \{X_j\}_{j \neq i})$.

Assumption 2. The function u is non-negative, is continuous, and is non-decreasing in its variables.

Assumption 3. f is symmetric in its last n arguments.

Assumption 4. The variables $S_1, \dots, S_m, X_1, \dots, X_n$ are affiliated.

Now we add the following feature to the above model of Milgrom and Weber (1982): we assume that each bidder i gets an independent random draw W_i from a common distribution G , which represents her available bidding budget. The corresponding density is denoted by g . We assume that $g(\cdot)$ is strictly positive on the support of W_i . Let $W = (W_1, \dots, W_n)$ be the random vector of bidding budgets for the n bidders. The following

assumptions are maintained:

Assumption 5. For each i , W_i is independent of (S, X, W_{-i}) .

Assumption 6. For $i = 1, \dots, n$, $\text{supp}(W_i) = [\underline{w}, \bar{w}]$ and $\text{supp}(X_i) = [\underline{x}, \bar{x}]$, and for $k = 1, \dots, m$, $\text{supp}(S_k) = [\underline{s}_k, \bar{s}_k]$. The bounded support assumption on X_i and S_k , together with Assumption 2 implies that for each i , $E[V_i]$ is finite.

Assumption 7. $\bar{w} > u(\bar{s}, \bar{x}, \bar{x}, \dots, \bar{x}) > \underline{w}$. This assumption implies that, first, with probability one a bidder with budget \bar{w} is not constrained; second, *ex ante* there is a positive probability that any bidder will be financially constrained.

3. DERIVING THE FIRST-ORDER CONDITION

We will analyze the equilibrium for the case $n = 2$. We will focus on the symmetric equilibrium in which the bidding function takes the form that for each $i = 1, 2$,

$$B(w_i, x_i) = \min\{w_i, b(x_i)\}, \quad (1)$$

where $b(\cdot)$ is a strictly increasing and piecewise differentiable function. We can think of $b(x_i)$ as the bid of bidder i who has a signal x_i , who is not herself financially constrained but understands that her opponents may be constrained and so on. We will refer to $b(\cdot)$ as the “*unconstrained*” *bidding function*. For expositional ease we will refer to bidder 1 as “she” and bidder 2 as “he.”

Suppose that bidder 2 follows the bidding rule $B(\cdot, \cdot)$ as in (1), and consider the optimal strategy for bidder 1 who has a budget \bar{w} and a signal x_1 . By Assumption 7, bidder 1 will not herself be financially constrained, but she understands that her opponent may be constrained and may expect him to be constrained with positive probability, and so on.

To analyze bidder 1’s best response, we write $Q(b, w_2, x_2)$ and $P(b, w_2, x_2)$ as, respectively, bidder 1’s probability of winning the object and her expected payment, if she bids b and if bidder 2 is of type (w_2, x_2) and is following the bidding strategy (1). By the rules of SPA, $Q(b, w_2, x_2)$ and $P(b, w_2, x_2)$ can be expressed respectively, as,

$$Q(b, w_2, x_2) = 1\{b > B(w_2, x_2)\} = \begin{cases} 0 & \text{if } b < B(w_2, x_2) \\ 1 & \text{if } b > B(w_2, x_2) \end{cases} \quad (2)$$

$$\begin{aligned} P(b, w_2, x_2) &= B(w_2, x_2)1\{b > B(w_2, x_2)\} \\ &= \begin{cases} 0 & \text{if } b < B(w_2, x_2) \\ w_2 & \text{if } b > B(w_2, x_2) = w_2 \\ b(x_2) & \text{if } b > B(w_2, x_2) = b(x_2). \end{cases} \end{aligned} \quad (3)$$

By Assumption 7, a type- (\bar{w}, x_1) bidder 1 will be unconstrained and hence she solves the following problem:

$$\max_{b \geq 0} \Pi(b; x_1) \equiv \int_{\underline{w}}^{\bar{w}} \int_{\underline{x}}^{\bar{x}} \{E[u(S, X_1, X_2)|X_1 = x_1, X_2 = x_2]Q(b, w_2, x_2) - P(b, w_2, x_2)\} dF_{X_2|X_1}(x_2|x_1) dG(w_2). \quad (4)$$

To further analyze (4), it is useful to define the “generalized inverse” of $b(\cdot)$, which we denote by ψ , as

$$\psi(b) = \sup\{x : b(x) \leq b\}. \quad (5)$$

It is clear that if $b(\cdot)$ is everywhere continuous then we will simply have $\psi(z) = b^{-1}(z)$, which follows directly from the assumed strict monotonicity of $b(\cdot)$.

Using (2) and (3), we can rewrite bidder 1’s objective function (4) as

$$\begin{aligned} \Pi(b; x_1) &= \int_{\underline{x}}^{\psi(b)} [1 - G(b(x_2))]\{E[u(S, X_1, X_2)|X_1 = x_1, X_2 = x_2] \\ &\quad - b(x_2)\} dF_{X_2|X_1}(x_2|x_1) \\ &\quad + \int_{\underline{w}}^b \{E[u(S, X_1, X_2)|X_1 = x_1, X_2 \geq \psi(w_2)] - w_2\} \\ &\quad \times [1 - F_{X_2|X_1}(\psi(w_2)|x_1)] dG(w_2). \end{aligned} \quad (6)$$

The right-hand side of (6) is the sum of bidder 1’s expected surplus under the two events when her bid, b wins the object (see Fig. 1 for a graphical illustration.) The first term is her expected surplus in event I, when she wins the object and pays bidder 2’s bid which is $b(x_2)$. Note that in this

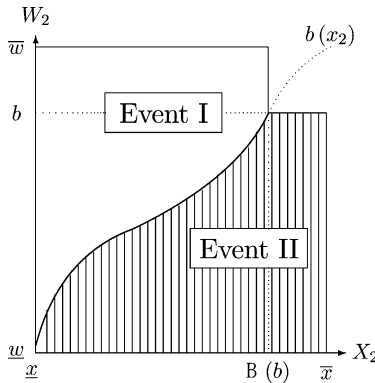


FIG. 1. The two events in which bidder 1 wins the object with a bid b . Event I: $\{(w_2, x_2) : b > \min\{w_2, b(x_2)\} = b(x_2)\}$; event II: $\{(w_2, x_2) : b > \min\{b(x_2), w_2\} = w_2\}$.

TABLE I
Notations

$\nu(x, z)$:	$E[u(S, X_i, X_j) X_i = x, X_j = z],$	$j \neq i$
$\varphi(x, z)$:	$E[u(S, X_i, X_j) X_i = x, X_j \geq z],$	$j \neq i$
$\lambda(z x)$:	$\frac{f_{X_j X_i}(z x)}{1 - F_{X_j X_i}(z x)},$	$j \neq i$
$\gamma(w)$:	$\frac{g(w)}{1 - G(w)}$	

event, conditional on bidder 2's signal x_2 , the probability that w_2 is higher than $b(x_2)$ is $1 - G(b(x_2))$. The second term is her expected surplus in event II, when she wins the object and pays bidder 2's bid, which is w_2 . Notice that in this event, bidder 1 can only infer from her winning the object that $X_2 \geq \psi(w_2)$, which, conditional on her own signal, occurs with probability $1 - F_{X_2|X_1}(\psi(w_2)|x_1)$.

We now introduce in Table I some notation that eases exposition. The interpretation of the four terms in Table I is as follows:

- $\nu(x, z)$ is the expected valuation of the object to bidder i when i and j 's ($j \neq i$) signals are, respectively, x and z ;
- $\varphi(x, z)$ is the expected valuation of the object to bidder i when i 's signal is x and j 's ($j \neq i$) signal is at least z ;
- $\lambda(z|x)$ is the hazard function of j 's signal conditional on i 's signal x ;
- $\gamma(w)$ is the hazard function of the budget constraint distribution.

Using the above notation, we can now differentiate $\Pi(b; x_1)$ with respect to b and obtain, after applying Leibniz's rule and some simplifications,

$$\begin{aligned} \frac{\partial \Pi(x_1, b)}{\partial b} &= [1 - G(b)][\nu(x_1, \psi(b)) - b]f_{X_2|X_1}(\psi(b)|x_1)\psi'(b) \\ &\quad + [\varphi(x_1, \psi(b)) - b][1 - F_{X_2|X_1}(\psi(b)|x_1)]g(b). \end{aligned} \quad (7)$$

It will prove more convenient to verify the second-order condition if we interpret the auction as a *revelation mechanism*: each bidder $i = 1, 2$ reports his or her type to a mediator who will submit a bid for i according to (1). Assume truth-telling by bidder 2. Then the expected payoff for bidder 1 of type (\bar{w}, x_1) if she reports z , assuming that she reports her budget type truthfully (which is verified in the proof of Theorem 1), is given by $\Pi(b(z); x_1)$ because the mediator will bid $b(z)$ for her. The first-order condition for truth-telling to be an equilibrium is that the derivative of

$\Pi(b(z); x_1)$ with respect to z evaluated at x_1 is equal to zero. Note that

$$\begin{aligned}
 \frac{\partial \Pi(b(z), x_1)}{\partial z} &= \frac{\partial \Pi(b(z); x_1)}{\partial b} b'(z) \\
 &= [1 - G(b(z))] [\nu(x_1, z) - b(z)] f_{X_2|X_1}(z|x_1) \\
 &\quad + [\varphi(x_1, z) - b(z)] [1 - F_{X_2|X_1}(z|x_1)] g(b(z)) b'(z) \\
 &= [1 - G(b(z))] [1 - F_{X_2|X_1}(z|x_1)] \\
 &\quad \cdot \{[\nu(x_1, z) - b(z)] \lambda(z|x_1) + \gamma(b(z)) [\varphi(x_1, z) \\
 &\quad - b(z)] b'(z)\}. \tag{8}
 \end{aligned}$$

We denote expression (8) as $\text{FOC}(z|x_1)$. In order for truth-telling to be optimal, we require that

$$\text{FOC}(x_1|x_1) = 0.$$

Noting that $[1 - G(b(x_1))] [1 - F_{X_2|X_1}(x_1|x_1)]$ is positive for all $x_1 < \bar{x}$, we can simplify the first-order condition as

$$[\nu(x_1, x_1) - b(x_1)] \lambda(x_1|x_1) + \gamma(b(x_1)) [\varphi(x_1, x_1) - b(x_1)] b'(x_1) = 0. \tag{9}$$

It is easier to interpret the first-order condition if we rewrite (9) as

$$\begin{aligned}
 &\overbrace{[1 - F_{X_2|X_1}(x_1|x_1)] g(b(x_1)) b'(x_1) \Delta x}^{\text{Term 1}} \quad \overbrace{[\varphi(x_1, x_1) - b(x_1)]}^{\text{Term 2}} \\
 &= \overbrace{[1 - G(b(x_1))] f_{X_2|X_1}(x_1|x_1) \Delta x}^{\text{Term 3}} \quad \overbrace{[b(x_1) - \nu(x_1, x_1)]}^{\text{Term 4}}. \tag{10}
 \end{aligned}$$

The left- and right-hand sides of Eq. (10) are, respectively, bidder 1's expected net benefit and expected net cost if she marginally raises the announcement of her type by Δx . To see this, note that term 1 in Eq. (10) is the increase in the probability of bidder 1 winning the object when her opponent was a marginal winner who was previously bidding his budget, and term 2 is the accrued net surplus to bidder 1 from winning the object in such events, term 3 is the increase in the probability of winning the object when her opponent was a marginal winner who was previously unconstrained (hence he has a signal no more than x_1), and term 4 is the accrued net cost of winning in such events. The optimality condition requires that the expected net benefit and cost be exactly balanced when bidder 1 announces her true signal.

Notice that if the bidders' valuations are private, that is, if $u(S, X_i, X_{-i})$ does not depend on X_{-i} , then it is immediate that $\nu(x, x) = \varphi(x, x)$. Hence the first order condition (9) can be satisfied only if $b(x_1) = \nu(x_1, x_1)$. In what follows we will consider the *interdependent value case* in which $u(S, X_i, X_{-i})$ is strictly increasing in all of its arguments.

4. EXISTENCE OF EQUILIBRIUM

In this section, we first obtain the properties of the “unconstrained” bidding function $b(\cdot)$ by examining the necessary first order condition (9), then we show the existence of solutions to (9), and finally we provide a technical condition under which condition (9) is also sufficient for $b(\cdot)$ to be part of an equilibrium.

4.1. *Properties of $b(\cdot)$*

The first property of $b(\cdot)$ is that, if $b(x) < \underline{w}$, then $b(x) = \nu(x, x)$. There are two ways to understand this property. The first is to examine Eq. (9) and note that the term $\gamma(b(x))$ is equal to zero when $b(x) < \underline{w}$, which entails that condition (9) can be satisfied only if $b(x) = \nu(x, x)$. Alternatively, when $b(x) < \underline{w}$, bidder 1 knows that her marginal losing opponent is not constrained. Hence the budget becomes irrelevant to her marginal benefit–cost trade-off in Eq. (10), which implies that she should bid exactly the same amount as in Milgrom and Weber’s (1982) second price auction with no financial constraints.

The second property of $b(\cdot)$ is that, if $b(x) \geq \underline{w}$ and $x < \bar{x}$, then function $b(\cdot)$ must satisfy the ordinary differential equation (ODE)

$$b'(x) = \frac{\lambda(x|x) [b(x) - \nu(x, x)]}{\gamma(b(x)) [\varphi(x, x) - b(x)]}, \quad (11)$$

which is a re-arrangement of condition (9).

To summarize, bidders who place bids lower than the minimum budget shall ignore the constraints and bid the same amount as that prescribed by the equilibrium with no financial constraints; bidders who place bids higher than the minimum budget shall bid according to the solution of ODE (11). We will denote a solution to ODE (11) by $b^*(x)$. From the above discussion, it is worth remarking that the solution to ODE (11) determines the *domain* on which the equation itself is defined: $\{x \in [\underline{x}, \bar{x}]: b^*(x) \geq \underline{w}\}$.

For the solution to ODE (11) to be monotonically increasing as postulated in (1), it must satisfy that, for all $x < \bar{x}$,

$$\nu(x, x) < b^*(x) < \varphi(x, x), \quad (12)$$

because the hazard functions $\lambda(x|x)$ and $\gamma(b(x))$ are both positive and $\nu(x, x) < \varphi(x, x)$ for $x < \bar{x}$ in the interdependent value case.

Finally, since $\nu(\bar{x}, \bar{x}) = \varphi(\bar{x}, \bar{x})$, it must be the case that if $b(\cdot)$ is continuous at \bar{x} , then

$$b(\bar{x}) = \varphi(\bar{x}, \bar{x}) = \nu(\bar{x}, \bar{x}). \quad (13)$$

We will refer to (13) as the *boundary condition*.

4.2. Existence of Monotonic Solutions to ODE (11)

Here we prove the existence of increasing solutions to ODE (11) that satisfy the boundary condition (13) and provide a local condition on the hazard function of the budget distribution that guarantees the uniqueness of the solution.

PROPOSITION 1. *There exist solutions to ODE (11) that satisfy the monotonicity condition (12) and the boundary condition (13). Furthermore, if $\gamma(\cdot)$ is non-increasing around a neighborhood of $\nu(\bar{x}, \bar{x})$, then the solution is unique.*

Proof (Existence). The main difficulty to prove existence lies in the fact that in the proximity of curve φ or point \bar{x} , ODE (11) does not satisfy the Lipschitz condition. Moreover, boundary condition (13) requires that any solution to ODE (11) get arbitrarily close to φ as x converges to \bar{x} . The idea of the proof is to first apply the standard existence and uniqueness theorem for a compact region strictly below φ and away from \bar{x} , where the Lipschitz condition holds and then exploit the particular structure of our problem to extend the solution to the whole domain D given by

$$D = \{(x, b) : \nu(x, x) \leq b < \varphi(x, x), b \geq \underline{w} \text{ and } \underline{x} \leq x < \bar{x}\}.$$

Step 1. We first consider a subset of D uniformly bounded away from φ by $\varepsilon > 0$ and from \bar{x} by $\delta > 0$:

$$D(\varepsilon, \delta) = \{(x, b) \in D : b \leq \varphi(x, x) - \varepsilon, x \leq \bar{x} - \delta\}.$$

Clearly ODE (11) satisfies the Lipschitz condition on $D(\varepsilon, \delta)$. By standard results in ordinary differential equations (see, e.g., Theorem 6 of Birkoff and Rota, 1969, p. 23), for any point $(x, b) \in D(\varepsilon, \delta)$, there exists a unique flow curve starting from (x, b) . Moreover, since ε is arbitrary, we can extend such a flow curve by continuity to the boundary of $D(0, \delta)$ (for the extension technique, see, e.g., Cronin, 1980). Also by taking δ to zero, we can extend the flow curves to the whole domain D . But note that the extension does not need to satisfy ODE (11) at the boundary of D . To summarize, we have shown that for any point in the *interior* of D there is a unique flow curve induced by ODE (11).

Step 2. To prove existence it now suffices to show that there is a flow curve that hits $(\bar{x}, \varphi(\bar{x}, \bar{x}))$. Let $F = \{(\underline{x}, b) : \underline{w} \leq b \leq \varphi(\underline{x}, \underline{x})\} \cup \{(x, \underline{w}) : \nu(x, x) \leq \underline{w} \leq \varphi(x, x)\}$. Note that F is a closed connected set. We then define

$$F^\nu = \{(x, b) \in F : \text{the flow curve induced by ODE (11) starting at } (x, b) \text{ hits } \nu\}$$

$$F^\varphi = \{(x, b) \in F : \text{the flow curve induced by ODE (11) starting at } (x, b) \text{ hits } \varphi\}.$$

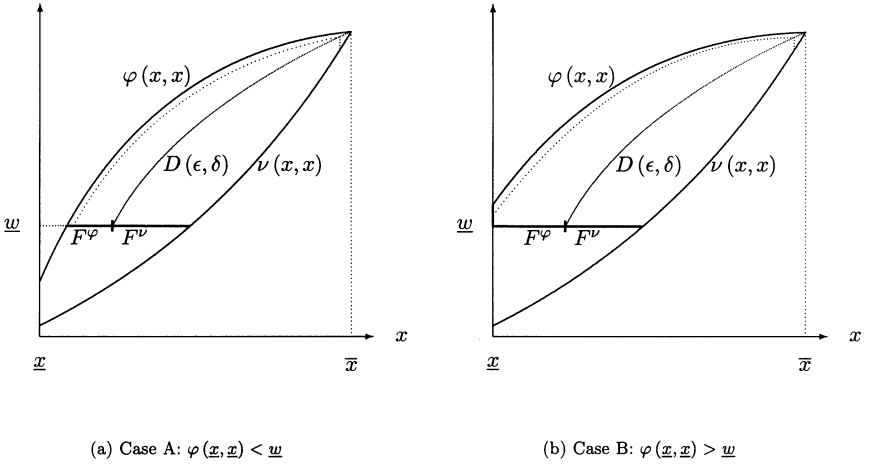


FIG. 2. Two possible configurations of the closed connected set $F = F^\nu \cup F^\varphi$.

We first establish that F^ν and F^φ are both non-empty (see Fig. 2 for an illustration of the two possible configurations of F). Take any point in F that is sufficiently close to φ , inspection of ODE (11) reveals that the flow curve from that point must intersect φ because the right-hand side of ODE (11) can be made sufficiently larger than the derivative of φ . Hence F^φ is non-empty. The same argument shows that F^ν is non-empty. Since D is bounded, it is immediate that $F = F^\nu \cup F^\varphi$. Since the end-point of a curve depends continuously on the starting point, the sets F^ν and F^φ are both closed. Because F is connected, F^ν and F^φ must have a non-empty intersection. Since the graphs of ν and φ intersect only at $(\bar{x}, \varphi(\bar{x}, \bar{x}))$, we have proved that there is at least one point from which the flow curve induced by ODE (11) satisfies boundary condition (13).

(Uniqueness). We know that the flow curves do not intersect each other in the interior of D , but to rule out the possibility that the flow curves intersect at the boundary point $(\bar{x}, \varphi(\bar{x}, \bar{x}))$ requires a different argument. Suppose that the solution is not unique and let $\tilde{b}(\cdot)$ and $\hat{b}(\cdot)$ be two distinct solutions. Pick any x' close to \bar{x} . Due to the local uniqueness in the interior, we can, without loss of generality, assume that $\tilde{b}(x') > \hat{b}(x')$. But then if $\gamma(\cdot)$ is non-increasing around a neighborhood of $\nu(\bar{x}, \bar{x})$, then direct inspection of ODE (11) reveals that $\tilde{b}'(x') > \hat{b}'(x')$, which implies that $\tilde{b}(\bar{x}) > \hat{b}(\bar{x})$, a contradiction. Hence the solution must be unique. Q.E.D.

The domain on which ODE (11) is defined is then given by the interval $[x_w, \bar{x}]$, where

$$x_w \equiv \inf\{x \in [x, \bar{x}]: b^*(x) \geq \underline{w}\}. \quad (14)$$

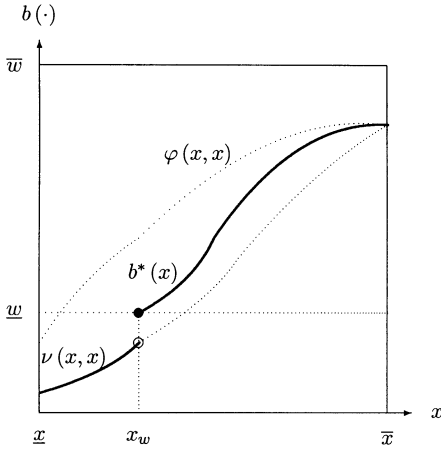


FIG. 3. An illustration of the “unconstrained” equilibrium bidding function $b(\cdot)$.

The “unconstrained” bidding function $b(\cdot)$ in a candidate symmetric equilibrium in the form of (1) will be given by

$$b(x) = \begin{cases} \nu(x, x) & \text{if } x < x_w \\ b^*(x) & \text{otherwise.} \end{cases} \quad (15)$$

The “unconstrained” bidding function $b(\cdot)$ is illustrated in Fig. 3.

The “unconstrained” bidding function (15) possesses, potentially, an interesting *discontinuity* feature. Note that $\nu(x_w, x_w) \leq \underline{w}$ holds by the definition of x_w . But there is no guarantee that $\nu(x_w, x_w) = \underline{w}$. If the strict inequality $\nu(x_w, x_w) < \underline{w}$ holds, then there is a discontinuity in the “unconstrained” bidding function. The empirical consequence of this discontinuity is that the bids in the interval $(\nu(x_w, x_w), \underline{w})$ will be observed with probability zero. This can be related to the explanation of *jump bidding* by Avery (1998). In his paper, two bidders are allowed to open an English auction by choosing to post a bid of 0 or $K > 0$. Avery shows that there exists an asymmetric equilibrium in which jump bidding strategies are used by bidders with higher signals to intimidate her opponents. However, by construction, the jump bidding in Avery’s paper can occur only in the beginning of the auction. In reality, jump bids often occur in the middle of an auction. For example, Cramton (1997) found that in the Federal Communications Commission (FCC) auction of the radio spectrum, “49% of all new high bids were jump bids . . . , 23% of these jump bids were raises of one’s own high bids.” Even though we study SPA in this paper, the equilibrium of the SPA is also an equilibrium of an English auction with two bidders. We can hence interpret the gap between $\nu(x_w, x_w)$ and $b^*(x_w)$ as a jump bid: as the standing bids are raised past $\nu(x_w, x_w)$, a bidder will realize that

her opponent may be financially constrained, and hence she will bid more aggressively if she herself is not yet financially constrained.

Finally, two features of Proposition 1 are worth noting. First, the existence of solutions to ODE (11) does not depend on the hazard function of the budget constraint distribution $\gamma(\cdot)$; second, the sufficient condition for the uniqueness is only a *local* condition: it requires that $\gamma(\cdot)$ be non-increasing in a neighborhood of $\nu(\bar{x}, \bar{x})$. Because it is a local condition on $\gamma(\cdot)$, one can always slightly perturb any budget density distribution $g(\cdot)$ around the neighborhood of $\nu(\bar{x}, \bar{x})$ into a new density $\tilde{g}(\cdot)$ that satisfies the local non-increasing condition.

4.3. Existence of Equilibrium

Proposition 1 establishes the existence of solutions to the first-order condition (9). To prove that the “unconstrained” bidding function identified in (15) constitutes a symmetric equilibrium in the form of (1), we need to further verify that the first-order condition is in fact sufficient for the optimality of problem (4).

We will show below that a sufficient condition that guarantees the sufficiency of the first-order condition for optimality is that $\text{FOC}(z|x_1)$ as defined in (8) is *strictly quasi-monotone* (SQM) in x_1 (see Lizzeri and Persico, 2000, for another application of the SQM condition).

DEFINITION 1. A function $H(y)$ is SQM in y if $H(y) \geq 0$ implies that $H(y') > 0$ for all $y' > y$.

In words, a function $H(y)$ is SQM in y if it crosses zero at most once from below. To guarantee that $\text{FOC}(z|x_1)$ is quasi-monotone in x_1 , we only need to ensure that the term in the brackets in (8), namely

$$[\nu(x_1, z) - b(z)]\lambda(z|x_1) + \gamma(b(z))[\varphi(x_1, z) - b(z)]b'(z), \quad (16)$$

is SQM in x_1 because, for all $z \in (\underline{x}, \bar{x})$, the term $[1 - G(b(z))][1 - F(z|x_1)]$ is strictly positive and SQM is preserved under positive multiplication.⁴ The second term in (16) is strictly increasing in x_1 due to the interdependent value assumption and the fact that $b'(\cdot) > 0$. However, the first term in (16) is not necessarily increasing in x_1 due to two competing forces: on the one hand, $\nu(x_1, z)$ is increasing in x_1 due to affiliation; on the other hand, the affiliation between X_1 and X_2 also implies that the hazard function $\lambda(z|x_1)$

⁴Note that SQM is *weaker* than strict monotonicity, moreover, strict monotonicity is *not* necessarily preserved by positive multiplication.

is non-increasing in x_1 .⁵Therefore we need to make the following sufficient assumption for $\text{FOC}(z|x_1)$ to be strictly quasi-monotone in x_1 .

Assumption 8. $\lambda(z|x_1)\nu(x_1, z)$ is non-decreasing in x_1 for all z .

This assumption is identical to that for Theorem 1 of Krishna and Morgan (1997) in their analysis of war of attrition. It requires that the affiliation between X_1 and X_2 is not so strong that it overwhelms the increase in the expected value of the object, $\nu(\cdot, z)$, resulting from a higher signal x_1 . It is clear that Assumption 8 ensures that $\text{FOC}(z|x_1)$ is SQM in x_1 for all $z \in (\underline{x}, \bar{x})$.

LEMMA 1. *If $\text{FOC}(z|x_1)$ is SQM in x_1 for all $z \in (\underline{x}, \bar{x})$, then the first-order condition (8) is sufficient for optimality.*

Proof. Consider SPA as a revelation mechanism. If an unconstrained bidder 1 of type x_1 reports a type $y < x_1$, then the first-order derivative of her objective with respect to her report, $\text{FOC}(y|x_1)$, will be strictly positive because, if $x_1 > y$, then SQM implies that $\text{FOC}(y|x_1) > 0$ since $\text{FOC}(y|y) = 0$. Similarly if she reports $y > x_1$, then $\text{FOC}(y|x_1) < 0$ since $\text{FOC}(y|y) = 0$. Hence SQM of $\text{FOC}(z|x_1)$ in x_1 entails that $\Pi(x_1, b(\cdot))$ is single peaked at x_1 for every $x_1 \in (x_w, \bar{x})$. Q.E.D.

Using Lemma 1, we can now prove the main result of this paper:

THEOREM 1. *Let $b(x)$ be given by (15). Then under Assumptions 1–8, $B(w, x) = \min\{w, b(x)\}$ is a symmetric equilibrium of the SPA. Moreover, let $\tilde{B}(w, x)$ be any symmetric equilibrium in the form of $\min\{w, \tilde{b}(x)\}$ for some strictly increasing and piecewise continuous function $\tilde{b}(\cdot)$; then $B(w, x) = \tilde{B}(w, x)$ except when $x = x_w$ or \bar{x} .*

Proof. Suppose that bidder 2 is bidding according to (1). If bidder 1's type (x, w) is such that $w > b(x)$, then by Lemma 1, her best response is to bid $b(x)$. Hence it suffices to show that, if bidder 1's type (x, w) is such that $w < b(x)$, her best response is to bid w . To show this, first note that, by the strict monotonicity of $b(\cdot)$, $b(x) > w \geq \underline{w}$ implies that $x > x_w$. Since $b^*(\cdot)$ is monotonically increasing, there exists a unique $z = b^{*-1}(w) < x$.

⁵To see this, let $x' > x, z' > z$; affiliation implies that

$$\frac{f_{X_j|X_i}(z'|x)}{f_{X_j|X_i}(z|x)} \leq \frac{f_{X_j|X_i}(z'|x')}{f_{X_j|X_i}(z|x')}.$$

Integrating over $z' \in (z, +\infty)$ yields

$$\frac{1 - F_{X_j|X_i}(z|x)}{f_{X_j|X_i}(z|x)} \leq \frac{1 - F_{X_j|X_i}(z|x')}{f_{X_j|X_i}(z|x')} \iff \lambda(z|x) \geq \lambda(z|x').$$

Since by Lemma 1 bidder 1's objective function $\Pi(x_1, b(\cdot))$ is single-peaked at x_1 , bidding w is then optimal. Thus her best response is again truth-telling: by reporting that her type is (w, x) , the mediator will submit a bid $\min\{w, b(x)\} = w$ for her.

However, the first-order condition (8) does not uniquely determine the best response of an unconstrained bidder 1 if she receives a signal $x_1 \in \{x_w, \bar{x}\}$. In fact, a bidder of type x_w is indifferent in placing any bid in the interval $[\nu(x_w, x_w), \underline{w}]$ because the probability that her opponent also bids in this interval is zero, which, under the rules of the SPA, implies that neither her probability of winning nor her expected payment depends on how much she bids in this interval. Conversely, her opponent's best response will not be affected when bidder 1 of type x_w changes her bids since she is of measure zero. Analogously, the expected surplus of an unconstrained bidder of type \bar{x} is constant for any bid above $\nu(\bar{x}, \bar{x})$. To see this, note that, by bidding more than $\nu(\bar{x}, \bar{x})$, a type \bar{x} bidder still wins the object with probability 1 and pays the opponent's bid. Conversely, her opponent's best response is not affected because the distribution of bids is not changed when a type \bar{x} bidder changes her bids. Q.E.D.

We now relate our results to the existing literature. As we discussed earlier, when the bidders' valuations are private, then $\nu(x, x) = \varphi(x, x)$ and the first-order condition (9) is satisfied only by $b(x) = \nu(x, x)$, confirming Che and Gale's (1998) results. When the bidders have interdependent values, we have $\nu(x, x) < \varphi(x, x)$; hence the "unconstrained" bid $b(x)$ is higher than $\nu(x, x)$ for all $x > x_w$. The intuition is simple: the presence of financial constraints attenuates the winner's curse. A buyer may win the object even if her competitors have higher signals, thus it is no longer true that the winner is the buyer who has most overestimated the value of object. Consequently, unconstrained bidders bid more aggressively.

4.4. A Linear Example

Here we present a simple linear example to illustrate that unconstrained bidders will bid more aggressively when it is possible that her opponents face financial constraints. For $i = 1, 2$, let X_i be bidder i 's signal, and let the common value of the object be $V = X_1 + X_2$. Suppose that X_1 and X_2 are independent and uniformly distributed on $[0, s]$, where $s > 0$. Let W_i be bidder i 's budget and suppose that W_1 and W_2 are independent and uniformly distributed on $[0, 2s]$. The following can be verified for this example:

$$\begin{aligned} \nu(x, z) &= x + z; & \varphi(x, z) &= x + \frac{s+z}{2}; \\ \lambda(z|x) &= \frac{1}{s-z}; & \gamma(w) &= \frac{1}{2s-w}. \end{aligned}$$

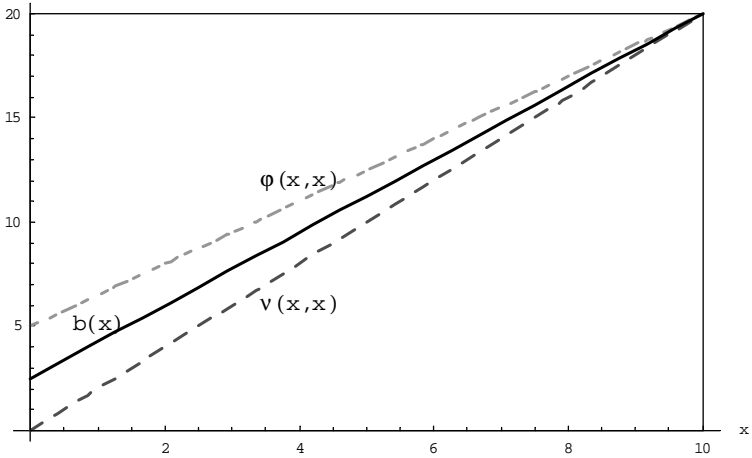


FIG. 4. The functions $b(x)$, $\nu(x, x)$ and $\varphi(x, x)$ in the linear example with $s = 10$.

Plugging these expressions into (11) we obtain

$$b'(x) = \frac{[2s - b(x)][b(x) - 2x]}{(s - x)[(3x + s)/2 - b(x)]};$$

together with the boundary condition that $b(s) = \nu(s, s) = \varphi(s, s) = 2s$, we obtain the unique solution

$$b(x) = \frac{7x}{4} + \frac{s}{4}.$$

The three curves, $\nu(x, x)$, $\varphi(x, x)$, and $b(x)$ are plotted in Fig. 4 for $s = 10$.

5. COMPARATIVE STATICS

In this section, we provide the comparative statics results of the equilibrium “unconstrained” bidding function with respect to public signals and the severity of the financial constraints.

5.1. Equilibrium with Public Signals

Suppose that, prior to the bidding, both bidders publicly observed a signal X_0 , which is a component of S . For notational ease, however, we write X_0 separately from S . The signal X_0 could be, for example, a signal publicly revealed by the seller. Bidders can now condition their bids both on their private signals and the public signal x_0 . We will write $b(x; x_0)$ as the “unconstrained” bid of a bidder with a private signal x and a public signal x_0 .

It is important to note that since x_0 is publicly observed, we can essentially treat x_0 as a parameter of the model. Analogous to the case without public signals, the “unconstrained” bidding function $b(\cdot; x_0)$ must satisfy the following differential equation if $b(x; x_0) > \underline{w}$,

$$b'(x; x_0) = \frac{\lambda(x|x; x_0) [b(x; x_0) - \nu(x, x; x_0)]}{\gamma(b(x; x_0)) [\varphi(x, x; x_0) - b(x; x_0)]}, \quad (17)$$

where

$$\varphi(x, z; x_0) = \mathbb{E}[u(S, X_1, X_2, X_0) | X_1 = x_1, X_2 \geq z, X_0 = x_0]$$

$$\nu(x, z; x_0) = \mathbb{E}[u(S, X_1, X_2, X_0) | X_1 = x_1, X_2 = z, X_0 = x_0]$$

$$\lambda(z|x; x_0) = \frac{f_{X_2|(X_1, X_0)}(z|x, x_0)}{1 - F_{X_2|(X_1, X_0)}(z|x, x_0)}.$$

Under conditions analogous to those in Theorem 1, we can prove the existence of a symmetric equilibrium. The proof is omitted. The following proposition shows how the public signal affects the “unconstrained” bidding function $b(\cdot)$.

PROPOSITION 2. *If the bidders have interdependent values and the signals are strictly affiliated, then the “unconstrained” bidding function $b(x; x_0)$ is increasing in the public signal x_0 for all $x \in (\underline{x}, \bar{x})$.*

Proof. Let $x_0^l < x_0^h$ be two public signals. The boundary condition requires that $b(\bar{x}; x_0^l) = \nu(\bar{x}; x_0^l) = \varphi(\bar{x}; x_0^l)$ and $b(\bar{x}; x_0^h) = \nu(\bar{x}; x_0^h) = \varphi(\bar{x}; x_0^h)$. By the interdependent values assumption, we have $b(\bar{x}; x_0^l) < b(\bar{x}; x_0^h)$. We want to show that $b(x; x_0^l) < b(x; x_0^h)$ for all $x \in (\underline{x}, \bar{x})$. Suppose that this is not so. Then the boundary condition implies that there exists some $\tilde{x} \in (\underline{x}, \bar{x})$, $b(\tilde{x}; x_0^l) = b(\tilde{x}; x_0^h)$. But inspection of ODE (17) reveals that $b'(\tilde{x}; x_0^l) > b'(\tilde{x}; x_0^h)$ since the strict affiliated signals assumption implies that $\lambda(x|x; x_0^l) > \lambda(x|x; x_0^h)$ and the interdependent values assumption implies that $\nu(x, x; x_0^h) > \nu(x, x; x_0^l)$ and $\varphi(x, x; x_0^h) > \varphi(x, x; x_0^l)$ for all $x \in (\underline{x}, \bar{x})$, which, in turn, implies that $b(x; x_0^l)$ can only intersect $b(x; x_0^h)$ from below. This subsequently implies that in the interval (\tilde{x}, \bar{x}) , $b(x; x_0^l)$ must always lie above $b(x; x_0^h)$, which then implies that $b(\bar{x}; x_0^l) \geq b(\bar{x}; x_0^h)$, a contradiction to the boundary condition. Q.E.D.

Proposition 2 is not a surprising result: the release of good news will intuitively make the unconstrained bidders bid more aggressively. The proof, however, is less trivial because we lack an explicit solution to differential equation (17).

5.2. The Severity of the Financial Constraints

Consider a parametric family of financial constraint distributions $G(\cdot; \theta)$ with $g(\cdot; \theta)$ being the corresponding probability density function. The parametric family of hazard function of the budget function, $\gamma(w; \theta)$, is denoted by

$$\gamma(w; \theta) = \frac{g(w; \theta)}{1 - G(w; \theta)}.$$

Suppose that the parametric family of hazard functions satisfies the *increasing hazard rate condition* (IHR): if $\theta_1 > \theta_2$, then $\gamma(w; \theta_1) > \gamma(w; \theta_2)$ for all w .⁶ It can be shown that IHR implies first-order stochastic dominance: if $\theta_1 > \theta_2$, then $G(w; \theta_2) \leq G(w; \theta_1)$ for all w . In other words, when the value of θ is higher, it is more likely that one's opponent is financially constrained. This means that an unconstrained bidder is more likely to win the object because of his or her opponent having a lower budget instead of having a lower signal. This attenuates the winner's curse and makes the "unconstrained" bidders bid more aggressively. Let $b^*(\cdot; \theta)$ denote the solution to ODE (11) when $G(\cdot; \theta)$ is the cumulative distribution function of the budget constraints.

PROPOSITION 3. *Suppose that IHR holds and that the bidders have interdependent values. Then $b^*(x; \theta_1) > b^*(x; \theta_2)$ for all x if $\theta_1 > \theta_2$.*

Proof. Suppose that the contrary is true. Then either $b^*(\tilde{x}; \theta_1) = b^*(\tilde{x}; \theta_2)$ for some $\tilde{x} < \bar{x}$, or $b^*(x; \theta_2) > b^*(x; \theta_1)$ for all $x < \bar{x}$ and $b^*(\bar{x}; \theta_1) = b^*(\bar{x}; \theta_2)$. We now show that neither is consistent with equilibrium.

Case I. If $b^*(\tilde{x}; \theta_1) = b^*(\tilde{x}; \theta_2)$ for some $\tilde{x} < \bar{x}$, then inspection of ODE (11) reveals that under IHR, $b^*(\tilde{x}; \theta_1) < b^*(\tilde{x}; \theta_2)$ must hold. Thus if $b^*(\cdot; \theta_2)$ and $b^*(\cdot; \theta_1)$ crosses $\tilde{x} < \bar{x}$ at some point, $b^*(\cdot; \theta_2)$ must cross $b^*(\cdot; \theta_1)$ from below, which implies that they can only cross once. However, the boundary condition (13) requires that $b^*(\bar{x}; \theta_1) = b^*(\bar{x}; \theta_2)$, a contradiction.

Case II. If $b^*(x; \theta_2) > b^*(x; \theta_1)$ for all $x < \bar{x}$ and $b^*(\bar{x}; \theta_1) = b^*(\bar{x}; \theta_2)$, we will rule out this possibility as follows: we show that if $b^*(\bar{x}; \theta_1) = b^*(\bar{x}; \theta_2)$, then there exists a neighborhood of \bar{x} in which

⁶Note that IHR is implied by the following condition: If $\theta_1 > \theta_2$ and $w_1 > w_2$, then $g(w_1; \theta_2)/g(w_2; \theta_2) > g(w_1; \theta_1)/g(w_2; \theta_1)$.

$b^*(x; \theta_1) > b^*(x; \theta_2)$. To this end, we first show the following lemma:

LEMMA 2. *Let $b^*(\cdot)$ be an equilibrium “unconstrained” bidding function that satisfies the boundary condition (13). Then the left derivative of $b^*(\cdot)$ at \bar{x} exists and is given by*

$$b^{*\prime}(\bar{x}) = \lim_{x \uparrow \bar{x}} \frac{b^*(\bar{x}) - b^*(x)}{\bar{x} - x} = \frac{d\nu}{dx}(\bar{x}, \bar{x}).$$

Proof. If $b^*(\cdot)$ is an equilibrium “unconstrained” bidding function, then $\nu(x, x) \leq b^*(x) \leq \varphi(x, x)$ for all $x < \bar{x}$. Together with the boundary condition (13) we have that, for all $x < \bar{x}$,

$$\frac{\varphi(\bar{x}, \bar{x}) - \varphi(x, x)}{\bar{x} - x} \leq \frac{b^*(\bar{x}, \bar{x}) - b^*(x, x)}{\bar{x} - x} \leq \frac{\nu(\bar{x}, \bar{x}) - \nu(x, x)}{\bar{x} - x}.$$

Let $\{x_n\}$ with $x_n < \bar{x}$ for all n be a sequence that converges to \bar{x} . We have then

$$\frac{d\varphi}{dx}(\bar{x}, \bar{x}) \leq \liminf_n \frac{b^*(\bar{x}) - b^*(x_n)}{\bar{x} - x_n} \leq \limsup_n \frac{b^*(\bar{x}) - b^*(x_n)}{\bar{x} - x_n} \leq \frac{d\nu}{dx}(\bar{x}, \bar{x}).$$

Now note that from ODE (11) we have

$$b^{*\prime}(x_n) = \frac{f(x_n|x_n)}{[1 - F(x_n|x_n)]\gamma(b^*(x_n))} \frac{b^*(x_n) - \nu(x_n, x_n)}{\varphi(x_n, x_n) - b^*(x_n)},$$

which can be rewritten as

$$b^{*\prime}(x_n)[1 - F(x_n|x_n)] = \frac{f(x_n|x_n)}{\gamma(b^*(x_n))} \frac{\frac{b^*(x_n) - b^*(\bar{x})}{\bar{x} - x_n} + \frac{\nu(\bar{x}, \bar{x}) - \nu(x_n, x_n)}{\bar{x} - x_n}}{\frac{\varphi(x_n, x_n) - \varphi(\bar{x}, \bar{x})}{\bar{x} - x_n} + \frac{b^*(\bar{x}) - b^*(x_n)}{\bar{x} - x_n}}. \quad (18)$$

Consider subsequences $\{x_{n_i}\}$ and $\{x_{n_j}\}$ along which the \liminf and \limsup of $[b^*(\bar{x}) - b^*(x_n)]/(\bar{x} - x_n)$, respectively, are obtained. Computing the limits of expression (18) along the subsequences $\{x_{n_i}\}$ and $\{x_{n_j}\}$, respectively, yields

$$\begin{aligned} 0 &= \frac{f(\bar{x}|\bar{x})}{\gamma(b^*(\bar{x}))} \left(\frac{d\nu}{dx}(\bar{x}, \bar{x}) - \liminf_n \frac{b^*(\bar{x}) - b^*(x_n)}{\bar{x} - x_n} \right) \\ 0 &= \frac{f(\bar{x}|\bar{x})}{\gamma(b^*(\bar{x}))} \left(\frac{d\nu}{dx}(\bar{x}, \bar{x}) - \limsup_n \frac{b^*(\bar{x}) - b^*(x_n)}{\bar{x} - x_n} \right), \end{aligned}$$

where the limit of the left-hand side of expression (18) is zero because $b^{*\prime}(x_n)$ must be bounded if x_n is close enough to \bar{x} and $1 - F(x_n|x_n)$ converges to zero. To see that $b^{*\prime}(x_n)$ is bounded in a neighborhood of \bar{x} , note that $b^*(x_n)$ lies between $\nu(x_n, x_n)$ and $\varphi(x_n, x_n)$, but they must meet at \bar{x} ;

hence in a neighborhood of \bar{x} , $b^{*'}(x_n)$ must be higher than $dv(x_n, x_n)/dx$ and lower than $d\varphi(x_n, x_n)/dx$. Therefore we have

$$\liminf_n \frac{b^*(\bar{x}) - b^*(x_n)}{\bar{x} - x_n} = \limsup_n \frac{b^*(\bar{x}) - b^*(x_n)}{\bar{x} - x_n} = \frac{dv}{dx}(\bar{x}, \bar{x});$$

hence,

$$b^{*'}(\bar{x}) = \lim_{x \uparrow \bar{x}} \frac{b^*(\bar{x}) - b^*(x)}{\bar{x} - x} = \frac{dv}{dx}(\bar{x}, \bar{x}).$$

Since the above lemma holds regardless of the distribution of the budget constraint, we know that

$$b^{*'}(\bar{x}; \theta_1) = b^{*'}(\bar{x}; \theta_2) = \frac{dv}{dx}(\bar{x}, \bar{x}).$$

For any $\tilde{x} < \bar{x}$, we know from ODE (11) that

$$\frac{b^{*'}(\tilde{x}; \theta_1)}{b^{*'}(\tilde{x}; \theta_2)} = \frac{\gamma(b^*(\tilde{x}; \theta_2); \theta_2) [b^*(\tilde{x}; \theta_1) - \nu(\tilde{x}, \tilde{x})] [\varphi(\tilde{x}, \tilde{x}) - b^*(\tilde{x}; \theta_2)]}{\gamma(b^*(\tilde{x}; \theta_1); \theta_1) [\varphi(\tilde{x}, \tilde{x}) - b^*(\tilde{x}; \theta_1)] [b^*(\tilde{x}; \theta_2) - \nu(\tilde{x}, \tilde{x})]}.$$

Taking limits as $\tilde{x} \uparrow \bar{x}$, we obtain

$$\begin{aligned} 1 &= \lim_{\tilde{x} \uparrow \bar{x}} \frac{b^{*'}(\tilde{x}; \theta_1)}{b^{*'}(\tilde{x}; \theta_2)} = \frac{\gamma(\nu(\bar{x}, \bar{x}); \theta_2)}{\gamma(\nu(\bar{x}, \bar{x}); \theta_1)} \\ &\quad \times \lim_{\tilde{x} \uparrow \bar{x}} \frac{[b^*(\tilde{x}; \theta_1) - \nu(\tilde{x}, \tilde{x})] [b^*(\tilde{x}; \theta_2) - \nu(\tilde{x}, \tilde{x})]}{[\varphi(\tilde{x}, \tilde{x}) - b^*(\tilde{x}; \theta_1)] [\varphi(\tilde{x}, \tilde{x}) - b^*(\tilde{x}; \theta_2)]}. \end{aligned}$$

Since IHR implies that

$$\frac{\gamma(\nu(\bar{x}, \bar{x}); \theta_2)}{\gamma(\nu(\bar{x}, \bar{x}); \theta_1)} < 1,$$

we must then have

$$\lim_{\tilde{x} \uparrow \bar{x}} \frac{[b^*(\tilde{x}; \theta_1) - \nu(\tilde{x}, \tilde{x})] [b^*(\tilde{x}; \theta_2) - \nu(\tilde{x}, \tilde{x})]}{[\varphi(\tilde{x}, \tilde{x}) - b^*(\tilde{x}; \theta_1)] [\varphi(\tilde{x}, \tilde{x}) - b^*(\tilde{x}; \theta_2)]} > 1,$$

which in turn implies that $b^*(\tilde{x}; \theta_1) > b^*(\tilde{x}; \theta_2)$ for \tilde{x} within a neighborhood of \bar{x} , a contradiction to $b^*(x; \theta_2) > b^*(x; \theta_1)$ for all $x < \bar{x}$. Q.E.D.

6. DISCUSSIONS

In this paper, we study an affiliated value SPA with two financially constrained bidders. We prove the existence of a symmetric equilibrium under quite general conditions. Comparative static results with respect to the release of public information and the severity of the budget constraints are provided. This paper serves as a step toward a better understanding of the interaction among bidding constraints, information revelation, and the winner's curse in auctions with affiliated values.

6.1. *The Possible Failure of the Linkage Principle*

The analysis of the equilibrium of the affiliated value auction with financially constrained bidders conducted in this paper allows one to study other interesting questions such as the linkage principle. In a companion article, Fang and Parreiras (2000), we provide an explicit example of a common value auction model with financially constrained bidders for which we analytically solve the differential equation (11). We then demonstrate that the linkage principle uncovered by Milgrom and Weber (1982) may not hold in auctions with financially constrained bidders. The intuition for the failure of the linkage principle with financially constrained bidders is that, in the presence of financial constraints, the extent of bidders' upward response to seller's good signals is limited by the financial constraints, while their downward response to seller's bad signals is not. The simplest example that demonstrates the bidders' asymmetric responses to good and bad news released by the seller is as follows. Suppose that two bidders compete for an object. Each bidder has a bidding budget of $3/4$. The common valuation is either 0 or 1. The common prior of the bidders and the seller is that 0 and 1 occur with equal probability. Bidders do not receive any private signal before they bid, while the seller will receive a signal that tells her the true value of the object. If the seller commits to a concealing policy, the bidders will both bid $1/2$ for the object. If the seller commits to a revealing policy, then the bidders will both bid 0 if the true value is 0 and $3/4$ if the true value is 1. Hence the seller's expected revenue under the concealing policy is $1/2$, while that under the revealing policy is $3/4 \times 1/2 = 3/8$, which is smaller. The reason is clear: when the low value 0 is revealed, the bidders both drop their bids unfettered by the budget, but when the high value 1 is revealed, the bidders can only increase their bids up to the bidding budget.

6.2. *Relation to Che and Gale (1998)*

It is interesting to relate our equilibrium characterization to Che and Gale's (1998) characterization of the first price IPV auction with financially constrained bidders. Their Lemma 1 states that under a technical condition there exists a *unique*, symmetric equilibrium in the form $B(w, x) = \min\{w, b(x)\}$ for some *continuous* and strictly increasing function $b(\cdot)$.⁷

Recall that in our characterization, the "unconstrained" bidding function $b(\cdot)$ could have a jump. The reason is that, in the first price auction, if

⁷However, in Fang and Parreiras (2001), we show, by way of a counterexample, that there is an important oversight in Che and Gale's analysis. Therefore the complete equilibrium characterization of the first price IPV auction with financially constrained bidders is still an open question.

there is a jump at, say x^* , such that $\lim_{x \uparrow x^*} b(x) = b^- < \lim_{x \downarrow x^*} b(x) = b^+$, then there must be a positive measure of bidders who submit bids in the interval (b^-, b^+) because otherwise the bidder who submits b^+ is not optimizing under the first price auction. However, this implication does not hold in a second price auction: a bidder could be optimizing with a bid b^+ when there is a zero measure of opponents bidding in (b^-, b^+) because her expected payment in the event of winning is determined solely by the distribution of other bidders' bids. This explains why in the second price auction the unconstrained bidding function $b(\cdot)$ could be discontinuous.

6.3. Extensions

A few extensions are worth pursuing. First, when there are more than two bidders, under what conditions can we obtain a similar symmetric equilibrium? Our preliminary analysis of the general n bidder case suggests that the basic features of two-bidder case may remain, even though the analysis becomes significantly more involved. Second, it is interesting to study the English and the first price affiliated value auctions with financially constrained bidders and to study how the revenue ranking of these auction mechanisms is altered by the presence of financial constraints. Third, what is the effect of seller financing on the seller's expected revenue in the affiliated value setting?

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