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Notes, Comments, and Letters to the Editor

## On the failure of the linkage principle with financially constrained bidders

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### Abstract

This paper provides a class of examples of two-bidder common value second price auctions in which bidders may be financially constrained and the seller has access to information about the common value. We show that the seller's expected revenue under a revelation policy may be lower than that under a concealing policy. The intuition for the failure of the linkage principle is as follows. In the presence of financial constraints, the bidders' upward response in their bids to the seller's good signals is limited by their financial constraints, while their downward response to bad signals is not.

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### 1. Introduction

One of the fundamental lessons from auction theory is the *linkage principle* which, loosely, states that the seller's expected revenue will be greater if she can undermine the privacy of the winning bidder's information.<sup>1</sup> This principle underlies the well-known results that a seller will prefer to conduct an ascending-bid auction rather

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<sup>1</sup>The linkage principle is first uncovered in Milgrom and Weber [9], also see Milgrom [8] for an excellent exposition.

than a sealed-bid auction, and for a given auction format she will prefer to publicly reveal her information to the bidders. Some assumptions are recognized to be crucial for the linkage principle to hold, such as risk neutrality, symmetric uncertainty about the bidders' valuations, and affiliation among the bidders' and the seller's signals.<sup>2</sup> In this paper, we show that for the linkage principle to hold it is also important that the bidders are not financially constrained. The importance of buyers' financial constraints in auctions has been demonstrated by a large body of empirical and anecdotal evidence.<sup>3</sup> For example, financial constraints are used by Cramton [4] to explain some bidders' exit decisions in the PCS auctions, and by Genesove [6] to explain the end-of-day drop in prices at used car auctions. Recently, governments in many parts of the world have aggressively sought to privatize once socially held assets. Given the magnitude of these privatization sales, it is often realistic to assume that buyers may run up against financial constraints. However, in practice, when called upon to choose an auction mechanism, one is often led by the faith placed in the linkage principle to employ an open auction format, even in scenarios where the bidders are most likely financially constrained.

The idea behind the failure of the linkage principle in common value auctions with financially constrained bidders can be conveyed by a simple example. Suppose that two bidders compete for an object in a first or a second price auction. Each bidder has a bidding budget of  $\frac{3}{4}$ . The common valuation is either 0 or 1. The common prior of the bidders and the seller is that 0 and 1 occur with equal probability. Bidders do not receive any private signal before they bid, while the seller will receive a signal that tells her the true value of the object. If the seller commits to a concealing policy, the bidders will both bid  $\frac{1}{2}$  for the object. If the seller commits to a revealing policy, then the bidders will both bid 0 if the true value is 0 and  $\frac{3}{4}$  if the true value is 1. Hence, the seller's expected revenue under the concealing policy is  $\frac{1}{2}$ , while that under the revealing policy is  $\frac{3}{4} \times \frac{1}{2} = \frac{3}{8}$ , which is smaller. The reason is as follows: when the low value 0 is revealed, the bidders both drop their bids unfettered by the budget; but when the high value 1 is revealed, the bidders can only increase their bids up to the bidding budget.

While this example demonstrates the intuition for the possible failure of the linkage principle, it is not rich enough to address some interesting questions. First, will the linkage principle uncovered by Milgrom and Weber [9] in the classical auction environment with no financial constraints still hold if the bidders' financial constraints are not very severe? Second, when bidders are financially constrained, how is the seller's expected revenue under a revealing policy affected by the accuracy of her signal? Will a seller be better off by revealing garbled signals (i.e., signals with less precision)? To this end, we consider a class of examples where budget constraints are determined by a family of Pareto distributions, and the seller receives signals

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<sup>2</sup>Ottaviani and Prat [10] extend the logic of the linkage principle to monopolistic price discrimination.

<sup>3</sup>In a seminal paper, Che and Gale [2] show that first-price auctions yield higher expected revenue and social surplus than second-price auctions when bidders are financially constrained. Also see Che and Gale [3] for more evidence of financial constraints in auctions.

about the value of the object with different accuracy. Importantly, when the parameter of the Pareto distribution converges to zero, our example converges to the classical auctions with no budget constraints; as we increase that parameter, the probability that a bidder's budget constraint binds increases. We compare the seller's expected revenue under a revealing and a concealing policy. We show that, indeed, the linkage principle still holds when the probability of the bidders' being financially constrained is small, and it fails when the budget constraints are severe enough. The intuition is quite simple. When the budget constraints are not severe, the increased competition between the bidders due to the released information on average outweighs the increase in the likelihood that the budget constraints bind, and hence the linkage principle still holds. On the other hand, when the financial constraints are severe, the extent to which a high signal revealed by the seller drives up the bids is now limited by the bidders' bidding constraints, but the extent to which a low signal drives down the bids is not limited. It is this asymmetry in the bidders' response toward good and bad news in the presence of budget constraints that causes the linkage principle to fail. We also show that the seller's expected revenue is non-monotonic in the accuracy of her signal.

## 2. The environment

Consider a common value auction with two bidders denoted by  $i = 1, 2$ . The common value of the object,  $S$ , is distributed according to an inverted Gamma distribution with support  $(0, +\infty)$  whose p.d.f. is<sup>4</sup>

$$f_S(s) = \frac{\beta^\alpha e^{-\beta/s}}{s^{\alpha+1} \Gamma(\alpha)}, \quad 0 < s < +\infty, \quad \text{where } \alpha > 1 \text{ and } \beta > 0.$$

The unconditional expected value of the object is  $ES = \beta/(\alpha - 1)$ .

Buyer  $i$  receives a signal  $X_i$ . We assume that conditional on the value of the object  $S$ , signals  $X_1$  and  $X_2$  are independently drawn from the same exponential distribution with parameter  $1/S$  and have support  $[0, +\infty)$ . That is, the p.d.f. of  $X_i$  conditional on  $s$ , is

$$f_{X_i|S}(x_i|s) = \frac{1}{s} e^{-x_i/s}, \quad 0 \leq x_i < +\infty, \quad \text{for } i = 1, 2.$$

Note that the buyer's signal is an unbiased estimate of the value of the object since  $E[X_i|S] = S$ .

The seller receives a signal  $X_0$  which, conditional on  $S = s$ , is independent of any other signals and follows a Gamma distribution with parameters  $k > 0$  and  $k/s$ . That is, the support of signal  $X_0$  is  $[0, +\infty)$  and its p.d.f. is

$$f_{X_0|S}(x_0|s) = \left(\frac{k}{s}\right)^k x_0^{k-1} \frac{\exp\left(-\frac{kx_0}{s}\right)}{\Gamma(k)}, \quad 0 \leq x_0 < +\infty.$$

<sup>4</sup>A random variable  $S$  has an inverted Gamma distribution if  $1/S$  has a Gamma distribution (see [1, p. 50]).

Notice that the seller's signal is also an unbiased estimate of the common value since  $E[X_0|S] = S$ .<sup>5</sup> The parameter  $k$  can be interpreted as a measure of the *precision* of the seller's signal because  $\text{Var}[X_0|S] = S^2/k$ .<sup>6</sup> It is important to note that as  $k \rightarrow 0$ , the seller's signal becomes completely uninformative. This allows us to interpret the concealing policy by the seller as a special case of a revealing policy except that the seller's signal is completely uninformative.

Assume that each bidder gets a draw of her available bidding budget,  $W$ , from a Pareto distribution with parameters  $w > 0$  and  $\theta > 0$ . That is, the bidding budget  $W$  has a support  $(w, +\infty)$  and its the c.d.f. is

$$G(w) = 1 - (w/w)^{\theta} \quad \text{if } w > w,$$

and we denote  $g(w)$  as its corresponding p.d.f. The parameter  $\theta$  measures the *severity* of budget constraint: the smaller  $\theta$  is, the less likely that a bidder may be financially constrained. As  $\theta \rightarrow 0$ , we will in the limit obtain the standard auction models without financial constraints. It can be verified that the hazard function of the budget distribution is  $\gamma(w) = g(w)/[1 - G(w)] = \theta/w$  if  $w > w$ , and 0 otherwise. For reasons to be explained after Proposition 1, we assume:

**Assumption 1.**  $0 < w < \beta/(\alpha + k + 1)$ .

The auction format employed by the seller is the Vickery auction. Each bidder submits a bid. The bidder who submits the higher bid wins the object and pays the low bid. Following Che and Gale [2], we assume that if a bidder bids more than his budget he will not win the object and will have to pay a small fine. Therefore, the strategy of bidding above one's budget is strictly dominated.

### 3. Equilibrium

Suppose that the seller commits to a policy of publicly revealing her signal to the bidders. We will focus on symmetric equilibrium in the following form: when the seller reveals a signal  $x_0$ , bidder  $i$  of budget  $w_i$  and signal  $x_i$  bids

$$B^k(x_i, w_i; x_0) = \min\{w_i, b^k(x_i; x_0)\}, \quad i = 1, 2, \quad (1)$$

where the superscript  $k$  indexes the precision of the seller's signal and  $b^k(\cdot; x_0)$  is a strictly increasing and piecewise differentiable function for all  $x_0$ . The term  $b^k(x_i; x_0)$  can be interpreted as the bid, when the seller reveals a signal  $x_0$  with precision  $k$ , of a

<sup>5</sup>In pure common value auctions with conditionally independent signals, the assumption that the bidders' and the seller's signals are unbiased estimates of the value of the object is without loss of generality since we can always transform any signals to satisfy this feature. We thank an Associate Editor for this observation.

<sup>6</sup>Alternatively, we can interpret the seller's signal as the mean of a random sample of  $k$  signals that are independently drawn from the same exponential distribution as that of the bidders. (It is easy to show that the sample mean of  $k$  independent exponential distribution with parameter  $1/s$  will have a Gamma distribution with  $\alpha = k$  and  $\beta = k/s$ .) Under this interpretation,  $k$  only takes positive integer values.

bidder with signal  $x_i$ , who is not herself financially constrained but understands that her opponents may be constrained. We will refer to  $b^k(\cdot; x_0)$  as the “unconstrained” bidding function.

In Appendix A, we show that whenever  $b^k(x; x_0) \geq w$  (which we verify below), the “unconstrained” bidding function  $b^k(\cdot; x_0)$  must satisfy the following differential equation:

$$b^k(x; x_0) = \frac{\lambda(x|x, x_0) [b^k(x; x_0) - v(x, x; x_0)]}{\gamma(b(x; x_0)) [\varphi(x, x; x_0) - b^k(x; x_0)]}, \tag{2}$$

where<sup>7</sup>

$$v(x, x; x_0) = E[S|X_1 = x, X_2 = x; X_0 = x_0] = \frac{2x + \beta + kx_0}{\alpha + k + 1}, \tag{3}$$

$$\varphi(x, x; x_0) = E[S|X_1 = x, X_2 \geq x; X_0 = x_0] = \frac{2x + \beta + kx_0}{\alpha + k}, \tag{4}$$

$$\lambda(x|x; x_0) = \frac{f_{X_i|(X_j, X_0)}(x|X_j = x; X_0 = x_0)}{1 - F_{X_i|(X_j, X_0)}(x|X_j = x; X_0 = x_0)} = \frac{\alpha + k + 1}{2x + \beta + kx_0},$$

where, as throughout the paper, we use  $F_{\bullet, \bullet}$  to denote the conditional c.d.f. of the signals. After substituting these expressions into Eq. (2) we obtain the following differential equation:

$$b^k(x; x_0) = \frac{\alpha + k + 1}{2x + \beta + kx_0} \frac{b^k(x; x_0)}{\theta} \left[ \frac{b^k(x; x_0) - (2x + \beta + kx_0)/(\alpha + k + 1)}{(2x + \beta + kx_0)/(\alpha + k) - b^k(x; x_0)} \right]. \tag{5}$$

Under the condition that  $b^k(\cdot; x_0)$  is positive for all  $x_0$ , we establish in Appendix B the following proposition:

**Proposition 1.** *The unique increasing solution to Eq. (5) is given by*

$$b^k(x; x_0) = \frac{(2\theta + \alpha + k)}{(2\theta + \alpha + k + 1)} \frac{(2x + \beta + kx_0)}{(\alpha + k)}. \tag{6}$$

Note that under Assumption 1, we have that

$$b^k(0; 0) = \frac{(2\theta + \alpha + k)}{(2\theta + \alpha + k + 1)} \frac{\beta}{(\alpha + k)} \geq \frac{\beta}{\alpha + k + 1} > w.$$

Since  $b^k(\cdot; \cdot)$  is increasing in both  $x$  and  $x_0$ , we indeed have  $b^k(x; x_0) \geq w$  for all  $x$  and  $x_0$ . Therefore, the equilibrium “unconstrained” bidding function  $b^k(\cdot; \cdot)$  is completely characterized by the solution to the ODE (5). As proved in Fang and Parreiras [5], a sufficient condition to guarantee that  $b^k(x; x_0)$  in (6) solves the maximization problem in Appendix A is that  $\lambda(z|x; x_0)v(x, z; x_0)$  is non-decreasing in  $x$  for all  $z$  and  $x_0$ , which is satisfied here because  $\lambda(z|x; x_0)v(x, z; x_0) = 1$  is a constant.

<sup>7</sup>The verification of these formulae is available from the authors upon request.

The equilibrium “unconstrained” bidding function (6) has the following intuitive properties:

- The coefficient of  $x$ ,  $2(2\theta + \alpha + k)/[(2\theta + \alpha + k + 1)(\alpha + k)]$ , is decreasing in  $k$ . That is, an unconstrained bidder becomes less sensitive to her own signal  $x$  as the seller’s signal becomes more precise.
- The coefficient of  $x_0$ ,  $k(2\theta + \alpha + k)/[(2\theta + \alpha + k + 1)(\alpha + k)]$ , is increasing in  $k$ . That is, the “unconstrained” bidding function becomes more sensitive to the seller’s signal as it becomes more precise.
- The coefficient of  $x_0$ ,  $k(2\theta + \alpha + k)/[(2\theta + \alpha + k + 1)(\alpha + k)]$ , is increasing in  $\theta$ . That is, an unconstrained bidder becomes more responsive to the seller’s signal as the opponent becomes more likely to be financially constrained. This effect is important because it implies that more severe budget constraints will entail more dramatic asymmetric responses of the bidders’ bids to favorable and unfavorable signals released by the seller. The asymmetric response effect, as we argue in Section 5, is what drives the failure of the linkage principle as  $\theta$  gets large.

Now we analyze the equilibrium of the auction when the seller follows a concealing policy. We focus on the symmetric equilibrium in which a bidder with budget  $w_i$  and signal  $x_i$  bids

$$B(w_i, x_i) = \min\{w_i, b(x_i)\}, \quad i = 1, 2,$$

where  $b(\cdot)$  is a strictly increasing and piecewise differentiable function. It can be shown that in equilibrium,

$$b(x_i) = \lim_{k \rightarrow 0} b^k(x_i; x_0) = \frac{(2\theta + \alpha)}{(2\theta + \alpha + 1)} \frac{(2x + \beta)}{\alpha}. \tag{7}$$

This is intuitive because, as we pointed out earlier, a concealing policy by the seller is equivalent to a revealing policy when her signal is completely uninformative, i.e., as  $k \rightarrow 0$ . The “unconstrained” bidding function under the concealing policy,  $b(\cdot)$ , above has the following intuitive properties:

- The coefficient of  $x$ ,  $2(2\theta + \alpha)/(2\theta + \alpha + 1)\alpha$  decreases in  $\alpha$  and increases in  $\theta$ . It decreases in  $\alpha$  because the expected value of the object is decreasing in  $\alpha$ ; and it increases in  $\theta$  because a higher  $\theta$  implies a higher probability that the opponent is financially constrained, and hence a less severe winner’s curse.
- As  $\theta \rightarrow 0$ ,  $b(x) \rightarrow (2x + \beta)/(\alpha + 1) = E[S|X_1 = x, X_2 = x]$ , which is the equilibrium bid obtained in Milgrom and Weber [9].

#### 4. Seller’s expected revenues and the linkage principle

Under the revealing policy, the price obtained by the seller, denoted by  $P^k$ , is a random variable given by

$$P^k = \min\{b^k(X_1; X_0), b^k(X_2; X_0), W_1, W_2\}.$$

We write  $M_N$  and  $M_1^k$ , respectively, as the seller’s expected revenue under a concealing and a revealing policy when the precision of her signal  $X_0$  is  $k$ , i.e.,  $M_1^k = E[P^k]$ ,  $M_N = \lim_{k \rightarrow 0} E[P^k]$ , where the latter equality follows from Eq. (7). We show in Lemma B.1 in Appendix C that

$$M_N = \frac{w}{2\theta - 1} \left\{ 2\theta - \frac{\alpha}{2\theta + \alpha - 1} \left[ \frac{\alpha(2\theta + \alpha + 1)w}{(2\theta + \alpha)\beta} \right]^{2\theta - 1} \right\}, \tag{8}$$

$$M_1^k = \frac{w}{2\theta - 1} \left\{ 2\theta - \frac{\Gamma(\alpha + k + 1)}{\Gamma(\alpha)} \frac{\Gamma(2\theta + \alpha - 1)}{\Gamma(2\theta + \alpha + k)} \times \left[ \frac{(\alpha + k)(2\theta + \alpha + k + 1)w}{(2\theta + \alpha + k)\beta} \right]^{2\theta - 1} \right\}. \tag{9}$$

Note that  $M_N \rightarrow \alpha\beta/[(\alpha - 1)(\alpha + 1)]$  and  $M_1^k \rightarrow (\alpha + k)\beta/[(\alpha - 1)(\alpha + k + 1)]$ , hence  $M_N < M_1^k$  as  $\theta \rightarrow 0$ . This is of course the linkage principle in common value auctions with no financial constraints. However, the following proposition shows that the linkage principle may fail when  $\theta$  is sufficiently large.

**Proposition 2.** *For any  $k > 0$ , there exist thresholds  $\underline{\theta}$  and  $\bar{\theta}$  with  $\underline{\theta} \leq \bar{\theta}$  such that  $M_1^k > M_N$  if  $\theta < \underline{\theta}$  and  $M_1^k < M_N$  if  $\theta > \bar{\theta}$ .*

Fig. 1 depicts the relationship between  $M_N - M_1^k$  and  $\theta$  in a numerical example in which we set  $\alpha = 2$ ,  $\beta = 6$ ,  $k = 2$ ,  $w = 0.2$ . As we increase  $\theta$ ,  $M_N - M_1^k$  changes from negative to positive. Moreover, as  $\theta$  goes to infinity, the difference converges to zero. This occurs because when  $\theta$  becomes large the budget constraints become so severe that the seller’s expected revenue approaches  $w$  under both the revelation and concealing policies. While in this numerical example,  $M_N - M_1^k$  as a function of  $\theta$  crosses zero only once from below, we are unable to show in general that there exists

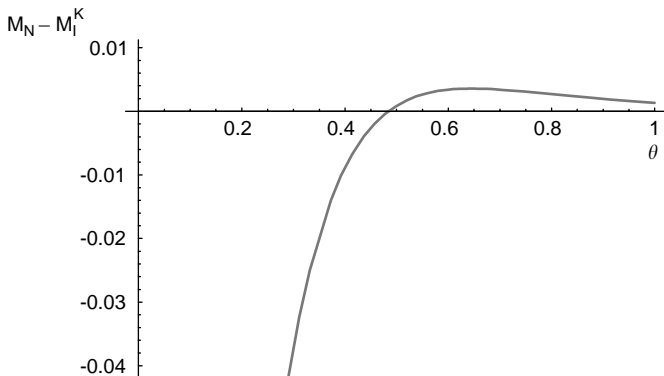


Fig. 1. The relationship between  $M_N - M_1^k$  and  $\theta$ :  $\alpha = 2$ ,  $\beta = 6$ ,  $k = 2$ ,  $w = 0.2$ .

a unique threshold value of  $\theta$  below which the linkage principle holds and above which it fails.

Now suppose that the bidders have access to a *public* signal, and the seller can choose the precision of the signal between  $k$  and  $k + 1$ , when would the seller’s expected revenue be higher with the more precise public signal?

**Proposition 3.** *For any  $k > 0$ , there exists a unique threshold level  $\theta_k^*$  such that  $M_1^k > M_1^{k+1}$  if and only if  $\theta > \theta_k^*$ .*

In Appendix B, we also explicitly show how the threshold  $\theta_k^*$  is determined. The main implication of Proposition 3 is that  $M_1^k$  may be *non-monotonic* in  $k$  for intermediate values of  $\theta$ . For example, suppose the actual severity of budget constraints  $\theta$  is such that  $\theta_1^* < \theta < \theta_2^*$ . By Proposition 3,  $\theta > \theta_1^*$  implies that  $M_1^1 > M_1^2$ , and  $\theta < \theta_2^*$  implies that  $M_1^2 < M_1^3$ . Fig. 2 depicts how  $M_1^k$  varies with  $k$  in a numerical example in which we set  $\alpha = 2, \beta = 6, w = 0.2$ , and  $\theta = 0.54$ . Indeed, in this example, it can be calculated that  $\theta_0^* \approx 0.458, \theta_1^* \approx 0.533, \theta_2^* \approx 0.588, \theta_3^* \approx 0.630$ , where  $\theta_0^*$  is the threshold above which  $M_N > M_1^1$ . Hence, when  $\theta = 0.54$ , we have  $M_N > M_1^1 > M_1^2$ , but  $M_1^2 < M_1^3 < M_1^4$ .

It is interesting to know the optimal precision level of the signal for the seller for any fixed level of  $\theta$ . From Proposition 3, it is clear that if  $\theta$  is sufficiently low, that is, when the budget constraint is not very severe, the seller will prefer the most informative signal; and if  $\theta$  is sufficiently high, she will prefer the least informative signal. What happens when  $\theta$  is in the intermediate range is less clear. The main difficulty is that we are unable to analytically establish that  $\theta_k^*$  is increasing in  $k$ . However, based on numerical examples we work with, we conjecture that this is indeed the case. If so, then for intermediate values of  $\theta$ ,  $M_1^k$  as a function of  $k$  will be convex (as illustrated in Fig. 2), hence the seller will follow a “bang–bang” policy: she will either prefer the least informative or the most informative signal.

The relationship between Propositions 2 and 3 is as follows. Proposition 2 compares the seller’s expected revenue between a concealing policy and a policy of revealing a signal with a fixed level of precision  $k$ . This leaves open the question

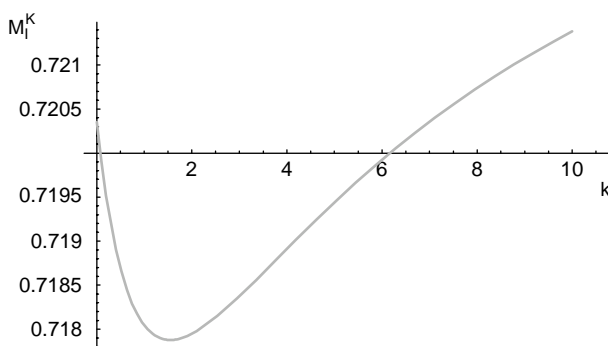


Fig. 2. The relationship between  $M_1^k$  and  $k$ :  $\alpha = 2, \beta = 6, w = 0.2, \theta = 0.54$ .



of whether the seller will benefit from revealing a garbled signal (i.e., a signal with less precision). Proposition 3 partially addresses this question. It implies that for levels of  $\theta$  that are sufficiently low or high, the policy of revealing a garbled signal will not be optimal for the seller, and if our conjecture that  $\theta_k^*$  increases with  $k$  holds, then a garbling policy is not optimal for intermediate levels of  $\theta$  either. Moreover, in the case when  $k$  is an integer, Proposition 3 provides one characterization of  $\underline{\theta}$  and  $\bar{\theta}$  in Proposition 2. Proposition 3 implies that  $M_N > M_1^1 > \dots > M_1^k$  for all  $\theta < \min\{\theta_0^*, \dots, \theta_k^*\}$ , and  $M_N < M_1^1 < \dots < M_1^k$  for all  $\theta > \max\{\theta_0^*, \dots, \theta_k^*\}$ . Hence, we can choose  $\underline{\theta} = \min\{\theta_0^*, \dots, \theta_k^*\}$  and  $\bar{\theta} = \max\{\theta_0^*, \dots, \theta_k^*\}$  as the thresholds in Proposition 2 though they may not be the tightest thresholds.

## 5. Conclusion

This paper provides a class of examples of two bidder common value second price auctions in which bidders may be financially constrained and the seller has access to information about the common value. We show that the seller's expected revenue under a revelation policy may be lower than that under a concealing policy. Furthermore, the seller's expected revenue under the revelation policy may not be monotonic in the accuracy of her information.

While we derived our results in a specific example with two bidders in a common value second price auction, we believe that the economic reasons underlying the possible failure of the linkage principle in the presence of budget constraints are general. A favorable signal revealed by the seller will raise, and an unfavorable signal will lower, the bidders' "unconstrained" bids. In the absence of budget constraints, these two effects operate on the actual bids in a symmetric manner. The presence of budget constraints, however, effectively imposes a limit on how much the bidders may raise their actual bids after seeing a favorable signals revealed by the seller; but no limit is imposed on how much the bidders can lower their actual bids after seeing an unfavorable signal. That is, with budget constraints, the seller has to fully absorb the negative impact of bad signals, while she can only partially reap the benefits of good signals. Importantly, as we see from Proposition 1, the higher  $\theta$  is, the more responsive the "unconstrained" bidding function is to the seller's signal, which implies that the asymmetry between the favorable and unfavorable signals becomes more dramatic as the severity of the financial constraints gets higher. As  $\theta$  is high enough, the increase in the likelihood that the budget constraints bind due to the asymmetric response effect will eventually dominate increased competition between the bidders as a result of the seller's released information, and cause the linkage principle to fail.

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### Appendix A. Derivation of ODE (2)

This derivation is adapted from that in Fang and Parreiras [5] which considers a more general environment. Suppose bidder 2 follows the bidding rule  $B^k(\cdot, \cdot; x_0)$  as specified in (1), and consider the best response for an unconstrained bidder 1. Define  $\psi^k(\cdot; x_0)$  as the inverse of  $b^k(\cdot; x_0)$ , i.e.,  $\psi^k(z; x_0) = [b^k]^{-1}(z; x_0)$ . Bidder 1's optimization problem is

$$\begin{aligned} \max \Pi(b; x_1, x_0) &= \int_x^{\psi^k(b; x_0)} [1 - G(b^k(x_2; x_0))] \\ &\times \{E[S|X_1 = x_1, X_2 = x_2; X_0 = x_0] - b^k(x_2; x_0)\} dF_{X_2|(X_1, X_0)}(x_2|x_1, x_0) \\ &+ \int_w^b \{E[S|X_1 = x_1, X_2 \geq \psi^k(w_2; x_0); X_0 = x_0] - w_2\} \\ &\times [1 - F_{X_2|(X_1, X_0)}(\psi^k(w_2; x_0)|x_1, x_0)] dG(w_2). \end{aligned}$$

In the above expression, the first term is bidder 1's expected surplus when bidder 2's budget  $w_2 > b^k(x_2; x_0)$ , which occurs with probability  $1 - G(b^k(x_2; x_0))$ , and his signal  $x_2$  is below  $\psi^k(b; x_0)$ . In this event, bidder 1 wins the object and pays  $b^k(x_2; x_0)$ . The second term is bidder 1's expected surplus when bidder 2's budget  $w_2$  is less than  $b$ , and he has a signal  $x_2$  above  $\psi^k(w_2; x_0)$ , which conditional on  $x_1$  and  $x_0$  occurs with probability  $1 - F_{X_2|(X_1, X_0)}(\psi^k(w_2; x_0)|x_1)$ . In this event, bidder 1 will win the object and pay bidder 2's bid  $w_2$ .

Differentiating  $\Pi(b; x_1, x_0)$  with respect to  $b$  and applying Leibniz's rule, we obtain, after some simplification,

$$\begin{aligned} \frac{d\Pi(b; x_1, x_0)}{db} &= [1 - G(b)]\{E[S|X_1 = x_1, X_2 = \psi^k(b; x_0); X_0 = x_0] - b\} \\ &\times f_{X_2|(X_1, X_0)}(\psi^k(b; x_0)|x_1, x_0)\psi^k(b; x_0) \\ &+ \{E[S|X_1 = x_1, X_2 \geq \psi^k(b; x_0); X_0 = x_0] - b\} \\ &\times (1 - F_{X_2|(X_1, X_0)}(\psi^k(b; x_0)|x_1, x_0))g(b). \end{aligned} \tag{A.1}$$

In a symmetric equilibrium, (A.1) can be simplified as

$$\begin{aligned}
 & [1 - G(b^k(x_1; x_0))][b^k(x_1; x_0) - v(x_1, x_1; x_0)]f_{X_2|(X_1, X_0)}(x_1|x_1, x_0) \\
 & + [1 - F_{X_2|(X_1, X_0)}(x_1|x_1, x_0)]g(b^k(x_1; x_0)) \\
 & \times [\varphi(x_1, x_1; x_0) - b^k(x_1; x_0)]b^k(x_1; x_0) = 0,
 \end{aligned}
 \tag{A.2}$$

where  $v(\cdot, \cdot; x_0)$  and  $\varphi(\cdot, \cdot; x_0)$  are defined by (3) and (4), respectively. Note that whenever  $b^k(x_1; x_0) > w$  we have  $g(b^k(x_1; x_0)) > 0$ , and subsequently we can rewrite (A.2) as (2). To verify the second-order condition, we need to show that  $\Pi(b; x_1, x_0)$  is a single-peaked function of  $b$  for every  $x_1$  and  $x_0$ . In Fang and Parreiras [5], we show that a sufficient condition for this to hold is that  $\lambda(z|x; x_0)v(x, z; x_0)$  is non-decreasing in  $x$  for all  $z$ .

**Appendix B. Proofs of main results**

**Proof of Proposition 1.** Suppose that  $b(\cdot)$  is an increasing solution of the ODE (2), it must be the case that  $v(x, x; x_0) < b^k(x; x_0) < \varphi(x, x; x_0)$ . Hence, there must be some weighting function  $z(\cdot)$ , with  $z(x) \in [0, 1]$  for all  $x$ , such that

$$\begin{aligned}
 b^k(x; x_0) &= [1 - z(x)]v(x, x; x_0) + z(x)\varphi(x, x; x_0) \\
 &= \frac{2x + \beta + kx_0}{(\alpha + k)(\alpha + k + 1)} [\alpha + k + z(x)].
 \end{aligned}
 \tag{B.1}$$

It suffices to prove the uniqueness of the weighting function  $z(\cdot)$  to show the uniqueness of increasing solution to ODE (5). Clearly the weighting function  $z(\cdot)$  implicitly defined by (B.1) is differentiable if  $b^k(x; x_0)$  is differentiable. Substituting expression (B.1) into (5), we obtain

$$\begin{aligned}
 & 2[\alpha + k + z(x)] + (2x + \beta + kx_0)z'(x) \\
 &= \frac{\alpha + k + 1}{\theta} [\alpha + k + z(x)] \frac{z(x)}{1 - z(x)}.
 \end{aligned}
 \tag{B.2}$$

We first consider the case that  $z'(\cdot) = 0$  everywhere. Then from Eq. (B.2) we obtain

$$z(x) = \frac{2\theta}{2\theta + \alpha + k + 1}.$$

Clearly, the above weighting function satisfies the range condition that  $z(x) \in [0, 1]$ . Plugging this weighting function into (B.1) immediately yields (6).

Now suppose that there is another solution  $z(\cdot)$  to (B.2) such that  $z'(\cdot) \neq 0$ . Because (B.2) is separable, we can show that they are implicitly given by the following

equation:

$$\begin{aligned} & \frac{1}{2} + \frac{\theta \log[\alpha + k + z(x)]}{(2\theta + \alpha + k) \log(2x + \beta + kx_0)} \\ & - \frac{\theta \log[(2\theta + \alpha + k + 1)z(x) - 2\theta]}{(2\theta + \alpha + k)(2\theta + \alpha + k + 1) \log(2x + \beta + kx_0)} \\ & = \frac{C}{\log(2x + \beta + kx_0)}, \end{aligned} \tag{B.3}$$

where  $C$  is an integration constant. The limit of the right-hand side of (B.3) is zero, and the limit of the second term in the left-hand side of (B.3) is 0 if  $z(x) \in [0, 1]$ ; therefore, we must have:

$$\lim_{x \rightarrow +\infty} \frac{\theta \log[(2\theta + \alpha + k + 1)z(x) - 2\theta]}{(2\theta + \alpha + k)(2\theta + \alpha + k + 1) \log(2x + \beta + kx_0)} = \frac{1}{2}.$$

This can be true only if  $(2\theta + \alpha + k + 1)z(x) - 2\theta$  goes to infinity as  $x$  goes to infinity, which violates the range condition that  $z(x) \in [0, 1]$  for all  $x$ .  $\square$

**Proof of Proposition 2.** We will use Lemma B.1 below. In order to make transparent the structure of the current proof, we relegate the proof of Lemma B.1, which is more technical, to Appendix C.

**Lemma B.1.** *The seller’s expected revenues under concealing and revealing policies are, respectively, given by (8) and (9).*

With a slight abuse of notation, we write  $M_N(\theta)$  and  $M_1^k(\theta)$  to denote, respectively, the seller’s expected revenue under concealing and revealing policies when the severity of the budget constraints is  $\theta$ . Using Lemma B.1, it is easy to see, after some manipulation, that if  $\theta > \frac{1}{2}$ , then  $M_N(\theta) > M_1^k(\theta)$  if and only if

$$\frac{\Gamma(\alpha + k + 1)\Gamma(2\theta + \alpha)}{\Gamma(2\theta + \alpha + k)\Gamma(\alpha + 1)} \left[ \frac{b(0)}{b^k(0; 0)} \right]^{2\theta - 1} > 1. \tag{B.4}$$

Note that

$$\frac{b(0)}{b^k(0; 0)} = \frac{(2\theta + \alpha)(\alpha + k)(2\theta + \alpha + k + 1)}{(2\theta + \alpha + 1)\alpha(2\theta + \alpha + k)} = \frac{\alpha + k}{\alpha} h(\theta),$$

where  $h(\theta)$  is defined by

$$h(\theta) = \frac{(2\theta + \alpha)(2\theta + \alpha + k + 1)}{(2\theta + \alpha + 1)(2\theta + \alpha + k)}.$$

It can be verified that  $h'(\theta) > 0$ . Therefore,

$$\frac{\alpha + k}{\alpha} = \frac{\alpha + k}{\alpha} h(\infty) > \frac{b(0)}{b^k(0; 0)} > \frac{\alpha + k}{\alpha} h(0) = \frac{\alpha + k + 1}{\alpha + 1} > 1.$$

Therefore, the term  $[b(0)/b^k(0;0)]^{2\theta-1}$  is at least of order  $[(\alpha + k + 1)/(\alpha + 1)]^{2\theta-1}$ . On the other hand, the term  $\Gamma(2\theta + \alpha)/\Gamma(2\theta + \alpha + k)$  is at the order of  $\theta^{-k}$  since

$$\frac{\Gamma(2\theta + \alpha)}{\Gamma(2\theta + \alpha + k)} = \frac{1}{(2\theta + \alpha) \cdots (2\theta + \alpha + k - 1)}.$$

Therefore, there exists a  $\bar{\theta}$ , which may depend on  $k$ , such that if  $\theta > \bar{\theta}$ , then inequality (B.4) holds, and hence  $M_N(\theta) > M_1^k(\theta)$ .

Analogously, one can show that if  $\theta < \frac{1}{2}$ , then  $M_N(\theta) < M_1^k(\theta)$  if and if inequality (B.4) holds. Note that as  $\theta \rightarrow 0$ , the left-hand side of inequality (B.4) is no less than 1. Therefore,  $\lim_{\theta \rightarrow 0} M_N(\theta) < \lim_{\theta \rightarrow 0} M_1^k(\theta)$ . By continuity, there exists some  $\underline{\theta}$  such that  $M_N(\theta) < M_1^k(\theta)$  for all  $\theta < \underline{\theta}$ .  $\square$

**Proof of Proposition 3.** After some simplification, one can show that  $M_1^{k+1} < M_1^k$  if and only if

$$\frac{\alpha + k + 1}{2\theta + \alpha + k} \left[ \frac{(\alpha + k + 1)(2\theta + \alpha + k)(2\theta + \alpha + k + 2)}{(\alpha + k)(2\theta + \alpha + k + 1)^2} \right]^{2\theta-1} \begin{cases} > 1 & \text{when } \theta > \frac{1}{2}, \\ < 1 & \text{when } \theta < \frac{1}{2}. \end{cases} \tag{B.5}$$

Define function  $m(\cdot)$  by

$$m(\theta) = \ln \left[ \left( \frac{\alpha + k + 1}{2\theta + \alpha + k + 1} \right)^\theta \left( \frac{2\theta + \alpha + k + 1}{2\theta + \alpha + k} \right)^{1-\theta} \left( \frac{\alpha + k}{2\theta + \alpha + k + 2} \right)^{\frac{1}{2}-\theta} \right].$$

We can then restate (B.5) as  $M_1^{k+1} < M_1^k$  if and only if  $m(\theta) > 0$  when  $\theta > \frac{1}{2}$  and  $m(\theta) < 0$  when  $\theta < \frac{1}{2}$ . It can be verified that  $m(\cdot)$  is strictly convex, and satisfies  $\lim_{\theta \rightarrow 0} m(\theta) > 0$ ,  $\lim_{\theta \rightarrow +\infty} m(\theta) = +\infty$ , and  $m(\frac{1}{2}) = 0$ . These imply that the equation  $m(\theta) = 0$  will either have a unique solution at  $\theta = \frac{1}{2}$  or have exactly one more solution besides  $\frac{1}{2}$ . We consider three cases:

*Case 1:*  $\theta = \frac{1}{2}$  is the unique solution to  $m(\theta) = 0$ . Since  $m(\cdot)$  is strictly convex, this implies that  $m(\theta) > 0$  for all  $\theta \neq \frac{1}{2}$ . Then we know that  $M_1^{k+1} < M_1^k$  for all  $\theta > \theta_k^* = \frac{1}{2}$ .

*Case 2:* Besides  $\theta = \frac{1}{2}$ , there is another solution  $\hat{\theta} < \frac{1}{2}$  to equation  $m(\theta) = 0$ . In this case,  $m(\theta) < 0$  for all  $\theta \in (\hat{\theta}, \frac{1}{2})$ ; and  $m(\theta) > 0$  for all  $\theta > \frac{1}{2}$ , hence  $M_1^{k+1} < M_1^k$  holds if  $\theta \in (\hat{\theta}, \frac{1}{2})$  or  $\theta > \frac{1}{2}$ . Then by continuity,  $M_1^{k+1} < M_1^k$  must also hold at  $\theta = \frac{1}{2}$ . Hence, in this case we can choose  $\theta_k^* = \hat{\theta}$ .

*Case 3:* Besides  $\theta = \frac{1}{2}$ , there is another solution  $\hat{\theta} > \frac{1}{2}$  to equation  $m(\theta) = 0$ . In this case, it is clear that  $M_1^{k+1} < M_1^k$  if  $\theta > \hat{\theta}$ , and  $M_1^{k+1} > M_1^k$  if  $\theta < \frac{1}{2}$  or if  $\theta \in (\frac{1}{2}, \hat{\theta})$ . By continuity  $M_1^{k+1} > M_1^k$  must also be true at  $\theta = \frac{1}{2}$ . Therefore, in this case the threshold is given by  $\theta_k^* = \hat{\theta}$ .  $\square$

**Appendix C. Proof of Lemma B.1**

In this appendix, we provide the proof of Lemma B.1 contained in Appendix B. Its proof uses the following intermediate results:

**Lemma C.1.** *Let  $H^k$  denote the c.d.f. of the selling price  $P^k$  when the seller reveals a signal with precision  $k$ .  $H^k$  is characterized by*

$$1 - H^k(p) = \begin{cases} [1 - G(p)]^2 & \text{if } p \leq b^k(0; 0), \\ [1 - G(p)]^2 \left\{ 1 - \sum_{i=1}^{\infty} \frac{\Gamma(\alpha+k+i)}{\Gamma(\alpha)\Gamma(k+i+1)} \left[\frac{b^k(0;0)}{p}\right]^\alpha \left[1 - \frac{b^k(0;0)}{p}\right]^{k+i} \right\} & \text{otherwise.} \end{cases}$$

**Proof.** Let  $H^k(p|s)$  denote c.d.f. of  $P^k$  conditional on  $S = s$ . Since

$$P^k = \min\{b^k(X_1; X_0), b^k(X_2; X_0), W_1, W_2\},$$

$1 - H^k(p|s)$  is given by

$$1 - H^k(p|s) = (1 - G(p))^2 \Pr[\min\{b(X_1; X_0), b(X_2; X_0)\} > p | S = s].$$

For notational simplicity, define  $\Phi(p|s) \equiv \Pr[\min\{b(X_1; X_0), b(X_2; X_0)\} > p | S = s]$ .

To calculate  $\Phi(p|s)$  we let  $\psi^k(\cdot; x_0)$  denote the inverse function of  $b^k(\cdot; x_0)$ :

$$\psi^k(p; x_0) = \frac{1}{2} \max \left\{ \frac{p(2\theta + \alpha + k + 1)(\alpha + k)}{2\theta + \alpha + k} - (\beta + kx_0), 0 \right\}.$$

Note that if  $\psi^k(p; x_0) > 0$ , then

$$\psi^k(p; x_0) = \psi^k(p; 0) - \frac{kx_0}{2}, \tag{C.1}$$

$$2\psi^k(p; 0) = \beta \left[ \frac{p}{b^k(0; 0)} - 1 \right]. \tag{C.2}$$

Using the inverse bidding function  $\psi^k(\cdot; x_0)$  we get

$$\begin{aligned} \Phi(p|s) &= \Pr[\min\{X_1, X_2\} > \psi^k(p; X_0) | S = s] \\ &= \int_0^{+\infty} \Pr[\min\{X_1, X_2\} > \psi^k(p; x_0) | s] dF_{X_0|S}(x_0|s) \\ &= \int_0^{+\infty} \Pr[X_i > \psi^k(p; su) | s]^2 dF_U(u), \end{aligned} \tag{C.3}$$

where the third equality follows from the fact that conditional on  $S = s$ , the seller’s signal  $X_0$  can be interpreted as the random variable  $sU$ , where  $U \sim \Gamma(k, k)$ , and furthermore  $X_1$  and  $X_2$  are independent conditional on  $S$ .

To compute the integral in (C.3), we divide the range of integration in two disjoint regions, namely,  $\Omega_+(p, s) = \{u: \psi^k(p; su) > 0\}$  and  $\Omega_0(p, s) = \{u: \psi^k(p; su) = 0\}$ . In words,  $\Omega_+(p, s)$  denotes the set of the noise  $u$  in the seller’s signal in which there is a

positive probability that an unconstrained bidder bids below  $p$ ; while  $\Omega_0(p, s)$  is the set of the noise  $u$  such that an unconstrained bidder will never find it optimal to bid less than  $p$ . It is easy to show that

$$\Omega_+(p, s) = \left\{ u: 0 \leq u < \frac{2\psi^k(p; 0)}{ks} \right\}, \quad \Omega_0(p, s) = \left\{ u: u \geq \frac{2\psi^k(p; 0)}{ks} \right\}.$$

Therefore, we have

$$\begin{aligned} \Phi(p|s) &= \int_0^{\frac{2\psi^k(p; 0)}{ks}} \Pr[X_i > \psi^k(p; su)|s]^2 dF_U(u) \\ &\quad + \int_{\frac{2\psi^k(p; 0)}{ks}}^{+\infty} \Pr[X_i > \psi^k(p; su)|s]^2 dF_U(u). \end{aligned}$$

We denote the first integral in the above expression by  $I_1$  and the second by  $I_2$ :

$$\begin{aligned} I_1 &= \int_0^{\frac{2\psi^k(p; 0)}{ks}} [1 - F_{X_i|S}(\psi^k(p; su)|s)]^2 dF_U(u) \\ &= \int_0^{\frac{2\psi^k(p; 0)}{ks}} \exp\left(-\frac{2\psi^k(p; su)}{s}\right) dF_U(u) \\ &= \int_0^{\frac{2\psi^k(p; 0)}{ks}} \exp\left(-\frac{2\psi^k(p; 0) - ksu}{s}\right) dF_U(u) \\ &= \exp\left(-\frac{2\psi^k(p; 0)}{s}\right) \int_0^{\frac{2\psi^k(p; 0)}{ks}} \exp(ku) dF_U(u) \\ &= \frac{\exp\left(-\frac{2\psi^k(p; 0)}{s}\right) \left[\frac{2\psi^k(p; 0)}{s}\right]^k}{\Gamma(k + 1)}. \end{aligned}$$

Now we compute  $I_2$ . Noting that  $\psi^k(p; su) = 0$  if  $u \in \Omega_0(p, s)$ , we obtain

$$\begin{aligned} I_2 &= \int_{\frac{2\psi^k(p; 0)}{ks}}^{+\infty} 1 dF_U(u) = 1 - \Pr\left[U \leq \frac{2\psi^k(p; 0)}{ks}\right] \\ &= 1 - \exp\left(-\frac{2\psi^k(p; 0)}{s}\right) \sum_{i=0}^{+\infty} \frac{\left[\frac{2\psi^k(p; 0)}{s}\right]^{k+i}}{\Gamma(k + i + 1)}, \end{aligned}$$

where the last equality follows from the formula for the c.d.f. of a Gamma random variable since  $U \sim \Gamma(k, k)$  (see [7, p. 173]). Thus,

$$\Phi(p|s) = I_1 + I_2 = 1 - \exp\left(-\frac{2\psi^k(p; 0)}{s}\right) \sum_{i=1}^{\infty} \frac{\left[\frac{2\psi^k(p; 0)}{s}\right]^{k+i}}{\Gamma(k + i + 1)}. \tag{C.4}$$

Finally, to obtain the unconditional distribution of the seller’s revenue we take expectations with respect to  $S$ . We will first provide a useful lemma which is proved by simple algebra:

**Lemma C.2.** *Suppose  $S$  is distributed according an inverted Gamma with parameters  $\alpha$  and  $\beta$ , and let  $z > 0, t > 0$  be constants. Then*

$$E\left[S^{-z} \exp\left(-\frac{t}{S}\right)\right] = \frac{\beta^z}{(\beta + t)^{\alpha+z}} \frac{\Gamma(\alpha + z)}{\Gamma(\alpha)}.$$

If  $p \leq b^k(0; 0)$ , then we know that  $\Phi(p|s) = 1$  for all  $s$ . Hence,

$$1 - H^k(p) = [1 - G(p)]^2.$$

If  $p > b^k(0; 0)$ , then

$$\begin{aligned} 1 - H^k(p) &= [1 - G(p)]^2 E\Phi(p|S) \\ &= 1 - E\left\{ \exp\left(-\frac{2\psi^k(p; 0)}{S}\right) \sum_{i=1}^{\infty} \frac{1}{\Gamma(k + i + 1)} \left[\frac{2\psi^k(p; 0)}{S}\right]^{k+i} \right\} \\ &= 1 - \sum_{i=1}^{\infty} \frac{[2\psi^k(p; 0)]^{k+i}}{\Gamma(k + i + 1)} E\left[\exp\left(-\frac{2\psi^k(p; 0)}{S}\right) S^{-(k+i)}\right] \\ &= 1 - \sum_{i=1}^{\infty} \frac{\Gamma(\alpha + k + i)}{\Gamma(\alpha)\Gamma(k + i + 1)} \frac{\beta^\alpha [2\psi^k(p; 0)]^{k+i}}{[2\psi^k(p; 0) + \beta]^{\alpha+k+i}} \\ &= 1 - \sum_{i=1}^{\infty} \frac{\Gamma(\alpha + k + i)}{\Gamma(\alpha)\Gamma(k + i + 1)} \left[\frac{b^k(0; 0)}{p}\right]^\alpha \left[1 - \frac{b^k(0; 0)}{p}\right]^{k+i}, \end{aligned}$$

where in the first equality we use Lemma C.2; in the third equality we exchange the expectation with the summation since the series is absolutely convergent; and in the fourth equality we use (C.2). □

**Proof of Lemma B.1 continued.** By Lemma C.1, we have

$$\begin{aligned} M_1^k &= E[P^k] = \int_0^{+\infty} [1 - H^k(p)] dp \\ &= w + \int_w^{b^k(0;0)} \left[\frac{w}{p}\right]^{2\theta} dp + \int_{b^k(0;0)}^{+\infty} \left[\frac{w}{p}\right]^{2\theta} \\ &\quad \times \left\{ 1 - \sum_{i=1}^{\infty} \frac{\Gamma(\alpha + k + i)}{\Gamma(\alpha)\Gamma(k + i + 1)} \left[\frac{b^k(0; 0)}{p}\right]^\alpha \left[1 - \frac{b^k(0; 0)}{p}\right]^{k+i} \right\} dp. \end{aligned}$$



For notational simplicity, we write the sum of the first two terms as  $T_1$  and the last term as  $T_2$ . One can show that

$$T_1 = w + \frac{w}{2\theta - 1} \left\{ 1 - \left[ \frac{w}{b^k(0; 0)} \right]^{2\theta - 1} \right\}.$$

To calculate  $T_2$  we need to proceed with some caution. First we note that  $E[P^k] < \infty$  for all  $\theta$ . Therefore, we know that  $T_2 < \infty$  for all  $\theta$ . However, when  $\theta < \frac{1}{2}$ ,

$$\int_{b^k(0; 0)}^{+\infty} \left[ \frac{w}{p} \right]^{2\theta} dp = +\infty,$$

therefore when  $\theta < \frac{1}{2}$ ,

$$\int_{b^k(0; 0)}^{+\infty} \left[ \frac{w}{p} \right]^{2\theta} \sum_{i=1}^{\infty} \frac{\Gamma(\alpha + k + i)}{\Gamma(\alpha)\Gamma(k + i + 1)} \left[ \frac{b^k(0; 0)}{p} \right]^{\alpha} \left[ 1 - \frac{b^k(0; 0)}{p} \right]^{k+i} dp = +\infty.$$

To deal with this problem, we use the following expansion of 1:

$$1 \equiv \sum_{i=0}^{\infty} \frac{\Gamma(\alpha + i)}{\Gamma(\alpha)\Gamma(i + 1)} \left[ \frac{b^k(0; 0)}{p} \right]^{\alpha} \left[ 1 - \frac{b^k(0; 0)}{p} \right]^i.$$

This result can be found in Rudin [11, Exercise 22, p. 201]. Substitute this expansion and make the change of variable by defining  $z = b^k(0; 0)/p$  (which implies that  $dp = -[b^k(0; 0)/z^2] dz$ ), we after some manipulation can re-write  $T_2$  as:

$$T_2 = \frac{w^{2\theta}}{b^k(0; 0)^{2\theta - 1}} \int_0^1 \left\{ \sum_{i=0}^{\infty} \frac{\Gamma(\alpha + i)}{\Gamma(\alpha)\Gamma(i + 1)} z^{2\theta + \alpha - 2} (1 - z)^i - \sum_{i=1}^{\infty} \frac{\Gamma(\alpha + k + i)}{\Gamma(\alpha)\Gamma(k + i + 1)} z^{2\theta + \alpha - 2} (1 - z)^{k+i} \right\} dz. \tag{C.5}$$

Since we know that  $T_2$  has to be finite, we can then commute summation and integration, and get

$$\begin{aligned} T_2 &= \sum_{i=0}^{\infty} \frac{\Gamma(\alpha + i)}{\Gamma(\alpha)\Gamma(i + 1)} \int_0^1 z^{2\theta + \alpha - 2} (1 - z)^i dz \\ &\quad - \sum_{i=1}^{\infty} \frac{\Gamma(\alpha + k + i)}{\Gamma(\alpha)\Gamma(k + i + 1)} \int_0^1 z^{2\theta + \alpha - 2} (1 - z)^{k+i} dz \\ &= \frac{w^{2\theta}}{b^k(0; 0)^{2\theta - 1}} \left\{ \begin{aligned} &\sum_{i=0}^{\infty} \frac{\Gamma(\alpha + i)}{\Gamma(\alpha)\Gamma(i + 1)} \frac{\Gamma(2\theta + \alpha - 1)\Gamma(i + 1)}{\Gamma(2\theta + \alpha + i)} \\ &- \sum_{i=1}^{\infty} \frac{\Gamma(\alpha + k + i)}{\Gamma(\alpha)\Gamma(k + i + 1)} \frac{\Gamma(2\theta + \alpha - 1)\Gamma(k + i + 1)}{\Gamma(2\theta + \alpha + k + i)} \end{aligned} \right\} \\ &= \frac{w^{2\theta}}{b^k(0; 0)^{2\theta - 1}} \frac{\Gamma(2\theta + \alpha - 1)}{\Gamma(\alpha)} \left\{ \frac{\Gamma(\alpha)}{\Gamma(2\theta + \alpha)} \right. \\ &\quad \left. + \sum_{i=1}^{\infty} \left[ \frac{\Gamma(\alpha + i)}{\Gamma(2\theta + \alpha + i)} - \frac{\Gamma(\alpha + k + i)}{\Gamma(2\theta + \alpha + k + i)} \right] \right\}, \end{aligned}$$

where the second equality follows from the definition of the Beta function (see, for example, [11, Theorem 8.20, p. 193]:

$$\int_0^1 x^a(1-x)^b dx = \frac{\Gamma(a+1)\Gamma(b+1)}{\Gamma(a+b+2)}.$$

Using the following identity (which can be verified in Maple<sup>®</sup>),

$$\begin{aligned} & \frac{1}{(2\theta-1)} \left[ \frac{\Gamma(1+\alpha)}{\Gamma(2\theta+\alpha)} - \frac{\Gamma(1+\alpha+k)}{\Gamma(2\theta+\alpha+k)} \right] \\ &= \sum_{i=1}^{\infty} \left[ \frac{\Gamma(\alpha+i)}{\Gamma(2\theta+\alpha+i)} - \frac{\Gamma(\alpha+k+i)}{\Gamma(2\theta+\alpha+k+i)} \right] \end{aligned}$$

we obtain

$$\begin{aligned} T_2 &= \frac{w^{2\theta}}{b^k(0;0)^{2\theta-1}} \frac{\Gamma(2\theta+\alpha-1)}{\Gamma(\alpha)} \\ &\quad \times \left\{ \frac{\Gamma(\alpha)}{\Gamma(2\theta+\alpha)} + \frac{1}{2\theta-1} \left[ \frac{\Gamma(1+\alpha)}{\Gamma(2\theta+\alpha)} - \frac{\Gamma(1+\alpha+k)}{\Gamma(2\theta+\alpha+k)} \right] \right\} \\ &= \frac{w^{2\theta}}{b^k(0;0)^{2\theta-1}} \left\{ \frac{1}{2\theta+\alpha-1} \right. \\ &\quad \left. + \frac{1}{2\theta-1} \left[ \frac{\alpha}{2\theta+\alpha-1} - \frac{\Gamma(2\theta+\alpha-1)}{\Gamma(\alpha)} \frac{\Gamma(1+\alpha+k)}{\Gamma(2\theta+\alpha+k)} \right] \right\} \\ &= \frac{w}{2\theta-1} \left\{ 1 - \frac{\Gamma(2\theta-1+\alpha)}{\Gamma(\alpha)} \frac{\Gamma(1+\alpha+k)}{\Gamma(2\theta+\alpha+k)} \right\} \left[ \frac{w}{b^k(0,0)} \right]^{2\theta-1}. \end{aligned}$$

Therefore, after substituting  $b^k(0;0)$  given by (6) into the expressions of  $T_1$  and  $T_2$ , we get after simplification

$$\begin{aligned} M_1^k &= \frac{w}{2\theta-1} \left\{ 2\theta - \frac{\Gamma(\alpha+k+1)}{\Gamma(\alpha)} \frac{\Gamma(2\theta+\alpha-1)}{\Gamma(2\theta+\alpha+k)} \right. \\ &\quad \left. \times \left[ \frac{(\alpha+k)(2\theta+\alpha+k+1)w}{(2\theta+\alpha+k)\beta} \right]^{2\theta-1} \right\}. \end{aligned}$$

Since  $M_N = \lim_{k \rightarrow 0} E[P^k] = \lim_{k \rightarrow 0} M_1^k$  due to Eq. (7), formula (8) immediately follows.  $\square$

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