Additional Materials for Macro-Finance Decoupling: Robust Evaluations of Macro Asset Pricing Models

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Abstract

This note contains additional technical details of Cheng, Dou, and Liao (2020). Section A provides the proofs of several lemmas on the asymptotic convergence of the random components in the test statistic $\mathcal{T}$ and the conditional critical value $c_\alpha(\hat{d})$. Section B verifies the bounded Lipschitz properties of the test statistic and the conditional critical value, which are used to show their weak convergence in large sample. Section C includes some auxiliary lemmas. Section D derives the Euler equations that serve as the asset pricing moment conditions in the long-run risk model and the disaster risk model. Section E considers the long-run risk model and shows that the Gaussian limit is an innocuous assumption. Section F provides derivations for the time-varying disaster risk model in the empirical application of the paper.

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A Proofs for Asymptotic Convergence

This section provides the proofs of Lemmas SA1, SA2, SA6 and SA7 in the Supplemental Appendix to Cheng, Dou, and Liao (2020) under their Assumptions 1, 2, 3 and 4. The proofs of Lemmas SA3, SA4 and SA5 in the Supplemental Appendix to Cheng, Dou, and Liao (2020) are in Sections B and C of this appendix. Throughout this appendix, we use $\lambda_{min}(A)$ to denote the minimal eigenvalue of a real symmetric matrix A, and $\|\cdot\|$ denotes the matrix Frobenius norm.

Proof of Lemma SA1. (a) Define $C_{1,n} \equiv \sup_{\theta\in\Theta} |\bar{g}(\theta)'(\hat{\Omega}(\theta))^{-1}\bar{g}(\theta) - G(\theta)'(\Omega(\theta))^{-1}G(\theta)|$ where $G(\theta) \equiv \mathbb{E}[\bar{g}(\theta)]$. Then by Assumptions 1(i, iii), 2(i) and 3(iii),

$$C_{1,n} = o_p(1), \tag{A.1}$$

uniformly over $\mathbb{P} \in \mathcal{P}_0$. Consider any $\varepsilon > 0$. By the definition of $\hat{\theta}$ and θ_0,

$$\begin{aligned} &\sup_{\mathbb{P}\in\mathcal{P}_0} \mathbb{P}\left(||\hat{\theta} - \theta_0|| \geq \varepsilon\right) \\ &\leq \sup_{\mathbb{P}\in\mathcal{P}_0} \mathbb{P}\left(\min_{\theta\in B^c_\varepsilon(\theta_0)} \bar{g}(\theta)'(\hat{\Omega}(\theta))^{-1}\bar{g}(\theta) \leq \bar{g}(\theta_0)'(\hat{\Omega}(\theta_0))^{-1}\bar{g}(\theta_0)\right) \\ &\leq \sup_{\mathbb{P}\in\mathcal{P}_0} \mathbb{P}\left(\min_{\theta\in B^c_\varepsilon(\theta_0)} G(\theta)'(\Omega(\theta))^{-1}G(\theta) \leq 2C_{1,n}\right) \leq \sup_{\mathbb{P}\in\mathcal{P}_0} \mathbb{P}\left(c_\lambda^{-1}\delta_\varepsilon^2 \leq 2C_{1,n}\right), \end{aligned} \tag{A.2}$$

where the second inequality is by the definition of $C_{1,n}$ and the third inequality is by Assumption 3. Combining the results in (A.1) and (A.2), we deduce that

$$\lim_{n\to\infty} \sup_{\mathbb{P}\in\mathcal{P}_0} \mathbb{P}\left(||\hat{\theta} - \theta_0|| \geq \varepsilon\right) = 0 \text{ for any } \varepsilon > 0. \tag{A.3}$$

Let $C_{2,n} \equiv \sup_{\theta\in\Theta} \|q(\theta) - Q(\theta)\|$. Then by Assumption 1(ii),

$$C_{2,n} = o_p(1) \text{ uniformly over } \mathbb{P} \in \mathcal{P}. \tag{A.4}$$

Applying the first order expansion, we get

$$g(\hat{\theta}) = g(\theta_0) + q(\tilde{\theta})n^{1/2}(\hat{\theta} - \theta_0), \tag{A.5}$$

where $\tilde{\theta}$ is the mean value between θ_0 and $\hat{\theta}$ and it may vary across rows. By Assumption 1(ii), the consistency of $\hat{\theta}$ and (A.4),

$$q(\tilde{\theta}) = Q(\tilde{\theta}) + o_p(1) = Q + o_p(1) = O_p(1), \text{ uniformly over } \mathbb{P} \in \mathcal{P}_0. \tag{A.6}$$

Similarly, we can show that

$$q(\hat{\theta}) = Q(\hat{\theta}) + o_p(1) = Q + o_p(1) = O_p(1), \text{ uniformly over } \mathbb{P} \in \mathcal{P}_0. \tag{A.7}$$

By Assumption 2(i, ii) and the consistency of $\hat{\theta}$,

$$\hat{\Omega} \equiv \hat{\Omega}(\hat{\theta}) = \Omega + o_p(1) \tag{A.8}$$

uniformly over $\mathbb{P} \in \mathcal{P}_0$. Applying the chain rule, we get the first order condition of $\hat{\theta}$:

$$0_{d_\theta \times 1} = 2q(\hat{\theta})'\hat{\Omega}^{-1}g(\hat{\theta}) - \left(n^{-1/2} g(\hat{\theta})'\hat{\Omega}^{-1} \frac{\partial \hat{\Omega}(\hat{\theta})}{\partial \theta_j} \hat{\Omega}^{-1} g(\hat{\theta}) \right)_{j=1,\ldots,d_\theta}, \tag{A.9}$$

where $(a_j)_{j=1,\ldots,d_\theta} \equiv (a_1, \ldots, a_{d_\theta})'$ for any real numbers $a_1, \ldots, a_{d_\theta}$. By Assumptions 1(i, iii), 2(iv) and 3(iii), the consistency of $\hat{\theta}$, (A.5), (A.6), (A.7) and (A.8),

$$\begin{aligned} 0_{d_\theta \times 1} &= q(\hat{\theta})'\hat{\Omega}^{-1}g(\hat{\theta}) = q(\hat{\theta})'\hat{\Omega}^{-1}g(\theta_0) + q(\hat{\theta})'\hat{\Omega}^{-1}q(\tilde{\theta})n^{1/2}(\hat{\theta} - \theta_0) \\ &= Q'\Omega^{-1}g(\theta_0) + \left(Q'\Omega^{-1}Q + o_p(1)\right) n^{1/2}(\hat{\theta} - \theta_0) + o_p(1) \end{aligned} \tag{A.10}$$

uniformly over $\mathbb{P} \in \mathcal{P}_0$. Similarly, we can show that for $j = 1, \ldots, d_\theta$,

$$n^{-1/2} g(\hat{\theta})'\hat{\Omega}^{-1} \frac{\partial \hat{\Omega}(\hat{\theta})}{\partial \theta_j} \hat{\Omega}^{-1} g(\hat{\theta}) = n^{1/2}(\hat{\theta} - \theta_0)o_p(1) + o_p(1) \text{ uniformly over } \mathbb{P} \in \mathcal{P}_0. \tag{A.11}$$

Combining the results in (A.9), (A.10), (A.11), and applying Assumptions 1(iii) and 3(iii), we deduce that

$$n^{1/2}(\hat{\theta} - \theta_0) = -(Q'\Omega^{-1}Q)^{-1}Q'\Omega^{-1}g(\theta_0) + o_p(1) = O_p(1) \tag{A.12}$$

uniformly over $\mathbb{P} \in \mathcal{P}_0$, which proves the first claim of the lemma.

(b) This claim follows by Assumptions 1 and 3(iii), (A.5), (A.6) and (A.12).

(c) This claim has been proved in (A.7).

(d) By Assumptions 2(iv) and 3(iii), and (A.8),

$$\lim_{n\to\infty} \inf_{\mathbb{P}\in\mathcal{P}_0} \mathbb{P}\left(K^{-1} \leq \lambda_{\min}(\hat{\Omega}) \leq \lambda_{\max}(\hat{\Omega}) \leq K \right) = 1. \tag{A.13}$$

By Assumptions 1(iii) and 3(iii), (A.7) and (A.8), we have

$$q(\hat{\theta})'\hat{\Omega}^{-1}q(\hat{\theta}) = Q'\Omega^{-1}Q + o_p(1) \text{ uniformly over } \mathbb{P} \in \mathcal{P}_0, \tag{A.14}$$

which together with Assumption 3(iii) implies that

$$\lim_{n\to\infty}\inf_{\mathbb{P}\in\mathcal{P}_0}\mathbb{P}\left(K^{-1}\leq\lambda_{\min}(q(\hat{\theta})'(\hat{\Omega})^{-1}q(\hat{\theta}))\leq\lambda_{\max}(q(\hat{\theta})'(\hat{\Omega})^{-1}q(\hat{\theta}))\leq K\right)=1. \tag{A.15}$$

Let $\|\cdot\|_S$ denote the matrix operator norm. By Exercise 7.2.18 in Horn and Johnson (1990),

$$\left\|\hat{\Omega}^{1/2}-\Omega^{1/2}\right\|_S\leq||\hat{\Omega}-\Omega||_S\left\|\Omega^{-1}\right\|_S, \tag{A.16}$$

which together with Assumption 3(iii) and (A.8) implies that

$$\left\|\hat{\Omega}^{1/2}-\Omega^{1/2}\right\|_S=o_p(1)\text{ uniformly over }\mathbb{P}\in\mathcal{P}_0. \tag{A.17}$$

By (A.17) and the relation between the operator norm and the Frobenius norm,

$$\left\|\hat{\Omega}^{1/2}-\Omega^{1/2}\right\|=o_p(1)\text{ uniformly over }\mathbb{P}\in\mathcal{P}_0. \tag{A.18}$$

By Assumptions 1(iii) and 3(iii), (A.7) and (A.18),

$$\hat{\Omega}^{-1/2}q(\hat{\theta})=\Omega^{-1/2}Q+o_p(1)=O_p(1) \tag{A.19}$$

uniformly over $\mathbb{P}\in\mathcal{P}_0$. The claim in the lemma follows by (A.14), (A.15) and (A.19).

(e) By Assumption 2(i, ii) and the consistency of $\hat{\theta}$,

$$\sup_{\theta\in\Theta}\left\|\hat{\Omega}(\theta,\hat{\theta})-\Omega(\theta,\theta_0)\right\|=o_p(1)\text{ uniformly over }\mathbb{P}\in\mathcal{P}_0. \tag{A.20}$$

By (A.13) and (A.20),

$$\sup_{\theta\in\Theta}\left\|(\hat{\Omega}(\theta,\hat{\theta})-\Omega(\theta,\theta_0))\hat{\Omega}^{-1}\right\|\leq(\lambda_{\min}(\hat{\Omega}))^{-1}\sup_{\theta\in\Theta}\left\|\hat{\Omega}(\theta,\hat{\theta})-\Omega(\theta,\theta_0)\right\|=o_p(1) \tag{A.21}$$

uniformly over $\mathbb{P}\in\mathcal{P}_0$, where the inequality is by the Cauchy-Schwarz inequality. Similarly,

$$\begin{aligned}\sup_{\theta\in\Theta}\left\|\Omega(\theta,\theta_0)(\hat{\Omega}^{-1}-\Omega^{-1})\right\|&=\sup_{\theta\in\Theta}\left\|\Omega(\theta,\theta_0)\hat{\Omega}^{-1}(\hat{\Omega}-\Omega)\Omega^{-1}\right\|\\&\leq(\lambda_{\min}(\hat{\Omega})\lambda_{\min}(\Omega))^{-1}\sup_{\theta,\tilde{\theta}\in\Theta}\left\|\Omega(\theta,\tilde{\theta})\right\|||\hat{\Omega}-\Omega||=o_p(1)\end{aligned} \tag{A.22}$$

uniformly over $\mathbb{P}\in\mathcal{P}_0$, where the inequality is by the Cauchy-Schwarz inequality, and the last equality is by Assumptions 2(iv) and 3(iii), (A.8) and (A.13). Collecting the results in (A.21) and

(A.22), we deduce that

$$\sup_{\theta\in\Theta}\left\|\hat{V}(\theta)-V(\theta)\right\| \tag{A.23}$$
$$\leq \sup_{\theta\in\Theta}\left\|(\hat{\Omega}(\theta,\hat{\theta})-\Omega(\theta,\theta_0))\hat{\Omega}^{-1}\right\| + \sup_{\theta\in\Theta}\left\|\Omega(\theta,\theta_0)(\hat{\Omega}^{-1}-\Omega^{-1})\right\| = o_p(1)$$

uniformly over $\mathbb{P}\in\mathcal{P}_0$. By Assumptions 2(iv) and 3(iii),

$$\sup_{\theta\in\Theta}\|V(\theta)\| \leq \sup_{\theta\in\Theta}\left\|S_0\Omega(\theta,\theta_0)(\Omega)^{-1}\right\| \leq (\lambda_{\min}(\Omega))^{-1}\sup_{\theta\in\Theta}\|\Omega(\theta,\theta_0)\| \leq c_\lambda^{-1}C_\Omega \tag{A.24}$$

which finishes the proof. *Q.E.D.*

Proof of Lemma SA2. To link $\hat{\xi}$ and ξ^*, we first define

$$\tilde{\xi} \equiv \left(\tilde{v}', \tilde{m}\left(\cdot\right)', \mathrm{vec}(V\left(\cdot\right))', \mathrm{vech}(\Omega)', \mathrm{vech}(\Omega_0\left(\cdot\right))', \mathrm{vech}(M)'\right)', \tag{A.25}$$

where $\tilde{v} \equiv \Omega^{1/2}M\Omega^{-1/2}g(\theta_0)$ and $\tilde{m}\left(\cdot\right) \equiv g_0\left(\cdot\right) - V\left(\cdot\right)\tilde{v}$. The difference between $\tilde{\xi}$ and ξ^* lie in the first two elements, where $\tilde{v}$ and $\tilde{m}(\cdot)$ in $\tilde{\xi}$ involve the empirical process $g\left(\cdot\right)$, but $\Omega^{1/2}Mv^*$ and $m^*(\cdot)$ in ξ^* involve the limiting Gaussian process $\psi(\cdot)$. Under Assumption 1(i), $g\left(\cdot\right) - \mathbb{E}[g\left(\cdot\right)]$ weakly converges to $\psi(\cdot)$, which is equivalent to

$$\lim_{n\to\infty}\sup_{\mathbb{P}\in\mathcal{P}_0}\sup_{f\in BL_1}\|\mathbb{E}\left[f(g-\mathbb{E}[g])\right] - E\left[f(\psi)\right]\| = 0. \tag{A.26}$$

Furthermore, Assumptions 1(iii), 2(iv) and 3(iii) imply that $\tilde{v}$ and $\tilde{m}(\cdot)$ are bounded and Lipschitz in $g(\cdot)$, which together with (A.26) implies that

$$\lim_{n\to\infty}\sup_{\mathbb{P}\in\mathcal{P}_0}\sup_{f\in BL_1}\left\|\mathbb{E}[f(\tilde{\xi})] - E\left[f(\xi^*)\right]\right\| = 0. \tag{A.27}$$

Next, note that the difference between $\hat{\xi}$ and $\tilde{\xi}$ is that $\hat{V}(\theta)$, $\hat{\Omega}$ and $\hat{M}$ in $\hat{\xi}$ are replaced by their probability limits in Lemma SA1(c, d, e) of Cheng, Dou, and Liao (2020). Thus, we have $\hat{\xi} = \tilde{\xi} + o_p(1)$ uniformly over $\mathbb{P}\in\mathcal{P}_0$ following Lemma SA1(c, d, e) in Cheng, Dou, and Liao (2020), which implies

$$\lim_{n\to\infty}\sup_{\mathbb{P}\in\mathcal{P}_0}\sup_{f\in BL_1}\left\|\mathbb{E}[f(\hat{\xi})] - \mathbb{E}[f(\tilde{\xi})]\right\| = 0. \tag{A.28}$$

The desirable result follows from (A.27), (A.28), and the triangle inequality. *Q.E.D.*

Proof of Lemma SA6. Since $0 \leq L(v; d^*) \leq v'Mv$ for any $v\in\mathbb{R}^k$ and $ut_C(u) \leq C$ for any

$u \geq 0$, we have

$$\left|L_C(\upsilon; d^*) - \overline{L}_C(\upsilon; d^*)\right| = L(\upsilon; d^*) t_C(\upsilon' M \upsilon) I\{\|\upsilon\|^2 > C\} \leq C I\{\|\upsilon\|^2 > C\} \tag{A.29}$$

for any $\upsilon \in \mathbb{R}^k$, which implies that

$$P\left(\left|L_C(\upsilon^*, d^*) - \overline{L}_C(\upsilon^*, d^*)\right| > \varepsilon\right) \leq P\left(I\{\|\upsilon^*\|^2 > C\} > \varepsilon/C\right) \leq P\left(\|\upsilon^*\|^2 > C\right). \tag{A.30}$$

Since $\|\upsilon^*\|^2$ follows the chi-square distribution with degree of freedom k, there exists a finite constant C_δ such that $P(\|\upsilon^*\|^2 > C_\delta) \leq \delta/4$ which together with (A.30) implies that for any $C \geq C_\delta$

$$P\left(\left|L_C(\upsilon^*, d^*) - \overline{L}_C(\upsilon^*, d^*)\right| > \varepsilon\right) \leq \delta/4. \tag{A.31}$$

By the union bound of probability and (A.31), we have for any $C \geq C_\delta$,

$$\begin{aligned} P\left(L_C(\upsilon^*, d^*) > c_{\alpha,C}(d^*) + \varepsilon\right) &\leq P\left(\overline{L}_C(\upsilon^*, d^*) + \left|L_C(\upsilon^*, d^*) - \overline{L}_C(\upsilon^*, d^*)\right| > c_{\alpha,C}(d^*) + \varepsilon\right) \\ &\leq P\left(\overline{L}_C(\upsilon^*, d^*) > c_{\alpha,C}(d^*)\right) + \delta/4 \leq \alpha + \delta/4, \end{aligned} \tag{A.32}$$

where the last inequality is by the definition of $c_{\alpha,C}(d^*)$. Since $R_C(\xi^*) = L_C(\upsilon^*, d^*)$ by definition, the claim of the lemma follows from (A.32). *Q.E.D.*

Proof of Lemma SA7. (a). Since $\upsilon^* = O_p(1)$, by Assumptions 1(iii), 2(iv) and 3(iii), Lemma SA1(b, d) in Cheng, Dou, and Liao (2020), and (A.18), we have uniformly over $\mathbb{P} \in \mathcal{P}_0$,

$$\hat{\Omega}^{1/2} \hat{M} \upsilon^* - g(\hat{\theta}) = \Omega^{1/2} M \Omega^{-1/2} (\Omega^{1/2} \upsilon^* - g(\theta_0)) + o_p(1) = O_p(1), \tag{A.33}$$

which together with Lemma SA1(e) in Cheng, Dou, and Liao (2020) implies that

$$\sup_{\theta \in \Theta} \left|\hat{V}(\theta)(\hat{\Omega}^{1/2} \hat{M} \upsilon^* - g(\hat{\theta}))\right| = O_p(1) \text{ uniformly over } \mathbb{P} \in \mathcal{P}_0. \tag{A.34}$$

Therefore by the triangle inequality, Assumption 1 and (A.34),

$$\begin{aligned} &\sup_{\theta \in \Theta} \left|n^{-1/2}(\hat{m}(\theta) + \hat{V}(\theta) \hat{\Omega}^{1/2} \hat{M} \upsilon^*) - \mathbb{E}[g_0(\theta)]\right| \\ &\leq \sup_{\theta \in \Theta} \left|n^{-1/2} g_0(\theta) - \mathbb{E}[g_0(\theta)]\right| + \sup_{\theta \in \Theta} \left|n^{-1/2} \hat{V}(\theta)(\hat{\Omega}^{1/2} \hat{M} \upsilon^* - g(\hat{\theta}))\right| = o_p(1) \end{aligned} \tag{A.35}$$

uniformly over $\mathbb{P} \in \mathcal{P}_0$. By Assumptions 2(i) and 4, and (A.35), we can apply similar arguments in the proof of (A.3) to deduce that

$$\hat{\theta}^* = \theta_0 + o_p(1) \tag{A.36}$$

uniformly over $\mathbb{P} \in \mathcal{P}_0 \cap \mathcal{P}_{00}$. Applying the chain rule, we get the first order condition of $\hat{\theta}^*$:

$$\begin{aligned}0_{d_\theta \times 1} = 2 & \left(\frac{\partial}{\partial \theta'}(\hat{m}(\hat{\theta}^*) + \hat{V}_S(\hat{\theta}^*)\upsilon^*)\right)' (\hat{\Omega}_0^*)^{-1}(\hat{m}(\hat{\theta}^*) + \hat{V}_S(\hat{\theta}^*)\upsilon^*) \\ & - \left((\hat{m}(\hat{\theta}^*) + \hat{V}_S(\hat{\theta}^*)\upsilon^*)'(\hat{\Omega}_0^*)^{-1}\frac{\partial \hat{\Omega}_0(\hat{\theta}^*)}{\partial \theta_j}(\hat{\Omega}_0^*)^{-1}(\hat{m}(\hat{\theta}^*) + \hat{V}_S(\hat{\theta}^*)\upsilon^*)\right)_{j=1,\ldots,d_\theta}, \qquad \text{(A.37)}\end{aligned}$$

where $\hat{\Omega}_0^* \equiv \hat{\Omega}_0^*(\hat{\theta}^*)$ and $\hat{V}_S(\hat{\theta}^*) \equiv \hat{V}(\hat{\theta}^*)\hat{\Omega}^{1/2}\hat{M}$. Using (A.36) and similar arguments for showing (A.5) and (A.6), we obtain

$$g_0(\hat{\theta}^*) = g_0(\theta_0) + n^{1/2}(\hat{\theta}^* - \theta_0)(Q_0 + o_p(1)) = O_p(1) + n^{1/2}(\hat{\theta}^* - \theta_0)O_p(1) \qquad \text{(A.38)}$$

uniformly over $\mathbb{P} \in \mathcal{P}_0$. By (A.34) and (A.38),

$$\hat{m}(\hat{\theta}^*) + \hat{V}_S(\hat{\theta}^*)\upsilon^* = g_0(\hat{\theta}^*) + \hat{V}(\hat{\theta}^*)(\hat{\Omega}^{1/2}\hat{M}\upsilon^* - g(\hat{\theta})) = O_p(1) + n^{1/2}(\hat{\theta}^* - \theta_0)O_p(1), \qquad \text{(A.39)}$$

which together with Assumptions 2 and 3(iii), (A.36) and the Cauchy-Schwarz inequality implies that

$$\begin{aligned}& n^{-1/2}(\hat{m}(\hat{\theta}^*) + \hat{V}_S(\hat{\theta}^*)\upsilon^*)'(\hat{\Omega}_0^*)^{-1}\frac{\partial \hat{\Omega}_0(\hat{\theta}^*)}{\partial \theta_j}(\hat{\Omega}_0^*)^{-1}(\hat{m}(\hat{\theta}^*) + \hat{V}_S(\hat{\theta}^*)\upsilon^*) \\ & = n^{1/2}(\hat{\theta}^* - \theta_0)o_p(1) + o_p(1), \text{ uniformly over } \mathbb{P} \in \mathcal{P}_0 \cap \mathcal{P}_{00}. \qquad \text{(A.40)}\end{aligned}$$

By Assumptions 2(iii, iv) and 3(iii),

$$\max_{1 \le j \le d_\theta} \sup_{\theta \in \Theta} \left\| \hat{V}(\theta)/\partial \theta_j \right\| = O_p(1) \text{ uniformly over } \mathbb{P} \in \mathcal{P}_0, \qquad \text{(A.41)}$$

which combined with (A.33) implies that

$$\frac{\partial}{\partial \theta'}\hat{V}(\hat{\theta}^*)(\hat{\Omega}^{1/2}\hat{M}\upsilon^* - g(\hat{\theta})) = O_p(1) \qquad \text{(A.42)}$$

uniformly over $\mathbb{P} \in \mathcal{P}_0 \cap \mathcal{P}_{00}$. By Assumption 1(ii) and (A.36),

$$n^{-1/2}\frac{\partial g_0(\hat{\theta}^*)}{\partial \theta'} = Q_0 + o_p(1) \text{ uniformly over } \mathbb{P} \in \mathcal{P}_0 \cap \mathcal{P}_{00}. \qquad \text{(A.43)}$$

Collecting the results in (A.42) and (A.43), we have

$$\begin{aligned} & n^{-1/2}\frac{\partial}{\partial\theta'}(\hat{m}(\hat{\theta}^*)+\hat{V}_S(\hat{\theta}^*)\upsilon^*) \\ &= n^{-1/2}\frac{\partial}{\partial\theta'}g_0(\hat{\theta}^*)+n^{-1/2}\frac{\partial}{\partial\theta'}\hat{V}(\hat{\theta}^*)(\hat{\Omega}^{1/2}\hat{M}\upsilon^*-g(\hat{\theta}))=Q_0+o_p(1) \end{aligned} \tag{A.44}$$

uniformly over $\mathbb{P}\in\mathcal{P}_0\cap\mathcal{P}_{00}$. Applying the first order expansion to get

$$\begin{aligned} & \hat{m}(\hat{\theta}^*)+\hat{V}_S(\hat{\theta}^*)\upsilon^* \\ &= g_0(\hat{\theta}^*)+\hat{V}(\hat{\theta}^*)(\hat{\Omega}^{1/2}\hat{M}\upsilon^*-g(\hat{\theta})) \\ &= g_0(\theta_0)+\hat{V}(\theta_0)(\hat{\Omega}^{1/2}\hat{M}\upsilon^*-g(\hat{\theta})) \\ &\quad+\frac{\partial g_0(\tilde{\theta}^*)}{\partial\theta'}(\hat{\theta}^*-\theta_0)+\frac{\partial\hat{V}(\tilde{\theta}^*)(\hat{\Omega}^{1/2}\hat{M}\upsilon^*-g(\hat{\theta}))}{\partial\theta'}(\hat{\theta}^*-\theta_0) \\ &= g_0(\theta_0)+\hat{V}(\theta_0)(\hat{\Omega}^{1/2}\hat{M}\upsilon^*-g(\hat{\theta}))+(Q_0+o_p(1))n^{1/2}(\hat{\theta}^*-\theta_0)+o_p(1) \end{aligned} \tag{A.45}$$

uniformly over $\mathbb{P}\in\mathcal{P}_0\cap\mathcal{P}_{00}$, where the third equality is by Assumption 1(ii), (A.33), (A.36) and (A.41). By $V(\theta_0)=S_0$, Lemma SA1(e) in Cheng, Dou, and Liao (2020) and (A.33),

$$\hat{V}(\theta_0)(\hat{\Omega}^{1/2}\hat{M}\upsilon^*-g(\hat{\theta}))=S_0\Omega^{1/2}M\Omega^{-1/2}(\Omega^{1/2}\upsilon^*-g(\theta_0))+o_p(1) \tag{A.46}$$

uniformly over $\mathbb{P}\in\mathcal{P}_0$. By (A.45) and (A.46), we have uniformly over $\mathbb{P}\in\mathcal{P}_0\cap\mathcal{P}_{00}$,

$$\begin{aligned} \hat{m}(\hat{\theta}^*)+\hat{V}_S(\hat{\theta}^*)\upsilon^* &= (Q_0+o_p(1))n^{1/2}(\hat{\theta}^*-\theta_0) \\ &\quad+Q_0(Q'\Omega^{-1}Q)^{-1}Q'\Omega^{-1}g(\theta_0)+S_0\Omega^{1/2}M\upsilon^*+o_p(1). \end{aligned} \tag{A.47}$$

By Assumptions 3(iii) and 4(ii), $Q_0'\Omega_0^{-1}Q_0$ is positive definite. Therefore collecting the results in (A.37), (A.40), (A.44), (A.47), and applying Assumption 2(i, ii) and (A.36) to $\hat{\Omega}_0^*$, we obtain

$$n^{1/2}(\hat{\theta}^*-\theta_0)=-(Q'\Omega^{-1}Q)^{-1}Q'\Omega^{-1}g(\theta_0)-\left(Q_0'\Omega_0^{-1}Q_0\right)^{-1}Q_0'\Omega_0^{-1}S_0\Omega^{1/2}M\upsilon^*+o_p(1) \tag{A.48}$$

uniformly over $\mathbb{P}\in\mathcal{P}_0\cap\mathcal{P}_{00}$, which proves part (a) of the lemma.

(b) By (A.48), Assumptions 3(iii) and 4,

$$n^{1/2}(\hat{\theta}^*-\theta_0)=O_p(1) \text{ uniformly over } \mathbb{P}\in\mathcal{P}_0\cap\mathcal{P}_{00}. \tag{A.49}$$

By Assumptions 2(iv), 3(iii) and 4, (A.47) and (A.48), we have uniformly over $\mathbb{P} \in \mathcal{P}_0 \cap \mathcal{P}_{00}$,

$$
\begin{aligned}
\hat{m}(\hat{\theta}^*) + \hat{V}_S(\hat{\theta}^*)\upsilon^* &= -Q_0 \left(Q_0'\Omega_0^{-1}Q_0\right)^{-1} Q_0'\Omega_0^{-1}S_0\Omega^{1/2}M\upsilon^* + S_0\Omega^{1/2}M\upsilon^* + o_p(1) \\
&= \Omega_0^{1/2}M_0\Omega_0^{-1/2}S_0\Omega^{1/2}M\upsilon^* + o_p(1) \\
&= \Omega_0^{1/2}M_0\Omega_0^{-1/2}S_0\Omega^{1/2}\upsilon^* + o_p(1) = O_p(1), \qquad \text{(A.50)}
\end{aligned}
$$

where $M_0 \equiv I_{k_0} - \Omega_0^{-1/2}Q_0 \left(Q_0'\Omega_0^{-1}Q_0\right)^{-1} Q_0'\Omega_0^{-1/2}$ and the last equality is by $M_0\Omega_0^{-1/2}Q_0 = 0_{k_0\times 1}$. By Assumptions 2(i, ii) and 3(iii), (A.36) and (A.50), we deduce that uniformly over $\mathbb{P} \in \mathcal{P}_0 \cap \mathcal{P}_{00}$,

$$
(\hat{m}(\hat{\theta}^*) + \hat{V}_S(\hat{\theta}^*)\upsilon^*)'(\hat{\Omega}_0^*)^{-1}(\hat{m}(\hat{\theta}^*) + \hat{V}_S(\hat{\theta}^*)\upsilon^*) = \upsilon^{*\prime}\tilde{M}_0\upsilon^* + o_p(1), \qquad \text{(A.51)}
$$

where $\tilde{M}_0 \equiv (\Omega_0^{-1/2}S_0\Omega^{1/2})'M_0(\Omega_0^{-1/2}S_0\Omega^{1/2})$. Since $\upsilon^* = O_p(1)$, Lemma SA1(e) implies that

$$
\upsilon^{*\prime}\hat{M}\upsilon^* = \upsilon^{*\prime}M\upsilon^* + o_p(1) \qquad \text{(A.52)}
$$

uniformly over $\mathbb{P} \in \mathcal{P}_0$, which together with (A.51) finishes the proof.

(c). Since $M \equiv I_k - \Omega^{-1/2}Q(Q'\Omega^{-1}Q)^{-1}Q'\Omega^{-1/2}$, $M^2 = M$. Moreover,

$$
\begin{aligned}
\tilde{M}_0^2 &= (\Omega_0^{-1/2}S_0\Omega^{1/2})'M_0(\Omega_0^{-1/2}S_0\Omega^{1/2})(\Omega_0^{-1/2}S_0\Omega^{1/2})'M_0(\Omega_0^{-1/2}S_0\Omega^{1/2}) \\
&= (\Omega_0^{-1/2}S_0\Omega^{1/2})'M_0(\Omega_0^{-1/2}S_0\Omega^{1/2}) = \tilde{M}_0 \qquad \text{(A.53)}
\end{aligned}
$$

and

$$
\begin{aligned}
\tilde{M}_0M &= \tilde{M}_0 - (\Omega_0^{-1/2}S_0\Omega^{1/2})'M_0(\Omega_0^{-1/2}S_0\Omega^{1/2})\Omega^{-1/2}Q(Q'\Omega^{-1}Q)^{-1}Q'\Omega^{-1/2} \\
&= (\Omega_0^{-1/2}S_0\Omega^{1/2})'M_0(\Omega_0^{-1/2}S_0\Omega^{1/2}) = \tilde{M}_0, \qquad \text{(A.54)}
\end{aligned}
$$

where the second equality is by $M_0\Omega_0^{-1/2}Q_0 = 0_{k_0\times 1}$. Similarly, $M\tilde{M}_0 = \tilde{M}_0$. Therefore, $(M - \tilde{M}_0)^2 = M^2 - M\tilde{M}_0 - \tilde{M}_0M + \tilde{M}_0^2 = M - \tilde{M}_0$ which implies that $M - \tilde{M}_0$ is an idempotent matrix. The rank of $M - \tilde{M}_0$ equals the trace of $M - \tilde{M}_0$ since $M - \tilde{M}_0$ is idempotent. By the definition of M and $\tilde{M}_0$,

$$
tr(M) = k - tr(\Omega^{-1/2}Q(Q'\Omega^{-1}Q)^{-1}Q'\Omega^{-1/2}) = k - d_\theta \qquad \text{(A.55)}
$$

and

$$
\begin{aligned}
tr(\tilde{M}_0) &= tr((\Omega_0^{-1/2}S_0\Omega^{1/2})'M_0(\Omega_0^{-1/2}S_0\Omega^{1/2})) \\
&= tr(M_0(\Omega_0^{-1/2}S_0\Omega^{1/2}\Omega^{1/2\prime}S_0'\Omega_0^{-1/2\prime})) = tr(M_0) = k_0 - d_\theta, \qquad \text{(A.56)}
\end{aligned}
$$

which implies that $tr(M - \tilde{M}_0) = k - k_0 = k_1$. Therefore, $M - \tilde{M}_0$ is an idempotent matrix with rank k_1 which together with $\upsilon^* \sim N(0, I_k)$ proves the claim (c). *Q.E.D.*

B Proofs for Bounded Lipschitz Conditions

This section contains the proofs of Lemmas SA4 and SA5 in Cheng, Dou, and Liao (2020) under their Assumptions 2 and 3, and some auxiliary results used to show them.

Lemma B1. *Consider $\xi \equiv (x', d')'$ where d satisfies Assumptions 2 and 3. Then we have for any given $C > 0$:*
(i) *$R(\xi)$ is bounded and Lipschitz in ξ on the set $\{\xi : R(\xi) \geq 0$ and $x'\Omega_d^{-1}x \leq C\}$;*
(ii) *$L(\upsilon; d)$ is bounded and Lipschitz in d on the set $\{(\upsilon, d) : L(\upsilon; d) \geq 0$ and $\|\upsilon\| \leq C\}$.*

Proof of Lemma B1. (i) Let $S_\xi \equiv \{\xi : R(\xi) \geq 0$ and $x'\Omega_d^{-1}x \leq C\}$. By the definition of $R(\xi)$, $R(\xi) \leq x'\Omega_d^{-1}x \leq C$ for any $\xi \in S_\xi$, which shows that $R(\xi)$ is bounded on S_ξ. Next, we want to show that for any $\xi_1, \xi_2 \in S_\xi$,

$$|R(\xi_1) - R(\xi_2)| \leq C_R \left\|\xi_1 - \xi_2\right\|_s \tag{B.57}$$

for some constant fixed C_R. By the triangle inequality,

$$|R(\xi_1) - R(\xi_2)| \leq R(\xi_1) + R(\xi_2) \leq x_2'\Omega_{d,2}^{-1}x_2 + x_1'\Omega_{d,1}^{-1}x_1 = 2C, \tag{B.58}$$

which implies that the claimed result holds with a Lipschitz constant $C_R = 2$ if $\left\|\xi_1 - \xi_2\right\|_s > C$. Thus, it is only necessary to consider the case that $\left\|\xi_1 - \xi_2\right\|_s \leq C$.

Define $A_j(\theta) \equiv m_{d,j}(\theta) + V_{d,j}(\theta)x_j$ for $j = 1, 2$. Consider any $\xi_1, \xi_2 \in S_\xi$, by the triangle inequality,

$$\begin{aligned}|R(\xi_1) - R(\xi_2)| \leq{}& \left|x_1'\Omega_{d,1}^{-1}x_1 - x_2'\Omega_{d,2}^{-1}x_2\right| \\ &+ \left|\min_{\theta\in\Theta} A_1(\theta)'(\Omega_{0,d,1}(\theta))^{-1}A_1(\theta) - \min_{\theta\in\Theta} A_2(\theta)'(\Omega_{0,d,2}(\theta))^{-1}A_2(\theta)\right|.\end{aligned} \tag{B.59}$$

By the triangle inequality, the Cauchy-Schwarz inequality, $x_1'\Omega_{d,1}^{-1}x_1 \leq C$ and $x_2'\Omega_{d,2}^{-1}x_2 \leq C$,

$$\begin{aligned}\left|x_1'\Omega_{d,1}^{-1}x_1 - x_2'\Omega_{d,2}^{-1}x_2\right| &\leq \left|(x_1 - x_2)'\Omega_{d,1}^{-1}x_1\right| + \left|x_2'\Omega_{d,2}^{-1}(\Omega_{d,1} - \Omega_{d,2})\Omega_{d,1}^{-1}x_1\right| + \left|x_2'\Omega_{d,2}^{-1}(x_1 - x_2)\right| \\ &\leq \left[\frac{(x_1'\Omega_{d,1}^{-1}x_1)^{1/2}}{(\lambda_{\min}(\Omega_{d,1}))^{1/2}} + \frac{(x_2'\Omega_{d,2}^{-1}x_2)^{1/2}}{(\lambda_{\min}(\Omega_{d,2}))^{1/2}}\right]\|x_1 - x_2\| \\ &\quad + \frac{(x_1'\Omega_{d,1}^{-1}x_1)^{1/2}(x_2'\Omega_{d,2}^{-1}x_2)^{1/2}}{(\lambda_{\min}(\Omega_{d,1})\lambda_{\min}(\Omega_{d,2}))^{1/2}}\|\Omega_{d,1} - \Omega_{d,2}\| \\ &\leq 2c_\lambda^{-1/2}C^{1/2}\|x_1 - x_2\| + c_\lambda^{-1}C\|\Omega_{d,1} - \Omega_{d,2}\| \leq C_1\left\|\xi_1 - \xi_2\right\|_s\end{aligned} \tag{B.60}$$

for some fixed constant C_1.

Let θ_j denote the minimizer of $A_j(\theta)'(\Omega_{0,d,j}(\theta))^{-1}A_j(\theta)$ for $j=1,2$. By the triangle inequality, we have

$$\begin{aligned}
&\left|\min_{\theta\in\Theta} A_1(\theta)'(\Omega_{0,d,1}(\theta))^{-1}A_1(\theta) - \min_{\theta\in\Theta} A_2(\theta)'(\Omega_{0,d,2}(\theta))^{-1}A_2(\theta)\right| \\
&\leq \max_{\{\theta_1,\theta_2\}} \left|A_1(\theta)'(\Omega_{0,d,1}(\theta))^{-1}A_1(\theta) - A_2(\theta)'(\Omega_{0,d,2}(\theta))^{-1}A_2(\theta)\right| \\
&\leq \max_{\{\theta_1,\theta_2\}} \left|(A_1(\theta)-A_2(\theta))'\,(\Omega_{0,d,1}(\theta))^{-1}A_1(\theta)\right| \\
&+ \max_{\{\theta_1,\theta_2\}} \left|A_2(\theta)'(\Omega_{0,d,1}(\theta))^{-1}\,(A_1(\theta)-A_2(\theta))\right| \\
&+ \max_{\{\theta_1,\theta_2\}} \left|A_2(\theta)'((\Omega_{0,d,1}(\theta))^{-1} - (\Omega_{0,d,2}(\theta))^{-1})A_2(\theta)\right|. \qquad \text{(B.61)}
\end{aligned}$$

We next investigate the three terms after the second inequality of (B.61) one by one. By the triangle inequality and the Cauchy-Schwarz inequality,

$$\begin{aligned}
&\max_{\{\theta_1,\theta_2\}} \|A_1(\theta)-A_2(\theta)\| \\
&\leq \sup_{\theta\in\Theta}\|m_{d,1}(\theta)-m_{d,2}(\theta)\| + \|x_1\|\sup_{\theta\in\Theta}\|V_{d,1}(\theta)-V_{d,2}(\theta)\| + \sup_{\theta\in\Theta}\|V_{d,2}(\theta)\|\,\|x_1-x_2\| \\
&\leq \|\xi_1-\xi_2\|_s + \lambda_{\max}(\Omega_{d,1})(x_1'\Omega_{d,1}^{-1}x_1)^{1/2}\,\|\xi_1-\xi_2\|_s + C_V\,\|\xi_1-\xi_2\|_s\,, \qquad \text{(B.62)}
\end{aligned}$$

where $C_V \equiv c_\lambda^{-1}C_\Omega$ and the second inequality is by the definition of $\|\xi_1-\xi_2\|_s$ and $\sup_{\theta\in\Theta}\|V_{2,d}(\theta)\| \leq c_\lambda^{-1}C_\Omega$ (which is proved in Lemma SA1 of Cheng, Dou, and Liao (2020)). Therefore, by Assumption 2(iv) and (B.62),

$$\max_{\{\theta_1,\theta_2\}} \|A_1(\theta)-A_2(\theta)\| \leq (1+(CC_\Omega)^{1/2}+C_V)\,\|\xi_1-\xi_2\|_s\,, \qquad \text{(B.63)}$$

which together with the triangle inequality, Assumption 2(iv) and $\|\xi_1-\xi_2\|_s \leq C$ implies that

$$\begin{aligned}
&\max_{\{\theta_1,\theta_2\}} \|A_1(\theta)\| \leq \|A_1(\theta_1)\| + \|A_1(\theta_2)\| \\
&\leq \|A_1(\theta_1)\| + \|A_2(\theta_2)\| + \|A_1(\theta_2)-A_2(\theta_2)\| \\
&\leq C_\Omega^{1/2}\left((A_1(\theta)'(\Omega_{d,1,0}(\theta))^{-1}A_1(\theta))^{1/2} + (A_2(\theta)'(\Omega_{d,2,0}(\theta))^{-1}A_2(\theta))^{1/2}\right) \\
&+ (1+(CC_\Omega)^{1/2}+C_V)\,\|\xi_1-\xi_2\|_s \\
&\leq 2(C_\Omega C)^{1/2} + (1+(CC_\Omega)^{1/2}+C_V)C. \qquad \text{(B.64)}
\end{aligned}$$

By the same arguments, the inequality in (B.64) applies to $\max_{\{\theta_1,\theta_2\}}||A_2(\theta)||$. By the Cauchy-

Schwarz inequality, Assumption 3(iii), (B.63) and (B.64),

$$\max_{\{\theta_1,\theta_2\}} \left|(A_1(\theta) - A_2(\theta))' (\Omega_{d,1,0}(\theta))^{-1} A_1(\theta)\right|$$
$$\leq (\lambda_{\min}(\Omega_{0,d,1}(\theta)))^{-1} \max_{\{\theta_1,\theta_2\}} \|A_1(\theta) - A_2(\theta)\| \max_{\{\theta_1,\theta_2\}} \|A_1(\theta)\| \leq C_2 \|\xi_1 - \xi_2\|_s \tag{B.65}$$

for some constant fixed C_2. Similarly, we can show that

$$\max_{\{\theta_1,\theta_2\}} \left|A_2(\theta)'(\Omega_{0,d,1}(\theta))^{-1} (A_1(\theta) - A_2(\theta))\right| \leq C_2 \|\xi_1 - \xi_2\|_s . \tag{B.66}$$

By the Cauchy-Schwarz inequality, Assumption 3(iii) and (B.64),

$$\max_{\{\theta_1,\theta_2\}} \left|A_2(\theta)'((\Omega_{0,d,1}(\theta))^{-1} - (\Omega_{0,d,2}(\theta))^{-1})A_2(\theta)\right|$$
$$\leq \frac{\max_{\{\theta_1,\theta_2\}} \|A_2(\theta)\|^2 \|\Omega_{0,d,1}(\theta) - \Omega_{0,d,2}(\theta)\|}{(\inf_{\theta\in\Theta} \lambda_{\min}(\Omega_{0,d,1}(\theta)) \inf_{\theta\in\Theta} \lambda_{\min}(\Omega_{0,d,2}(\theta)))^{-1}} \leq C_3 \sup_{\theta\in\Theta} \|\Omega_{0,d,1}(\theta) - \Omega_{0,d,2}(\theta)\| \tag{B.67}$$

for some constant C_3. Collecting the results in (B.61), (B.65), (B.66) and (B.67), we get

$$\left|\min_{\theta\in\Theta} A_1(\theta)'(\Omega_{0,d,1}(\theta))^{-1}A_1(\theta) - \min_{\theta\in\Theta} A_2(\theta)'(\Omega_{0,d,2}(\theta))^{-1}A_2(\theta)\right| \leq (2C_2 + C_3) \|\xi_1 - \xi_2\|_s , \tag{B.68}$$

which together with (B.59) and (B.60) implies that the lipschitz constant is $C_R = C_1 + 2C_2 + C_3$.

(ii) Note that under the condition $L(\upsilon; d) \geq 0$ and $\|\upsilon\| \leq C$, we have $0 \leq L(\upsilon; d) \leq \upsilon' M_d \upsilon \leq C^2$. To show $L(\upsilon; d)$ is Lipschitz in d, we write

$$L(\upsilon; d) \equiv \upsilon' M_d \upsilon - \min_{\theta\in\Theta} (m_d(\theta) + V_{d,S}(\theta)\upsilon)' (\Omega_{d,0}(\theta))^{-1} (m_d(\theta) + V_{d,S}(\theta)\upsilon) , \tag{B.69}$$

where $V_{d,S}(\cdot) \equiv V_d(\cdot)\, \Omega_d^{1/2} M_d$. This functional form is analogous to $R(\xi)$, with Ω_d and $V_d(\cdot)$ in $R(\xi)$ replaced by M_d and $V_{d,S}(\cdot)$, respectively. Given that $V_{d,S}(\cdot)$ is Lipschitz in d (established in Lemma C4 below) and $\sup_{\theta\in\Theta} \|V_{d,S}(\theta)\| \leq C_\Omega^{1/2} C_V$ (by Assumptions 2(iv) and 3(iii)), showing $L(\upsilon; d)$ is Lipschitz in d is analogous to showing $R(\xi)$ is Lipschitz in ξ. The only difference is that M_d is not a full rank matrix, unlike Ω_d, which is the reason that we have to bound $\|\upsilon\|$ directly instead of bounding $\upsilon' M_d \upsilon$. Because (B.60) in the proof of part (i) uses the full rank condition of Ω_d, we replace (B.60) with the following argument to show $\upsilon' M_d \upsilon$ is Lipschitz in M_d. Given $\|\upsilon\| \leq C$, we have

$$\left|\upsilon' M_{d,1} \upsilon - \upsilon' M_{d,2} \upsilon\right| = \left|\upsilon'(M_{d,1} - M_{d,2})\upsilon\right| \leq \|\upsilon\|^2 \|M_{d,1} - M_{d,2}\| \leq C^2 \|M_{d,1} - M_{d,2}\| \tag{B.70}$$

because $M_{d,1}$ and $M_{d,2}$ are both idempotent. The rest of the proof is analogous to those in the proof of Lemma B1(i) and hence is omitted. *Q.E.D.*

PROOF OF LEMMA SA4. The truncation function $t_C(u)$ satisfies the following properties: (i) for any $u \in \mathbb{R}^+$, $0 \leq t_C(u) \leq 1$ and $ut_C(u) \leq u$; (ii) for any u_1, $u_2 \in \mathbb{R}$, $|t_C(u_1) - t_C(u_2)| \leq C^{-1} |u_1 - u_2|$, which implies that $t_C(u)$ is Lipschitz. Therefore,

$$0 \leq R_C(\xi) \leq \left(x'\Omega_d^{-1}x\right) t_C(x'\Omega_d^{-1}x) \leq C \tag{B.71}$$

which means that $R_C(\xi)$ is bounded.

Next, we show that $R_C(\xi)$ is Lipschitz in ξ. That is for any ξ_j with $R(\xi_j) \geq 0$ $(j = 1, 2)$,

$$|R_C(\xi_1) - R_C(\xi_2)| \leq C_R \left\|\xi_1 - \xi_2\right\|_s, \tag{B.72}$$

where C_R is a finite constant. Without loss of generality, we assume that $x_2'\Omega_{d,2}^{-1}x_2 \leq x_1'\Omega_{d,1}^{-1}x_1$. By the triangle inequality,

$$|R_C(\xi_1) - R_C(\xi_2)| \leq \left|(R(\xi_1) - R(\xi_2))t_C(x_1'\Omega_{d,1}^{-1}x_1)\right| + \left|(t_C(x_1'\Omega_{d,1}^{-1}x_1) - t_C(x_2'\Omega_{d,2}^{-1}x_2))R(\xi_2)\right|. \tag{B.73}$$

We have (B.72) holds with $C_R = C_{R_1} + C_{R_2}$ if we can show that

$$\left|(R(\xi_1) - R(\xi_2))t_C(x_1'\Omega_{d,1}^{-1}x_1)\right| \leq C_{R_1} \left\|\xi_1 - \xi_2\right\|_s \tag{B.74}$$

and

$$\left|(t_C(x_1'\Omega_{d,1}^{-1}x_1) - t_C(x_2'\Omega_{d,2}^{-1}x_2))R(\xi_2)\right| \leq C_{R_2} \left\|\xi_1 - \xi_2\right\|_s \tag{B.75}$$

for some finite constants C_{R_1} and C_{R_2}.

We first consider (B.74). First, note that it holds trivially if $x_1'\Omega_{d,1}^{-1}x_1 > 2C$ because, in this case, $t_C(x_1'\Omega_{d,1}^{-1}x_1) = 0$. Second, given $x_2'\Omega_{d,2}^{-1}x_2 \leq x_1'\Omega_{d,1}^{-1}x_1 \leq 2C$, we deduce that

$$\left|(R(\xi_1) - R(\xi_2))t_C(x_1'\Omega_{d,1}^{-1}x_1)\right| \leq |R(\xi_1) - R(\xi_2)| \leq C_{R_1} \left\|\xi_1 - \xi_2\right\|_s, \tag{B.76}$$

where the first inequality is by property (i) of $t_C(u)$, and the second inequality is by Lemma B1(i).

Next, we show (B.75). First, note that it holds trivially if $2C < x_2'\Omega_{d,2}^{-1}x_2$. In this case,

$$t_C(x_1'\Omega_{d,1}^{-1}x_1) = t_C(x_2'\Omega_{d,2}^{-1}x_2) = 0 \tag{B.77}$$

following the definition of $t_C(u)$. Second, given $x_2'\Omega_{d,2}^{-1}x_2 \leq 2C$, we have

$$\left|(t_C(x_1'\Omega_{d,1}^{-1}x_1) - t_C(x_2'\Omega_{d,2}^{-1}x_2))R(\xi_2)\right| \leq |R(\xi_2)| \leq x_2'\Omega_{d,2}^{-1}x_2 \leq 2C. \tag{B.78}$$

Thus, (B.75) holds with $C_{R_2} = 1$ if $\|\xi_1 - \xi_2\|_s > 2C$. Third, it remains to consider the case where $x_2'\Omega_{d,2}^{-1}x_2 \leq 2C$ and $\|\xi_1 - \xi_2\|_s \leq 2C$. In this case, Lemma C3 in the Section C implies that $x_1'\Omega_{d,1}^{-1}x_1 \leq C^*$ for some finite constant C^*. In this case,

$$\begin{aligned} &\left|(t_C(x_1'\Omega_{d,1}^{-1}x_1) - t_C(x_2'\Omega_{d,2}^{-1}x_2))R(\xi_2)\right| \\ \leq &\left|t_C(x_1'\Omega_{d,1}^{-1}x_1) - t_C(x_2'\Omega_{d,2}^{-1}x_2)\right| x_2'\Omega_{d,2}^{-1}x_2 \leq 2C\left|x_1'\Omega_{d,1}^{-1}x_1 - x_2'\Omega_{d,2}^{-1}x_2\right| \end{aligned} \tag{B.79}$$

using $0 \leq R(\xi_2) \leq x_2'\Omega_{d,2}^{-1}x_2 \leq 2C$ and $|t_C(x_1) - t_C(x_2)| \leq C^{-1}|x_1 - x_2|$ which follows from property (ii) of $t_C(x)$. Then we can show $|x_1'\Omega_{d,1}^{-1}x_1 - x_2'\Omega_{d,2}^{-1}x_2| \leq C_{R_2}\|\xi_1 - \xi_2\|_S$ for some constant C_{R_2} by the same arguments that show (B.60) but with $x_1'\Omega_{d,1}^{-1}x_1 \leq 2C$ replaced by $x_1'\Omega_{d,1}^{-1}x_1 \leq C^*$. *Q.E.D.*

Lemma B2. *Given $L(\upsilon; d) \geq 0$, $\overline{L}_C(\upsilon; d)$ is bounded and Lipschitz in d.*

Proof of Lemma B2. The proof is analogous to that of Lemma SA4 with the truncation function $t_C(x'\Omega_d^{-1}x)$ replaced by $t_C(\upsilon' M_d \upsilon)I\{\|\upsilon\|^2 \leq C\}$ and Lemma B1(i) replaced by Lemma B1(ii), and hence is omitted. *Q.E.D.*

Proof of Lemma SA5. Since $0 \leq L(\upsilon; d) \leq \upsilon' M_d \upsilon$ for any $\upsilon \in \mathbb{R}^k$ and $ut_C(u) \leq C$ for any $u \geq 0$, we have

$$\overline{L}_C(\upsilon; d) = L(\upsilon; d)t_C(\upsilon' M_d \upsilon)I\{\|\upsilon\|^2 \leq C\} \leq L(\upsilon; d)t_C(\upsilon' M_d \upsilon) \leq C, \tag{B.80}$$

which implies that $c_{\alpha,C}(d)$ is bounded. For any υ, any d_1 and d_2, by Lemma B2 there exists a finite constant C_L such that

$$\left|\overline{L}_C(\upsilon; d_1) - \overline{L}_C(\upsilon; d_2)\right| \leq C_L \|d_1 - d_2\|_s. \tag{B.81}$$

Since $\overline{L}_C(\upsilon; d_1) \geq \overline{L}_C(\upsilon; d_2) - C_L \|d_1 - d_2\|_s$ for any υ and $P^*\left(\overline{L}_C(\upsilon^*; d_1) > c_{\alpha,C}(d_1)\right) \leq \alpha$, we have $P\left(\overline{L}_C(\upsilon^*; d_2) > c_{\alpha,C}(d_1) + C_L \|d_1 - d_2\|_s\right) \leq \alpha$, which implies that

$$c_{\alpha,C}(d_2) \leq c_{\alpha,C}(d_1) + C_L \|d_1 - d_2\|_s. \tag{B.82}$$

Similarly, we also have

$$c_{\alpha,C}(d_1) \leq c_{\alpha,C}(d_2) + C_L \|d_1 - d_2\|_s. \tag{B.83}$$

Combining (B.82) and (B.83), we get

$$|c_{\alpha,C}(d_1) - c_{\alpha,C}(d_2)| \leq C_L \|d_1 - d_2\|_s, \tag{B.84}$$

which shows the claim of the lemma. *Q.E.D.*

C Additional Auxiliary Lemmas

This section contains the proof of Lemma SA3 in Cheng, Dou, and Liao (2020) and some other auxiliary results.

PROOF OF LEMMA SA3. For any square matrices A_1 and A_2, let diag (A_1, A_2) denote the block diagonal matrix created by aligning the input matrices A_1 and A_2 along the diagonal. Since $\hat{\theta} \in \Theta$ is the minimizer of $g(\theta)'(\hat{\Omega}(\theta))^{-1}g(\theta)$,

$$\mathcal{T} \geq g(\hat{\theta})'\hat{\Omega}^{-1}g(\hat{\theta}) - g_0(\hat{\theta})'\hat{\Omega}_0^{-1}g_0(\hat{\theta}) = g(\hat{\theta})'\left(\hat{\Omega}^{-1} - \text{diag}\left(\hat{\Omega}_0^{-1}, 0_{k_1\times k_1}\right)\right)g(\hat{\theta}) \tag{C.85}$$

Let $\hat{\Omega}_{0,1}$ denote the upper-right $k_0 \times k_1$ submatrix of $\hat{\Omega}$ and $\hat{\Omega}_{1,0} \equiv \hat{\Omega}_{0,1}'$. Since $\hat{\Omega}$ is positive definite,

$$\hat{\Omega}(\hat{\Omega}^{-1} - \text{diag}(\hat{\Omega}_0^{-1}, 0_{k_1\times k_1}))\hat{\Omega} = \text{diag}(0_{k_0\times k_0}, \hat{\Omega}_1 - \hat{\Omega}_{1,0}\hat{\Omega}_0^{-1}\hat{\Omega}_{0,1}) \tag{C.86}$$

is a positive semi-definite matrix, which together with (C.85) proves the first claim of the lemma.

To prove the second claim, we first notice that $\hat{m}(\hat{\theta}) = g_0(\hat{\theta}) - \hat{V}(\hat{\theta})g(\hat{\theta}) = 0_{k_0\times 1}$ which implies that for any $\upsilon \in \mathbb{R}^k$

$$L(\upsilon; \hat{d}) \geq \upsilon'\hat{M}\upsilon - \upsilon'\hat{V}_S(\hat{\theta})'\hat{\Omega}_0^{-1}\hat{V}_S(\hat{\theta})\upsilon, \tag{C.87}$$

where $\hat{V}_S(\theta) \equiv S_0\hat{\Omega}^{1/2}\hat{M}$. Therefore

$$L(x; \hat{d}) \geq \upsilon'\hat{M}\hat{\Omega}^{1/2}\left(\hat{\Omega}^{-1} - \text{diag}(\hat{\Omega}_0^{-1}, 0_{k_1\times k_1})\right)\hat{\Omega}^{1/2}\hat{M}\upsilon \geq 0, \tag{C.88}$$

where the second inequality holds since the matrix in (C.86) is positive semi-definite. *Q.E.D.*

Lemma C3. *For any ξ_1 and ξ_2 with $x_2'\Omega_{d,2}^{-1}x_2 \leq C$ and $\|\xi_1 - \xi_2\|_s \leq C$, where C is a finite constant, we have $x_1\Omega_{d,1}^{-1}x_1 \leq C^*$ for some finite constant C^*, which depends the constant C of the lemma, C_Ω and c_λ in Assumptions 2(iv) and 3(iii) respectively.*

PROOF OF LEMMA C3. Since $\Omega_{d,1}^{-1}$ is symmetric and positive definite under Assumption 3(iii),

$$\begin{aligned}
x_1\Omega_{d,1}^{-1}x_1 &\leq 2x_2'\Omega_{d,1}^{-1}x_2 + 2(x_1 - x_2)'\Omega_{d,1}^{-1}(x_1 - x_2) \\
&= 2x_2'\Omega_{d,2}^{-1}x_2 + 2x_2'(\Omega_{d,1}^{-1} - \Omega_{d,2}^{-1})x_2 + 2(x_1 - x_2)'\Omega_{d,1}^{-1}(x_1 - x_2) \\
&\leq 2C + 2x_2'(\Omega_{d,1}^{-1} - \Omega_{d,2}^{-1})x_2 + 2(x_1 - x_2)'\Omega_{d,1}^{-1}(x_1 - x_2),
\end{aligned} \tag{C.89}$$

where the second inequality is by $x_2'\Omega_{d,2}^{-1}x_2 \leq C$ as assumed in the lemma. By Assumption 3(iii) and $\|\xi_1 - \xi_2\|_s \leq C$,

$$(x_1 - x_2)'\Omega_{d,1}^{-1}(x_1 - x_2) \leq (\lambda_{\min}(\Omega_{d,1}))^{-1}\|x_1 - x_2\|^2 \leq C^2c_\lambda^{-1}. \tag{C.90}$$

Similarly, by Assumption 3(iii) and $\|\xi_1 - \xi_2\|_s \leq C$,

$$\begin{aligned}\left|x_2'(\Omega_{d,1}^{-1} - \Omega_{d,2}^{-1})x_2\right|^2 &= \left|x_2'\Omega_{d,1}^{-1}(\Omega_{d,1} - \Omega_{d,2})\Omega_{d,2}^{-1}x_2\right|^2 \leq (x_2'\Omega_{d,1}^{-2}x_2)(x_2'\Omega_{d,2}^{-2}x_2)\left\|\Omega_{d,1} - \Omega_{d,2}\right\|^2 \\ &\leq \frac{\lambda_{\max}(\Omega_{d,1})(x_2'\Omega_{d,2}^{-1}x_2)^2}{\lambda_{\min}(\Omega_{d,2})(\lambda_{\min}(\Omega_{d,1}))^2}\left\|\Omega_{d,1} - \Omega_{d,2}\right\|^2 \leq C^4 c_\lambda^{-3} C_\Omega, \end{aligned} \tag{C.91}$$

where the last inequality is by $x_2'\Omega_{d,2}^{-1}x_2 \leq C$, Assumption 3(iii) and $\|\xi_1 - \xi_2\|_s \leq C$. The claim of the lemma follows from (C.89)-(C.91). *Q.E.D.*

Lemma C4. *For any ξ_1 and ξ_2, define $V_{d,S,j}(\theta) \equiv V_{d,j}(\theta)\,\Omega_{d,j}^{1/2} M_{d,j}$ for $j = 1, 2$. Then we have*

$$\sup_{\theta\in\Theta}\left\|V_{d,S,1}(\theta) - V_{d,S,2}(\theta)\right\| \leq C_\Omega^{1/2}(1 + c_\lambda^{-1}C_\Omega^{1/2} + c_\lambda^{-1}C_\Omega)\left\|\xi_1 - \xi_2\right\|_s, \tag{C.92}$$

where C_Ω and c_λ are in Assumptions 2(iv) and 3(iii) respectively.

Proof of Lemma C4. By definition,

$$\begin{aligned} V_{d,S,1}(\theta) - V_{d,S,2}(\theta) &= [V_{d,1}(\theta) - V_{d,2}(\theta)]\,\Omega_{d,1}^{1/2} M_{d,1} \\ &\quad + V_{d,2}(\theta)(\Omega_{d,1}^{1/2} - \Omega_{d,2}^{1/2})M_{d,1} + V_{d,2}(\theta)\Omega_{d,2}^{1/2}(M_{d,1} - M_{d,2}). \end{aligned} \tag{C.93}$$

By the properties of $\Omega_{d,j}$ and $M_{d,j}$,

$$\sup_{\theta\in\Theta}\left\|[V_{d,1}(\theta) - V_{d,2}(\theta)]\,\Omega_{d,1}^{1/2}M_{d,1}\right\| \leq C_\Omega^{1/2}\sup_{\theta\in\Theta}\left\|V_{d,1}(\theta) - V_{d,2}(\theta)\right\|. \tag{C.94}$$

By the properties of $V_{d,j}(\theta)$ and $M_{d,j}$, and Exercise 7.2.18 in Horn and Johnson (1990),

$$\begin{aligned}\sup_{\theta\in\Theta}\left\|V_{d,2}(\theta)(\Omega_{d,1}^{1/2} - \Omega_{d,2}^{1/2})M_{d,1}\right\| &\leq \sup_{\theta\in\Theta}\left\|V_{d,2}(\theta)\right\|\left\|\Omega_{d,1}^{1/2} - \Omega_{d,2}^{1/2}\right\|_S \\ &\leq c_\lambda^{-1}C_\Omega\left\|\Omega_{d,2}^{-1}\right\|_S\left\|\Omega_{d,1} - \Omega_{d,2}\right\| \leq c_\lambda^{-2}C_\Omega\left\|\Omega_{d,1} - \Omega_{d,2}\right\|. \end{aligned} \tag{C.95}$$

Similarly,

$$\sup_{\theta\in\Theta}\left\|V_{d,2}(\theta)\Omega_{d,2}^{1/2}(M_{d,1} - M_{d,2})\right\| \leq c_\lambda^{-1}C_\Omega^{3/2}\left\|M_{d,1} - M_{d,2}\right\|. \tag{C.96}$$

The the desirable result follows by (C.94)-(C.96) and the triangle inequality. *Q.E.D.*

D Derivation of Asset Pricing Moments in Examples

D.1 Equilibrium Solution of the Disaster Risk Model

The dividend-price ratio is constant in the equilibrium, denoted by C/P. The stock return is

$$R_t = (C_t + P_t)/P_{t-1} = (C/P + 1)C_t/C_{t-1} = (C/P + 1)e^{\sigma\varepsilon_t - \zeta_t}. \tag{D.1}$$

Thus, the expected log stock return is

$$\mathbb{E}_{t-1}\left[\ln R_t\right] = \ln(C/P + 1) - p\left(\underline{v} + 1/\alpha\right). \tag{D.2}$$

The equilibrium dividend-price ratio is

$$\begin{aligned} P/C &= \mathbb{E}_{t-1}\left[e^{-\delta}\left(C_t/C_{t-1}\right)^{-\gamma}\left(P_t/C_t + 1\right)C_t/C_{t-1}\right] \\ &= (P/C + 1)e^{-\delta}\mathbb{E}_{t-1}\left[\left(C_t/C_{t-1}\right)^{1-\gamma}\right] \\ &= (P/C + 1)e^{-\delta + \frac{1}{2}(1-\gamma)^2\sigma^2}\left[1 - p + pe^{-(1-\gamma)\underline{v}}\frac{\alpha}{\alpha - \gamma + 1}\right]. \end{aligned} \tag{D.3}$$

Thus, the term $\ln(C/P + 1)$ equals to

$$\begin{aligned} \ln(C/P + 1) &= \delta - \frac{1}{2}(1-\gamma)^2\sigma^2 - \ln\left[1 - p + pe^{-(1-\gamma)\underline{v}}\frac{\alpha}{\alpha - \gamma + 1}\right] \\ &\approx \delta - \frac{1}{2}(1-\gamma)^2\sigma^2 + p - pe^{-(1-\gamma)\underline{v}}\frac{\alpha}{\alpha - \gamma + 1}. \end{aligned} \tag{D.4}$$

Define the log return as $r_t \equiv \ln R_t$. Equations (D.4) and (D.2) imply the following relation:

$$\mathbb{E}_{t-1}\left[r_t\right] \approx \delta - \frac{1}{2}(1-\gamma)^2\sigma^2 - p\left(\underline{v} + 1/\alpha\right) + p - pe^{(\gamma-1)\underline{v}}\frac{\alpha}{\alpha - \gamma + 1}. \tag{D.5}$$

The equilibrium risk-free rate satisfies that

$$R_f = \mathbb{E}_{t-1}\left[e^{-\delta}(C_t/C_{t-1})^{-\gamma}\right]^{-1} = e^{\delta - \frac{1}{2}\gamma^2\sigma^2}\left[1 - p + pe^{\gamma\underline{v}}\frac{\alpha}{\alpha - \gamma}\right]^{-1}. \tag{D.6}$$

Define the log risk-free rate as $r_f \equiv \ln R_f$. Thus, the log risk-free rate satisfies

$$\begin{aligned} r_f &= \delta - \frac{1}{2}\gamma^2\sigma^2 - \ln\left[1 - p + pe^{\gamma\underline{v}}\frac{\alpha}{\alpha - \gamma}\right] \\ &\approx \delta - \frac{1}{2}\gamma^2\sigma^2 + p - pe^{\gamma\underline{v}}\frac{\alpha}{\alpha - \gamma}. \end{aligned}$$

The excess log return, defined as $r_t^e \equiv r_t - r_f$, is

$$\mathbb{E}_{t-1}\left[r_t^e\right] = \gamma\sigma^2 - \frac{1}{2}\sigma^2 - p(\underline{v} + 1/\alpha) + p\alpha\left[\frac{e^{\gamma\underline{v}}}{\alpha-\gamma} - \frac{e^{(\gamma-1)\underline{v}}}{\alpha-\gamma+1}\right]. \tag{D.7}$$

The excess log return r_t^e has the same conditional exposure to the shocks as the log return r_t since the log risk-free rate r_f has zero exposure to the shocks. Therefore, the excess log return in the equilibrium can be represented as follows:

$$r_t^e = \gamma\sigma^2 - \frac{1}{2}\sigma^2 - p(\underline{v} + 1/\alpha) + p\alpha\left[\frac{e^{\gamma\underline{v}}}{\alpha-\gamma} - \frac{e^{(\gamma-1)\underline{v}}}{\alpha-\gamma+1}\right] + \varepsilon_{\ell,t}^e, \tag{D.8}$$

where $\varepsilon_{\ell,t}^e \equiv \sigma\varepsilon_t - [x_t(\underline{v} + J_t) - p(\underline{v} + 1/\alpha)] + \sigma_r\varepsilon_{\ell,t}$. The shock of the consumption growth is ε_t, the jump shock of the consumption growth is $-[x_t(\underline{v} + J_t) - p(\underline{v} + 1/\alpha)]$, and the shock of the measurement error is $\varepsilon_{\ell,t}$. The measurement error $\varepsilon_{\ell,t}$ is i.i.d. standard normal and is independent of other shocks.

D.2 Equilibrium Solution of the Long-Run Risk Model

The stochastic discount factor (SDF) can be expressed as follows:

$$M_t = \delta^{\vartheta}\left(\frac{C_t}{C_{t-1}}\right)^{-\vartheta/\psi} R_{c,t}^{\vartheta-1}, \quad \text{with } \vartheta \equiv \frac{1-\gamma}{1-1/\psi}, \tag{D.9}$$

where $R_{c,t}$ is the return on the consumption claim. The log SDF can be written as

$$m_t = \vartheta\log\delta - \frac{\vartheta}{\psi}\Delta c_t + (\vartheta - 1)r_t. \tag{D.10}$$

The state variable in the simplest long-run risk model is x_t. To turn the system into an affine model, we first exploit the Campbell-Shiller log-linearization approximation:

$$r_t = \kappa_0 + \kappa_1 z_t + \Delta c_t - z_{t-1}, \tag{D.11}$$

where $z_{t-1} = \ln(W_{t-1}/C_{t-1})$ is the log wealth-consumption ratio and wealth is the "price" of consumption claims. The log-linearization constants are determined by long-run steady state:

$$\kappa_0 = \ln(1 + e^{\overline{z}}) - \kappa_1\overline{z} \quad \text{and} \quad \kappa_1 = \frac{e^{\overline{z}}}{1 + e^{\overline{z}}}, \tag{D.12}$$

where $\overline{z}$ is the mean of the log price-consumption ratio. Given the log-linearization approximation

(D.11) – (D.12), we can search the equilibrium characterized by

$$z_t = A_0 + A_1 x_t, \tag{D.13}$$

where the constants A_0 and A_1 are to be determined by the equilibrium conditions.

Thus, the log return on consumption claim can be written as

$$r_t = \kappa_0 + \kappa_1 \left(A_0 + A_1 x_t\right) + \Delta c_t - \left(A_0 + A_1 x_{t-1}\right). \tag{D.14}$$

Therefore, the log SDF can be re-written in terms of state variables and exogenous shocks

$$m_t = \Gamma_0 + \Gamma_1 x_{t-1} - \lambda_c \sigma_c \epsilon_{c,t} - \lambda_x \phi \epsilon_{x,t}, \tag{D.15}$$

where predictive coefficients are

$$\Gamma_0 = \ln \delta \quad \text{and} \quad \Gamma_1 = -\psi^{-1}, \tag{D.16}$$

and the market price of risk coefficients are

$$\lambda_c = \gamma \quad \text{and} \quad \lambda_x = \left(\gamma - \psi^{-1}\right) \frac{\kappa_1 \phi}{1 - \kappa_1 \rho}. \tag{D.17}$$

The coefficients A_j's are determined by the equilibrium condition (i.e., the Euler equation for the price of consumption claim) as follows:

$$1 = \mathbb{E}_{t-1}\left[M_t R_{c,t}\right] = \mathbb{E}_{t-1}\left[e^{m_t + r_t}\right], \tag{D.18}$$

where $\mathbb{E}_{t-1}[\cdot]$ denote the expectation given the information at time $t-1$. It leads to the equilibrium conditions:

$$A_0 = \frac{1}{1-\kappa_1}\left(\ln \delta + \kappa_0\right) \quad \text{and} \quad A_1 = \frac{1-\psi^{-1}}{1-\kappa_1 \rho}\phi. \tag{D.19}$$

The long-run mean $\overline{z}$ is also determined endogenously in the equilibrium. In the long-run steady state, we have

$$\overline{z} = A_0. \tag{D.20}$$

We first derive κ_1 in equilibrium. After taking log on the both sides of (D.12) and plugging (D.19) and (D.20) into the equation, we can obtain the following relation:

$$\begin{aligned} \ln \kappa_1 &= \overline{z} - \ln(1 + e^{\overline{z}}) = \overline{z} - \kappa_0 + \kappa_1 \overline{z} \\ &= (1 - \kappa_1)\overline{z} - \kappa_0 = \ln \delta. \end{aligned}$$

Thus, the equilibrium log-linearization coefficient is equal to the representative agent's time preference parameter; that is, $\kappa_1 = \delta$, in equilibrium.

From (D.14) and (D.19), it follows that

$$r_t - \mathbb{E}_{t-1}[r_t] = \beta_c \sigma_c \epsilon_{c,t} + \beta_x \epsilon_{x,t}, \tag{D.21}$$

where the betas are

$$\beta_c = 1 \quad \text{and} \quad \beta_x = \kappa_1 A_1. \tag{D.22}$$

The Euler equation for the log market return, denoted by r_t, and the risk free rate, denoted by $r_{f,t-1}$, can be written in one equation

$$\mathbb{E}_{t-1}\left[e^{m_t}\right] = \mathbb{E}_{t-1}\left[e^{m_t + r_t^e}\right], \tag{D.23}$$

where $r_t^e \equiv r_t - r_{f,t-1}$ is the excess log return.

The Euler equation (D.23) leads to

$$\begin{aligned}
\mathbb{E}_{t-1}[r_t^e] &= \lambda_c \beta_c \sigma_c^2 + \lambda_x \beta_x - \frac{1}{2}\left(\beta_c^2 \sigma_c^2 + \beta_x^2 \phi^2\right) \\
&= \gamma \sigma_c^2 - \frac{1}{2}\sigma_c^2 + \frac{1}{2}(2\gamma - \psi^{-1} - 1)(1 - \psi^{-1})\frac{\kappa_1^2 \phi^2}{(1 - \kappa_1 \rho)^2} \qquad \text{(D.24)} \\
&= \gamma \sigma_c^2 - \frac{1}{2}\sigma_c^2 + \frac{1}{2}(2\gamma - \psi^{-1} - 1)(1 - \psi^{-1})\frac{\phi^2}{(\delta^{-1} - \rho)^2}. \qquad \text{(D.25)}
\end{aligned}$$

The excess log return r_t^e has the same conditional exposure to the shocks as the log return r_t since the log risk-free rate $r_{f,t-1}$ has zero conditional exposure to the shocks $\varepsilon_{c,t}$ and $\varepsilon_{x,t}$. Therefore, combining (D.19), (D.22), and (D.25), we can obtain that the excess log return in the equilibrium can be represented as follows:

$$r_t^e = \gamma \sigma_c^2 - \frac{1}{2}\sigma_c^2 + \frac{1}{2}(2\gamma - \psi^{-1} - 1)(1 - \psi^{-1})\frac{\phi^2}{(\delta^{-1} - \rho)^2} + \varepsilon_{d,t}^e, \tag{D.26}$$

where $\varepsilon_{d,t}^e \equiv \sigma_c \varepsilon_{c,t} + (1 - \psi^{-1})(1 - \rho)^{-1} \phi \varepsilon_{x,t} + \sigma_r \varepsilon_{d,t}$. The shock of the contemporaneous consumption growth is $\varepsilon_{c,t}$, the shock of the low-frequency component is $\varepsilon_{x,t}$, and the shock of the measurement error is $\varepsilon_{d,t}$. The measurement error $\varepsilon_{d,t}$ is i.i.d. standard normal and is independent of other shocks.

E Gaussian Limit in the Long-Run Risk Model

For the long-run risk model, we show that the Gaussian limit for the moment conditions is an innocuous assumption even if $\rho_n \to 1$ and $\phi_n \to 0$ at any rate, as long as $\theta_n = (1 - \rho_n)/\phi_n$ is bounded from above and below by some finite positive constants, where the subscript n denotes the sample size n.

We start with the first moment condition. The observed process Δc_t satisfies

$$\begin{aligned}\Delta c_{t+1} - \rho \Delta c_t &= \phi_n \left(\rho_n x_{t-1} + \epsilon_{x,t}\right) + \sigma_c \epsilon_{c,t+1} - \rho \left(\phi_n x_{t-1} + \sigma_c \epsilon_{c,t}\right) \\ &= \phi_n(\rho_n - \rho) x_{t-1} + \phi_n \epsilon_{x,t} + \sigma_c \epsilon_{c,t+1} - \rho \sigma_c \epsilon_{c,t}. \end{aligned} \tag{E.1}$$

Multiply the first difference $\Delta c_{t+1} - \rho \Delta c_t$ in (E.1) by $\Delta c_{t-1} = \phi_n x_{t-2} + \sigma_c \epsilon_{c,t-1}$. We obtain

$$M_{1t} \equiv \Delta c_{t-1} \left(\Delta c_{t+1} - \rho \Delta c_t\right) = M_{1a,t} + M_{1b,t} + M_{1c,t} + M_{1d,t} + \mu_1, \tag{E.2}$$

where

$$\begin{aligned} \mu_1 &\equiv \mathbb{E}\left[\Delta c_{t-1} \left(\Delta c_{t+1} - \rho \Delta c_t\right)\right] = \phi_n^2 (\rho_n - \rho) \rho_n \mathbb{E}[x_{t-2}^2], \\ M_{1a,t} &\equiv \phi_n^2 (\rho_n - \rho) \rho_n \left(x_{t-2}^2 - \mathbb{E}[x_{t-2}^2]\right), \\ M_{1b,t} &\equiv \phi_n (\rho_n - \rho) \left(\sigma_c x_{t-1} \epsilon_{c,t-1} + \phi_n x_{t-2} \varepsilon_{x,t-1}\right), \\ M_{1c,t} &\equiv \phi_n x_{t-2} \left(\phi_n \epsilon_{x,t} + \sigma_c \epsilon_{c,t+1} - \rho \sigma_c \epsilon_{c,t}\right), \\ M_{1d,t} &\equiv \sigma_c \epsilon_{c,t-1} \left(\phi_n \epsilon_{x,t} + \sigma_c \epsilon_{c,t+1} - \rho \sigma_c \epsilon_{c,t}\right). \end{aligned} \tag{E.3}$$

The moment condition satisfies

$$M_1 \equiv n^{-1/2} \sum_{t=1}^{n} \left(M_{1t} - \mathbb{E}\left[M_{1t}\right]\right) = M_{1a} + M_{1b} + M_{1c} + M_{1d}, \tag{E.4}$$

where $M_{1j} \equiv n^{-1/2} \sum_{t=1}^{n} M_{1j,t}$ for $j = a, b, c, d$.

Below we consider three separate cases: (i) ρ_n is bounded away from 1; (ii) ρ_n converges to 1 at the rate slower than n^{-1}, i.e., $(1 - \rho_n)n \to \infty$; (iii) ρ_n converges to 1 at the rate n^{-1} or faster, i.e., $(1 - \rho_n)n \to c \in [0, \infty)$. In all three cases, $\theta_n = (1 - \rho_n)/\phi_n$ is bounded below from 0 and above from infinity. Therefore, ϕ_n always converges to 0 at the same rate at which ρ_n converges to 1.

In case (i), we can apply the central limit theorem (CLT) for weakly dependent triangular arrays and M_1 has a Gaussian limit.

Now we consider case (ii). We assume the initial condition satisfies $\mathbb{E}[x_0^2] = o(n^{1/2})$ as in GP. Following Lemma 1 of Giraitis and Phillips (2006) (hereafter GP),

$$\frac{(1 - \rho_n^2)^{1/2}}{n^{1/2}} \sum_{t=1}^{n} x_{t-1} \epsilon_{x,t} \to_d N(0, 1) \tag{E.5}$$

using $Var(\epsilon_{x,t}) = 1$. We assume the initial condition satisfies $\mathbb{E}[x_0^2] = o(n^{1/2})$ as in GP. Following the proof for Lemma 1 of GP we also have

$$\frac{(1-\rho_n^2)^{1/2}}{n^{1/2}} \sum_{t=1}^{n} x_{t-1}\epsilon_{x,t+1} \to_d N(0,1),$$
$$\frac{(1-\rho_n^2)^{1/2}}{n^{1/2}} \sum_{t=1}^{n} x_{t-1}\epsilon_{c,t+j} \to_d N(0,\sigma_c^2), \tag{E.6}$$

for any j following the independence of $\epsilon_{x,t}$ and $\epsilon_{c,t'}$ for any $t \neq t'$. Following equation (20) in the proof of Lemma 2 of GP,

$$n^{-1/2} \sum_{t=1}^{n} \left(x_{t-1}^2 - \mathbb{E}[x_{t-1}^2]\right) = \left(1-\rho_n^2\right)^{-3/2} 2\rho_n Z_1 + \left(1-\rho_n^2\right)^{-1} Z_2 + Z_3, \tag{E.7}$$

where

$$Z_1 \equiv \frac{(1-\rho_n^2)^{1/2}}{n^{1/2}} \sum_{t=1}^{n} x_{t-1}\epsilon_{x,t} \to_d N(0,1),$$
$$Z_2 \equiv n^{-1/2} \sum_{t=1}^{n} \left(\epsilon_{x,t}^2 - 1\right) \to_d N(0, V_{x^2}),$$
$$Z_3 \equiv \frac{x_0^2 - \mathbb{E}[x_0^2]}{n^{1/2}\left(1-\rho_n^2\right)} - \frac{x_n^2 - \mathbb{E}[x_n^2]}{n^{1/2}\left(1-\rho_n^2\right)}, \tag{E.8}$$

where the convergence for Z_1 holds by (E.5), the convergence for Z_2 follows from the CLT with V_{x^2} being the variance of $\epsilon_{x,t}^2$. Then, we have $M_{1a} = o_p(1)$ because

$$\phi_n^2 \left(1-\rho_n^2\right)^{-3/2} 2\rho_n Z_1 = o_p(1),$$
$$\phi_n^2 \left(1-\rho_n^2\right)^{-1} Z_2 = o_p(1),$$
$$\phi_n^2 Z_3 = o_p(1), \tag{E.9}$$

which in turn holds because $\theta_n = (1-\rho_n)/\phi_n$ is bounded, $x_0^2 = o_p(n^{1/2})$ by the initial condition, and $\phi_n^2 \left(1-\rho_n^2\right)^{-1} n^{-1/2} x_n^2 = o_p(1)$ by Lemma 3 of GP and the Markov inequality.

The other terms satisfy

$$M_{1b} = \phi_n O_p((1-\rho_n^2)^{-1/2}) = o_p(1),$$
$$M_{1c} = \phi_n O_p((1-\rho_n^2)^{-1/2}) = o_p(1), \tag{E.10}$$

by $\phi_n = O((1-\rho_n))$, (E.5) and (E.6). Then we can apply the CLT for triangular array of i.i.d. random variables to M_{1d} and the Gaussian limit holds for M_1.

Finally, we consider the case (iii), where $n(1-\rho_n) \to c$ for $c \in [0, \infty)$. Following Phillips (1987),

$$n^{-2} \sum_{t=1}^{n} x_{t-1}^2 = O_p(1), \;\; n^{-1} \sum_{t=1}^{n} x_{t-1}\epsilon_{x,t} = O_p(1). \tag{E.11}$$

We also have

$$n^{-1} \sum_{t=1}^{n} x_{t-1}\epsilon_{x,t+1} = O_p(1) \quad \text{and} \quad n^{-1} \sum_{t=1}^{n} x_{t-1}\epsilon_{c,t+j} = O_p(1) \tag{E.12}$$

for any j, because $\epsilon_{x,t}$ is i.i.d. with mean zero and it is independent of $\epsilon_{c,t+j}$, following results for a vector process, see Park and Phillips (1988). Therefore,

$$\begin{aligned} M_{1a} &= \phi_n^2 \left[O_p(n^{3/2}) - O(n^{1/2})(1-\rho_n^2)^{-1} \right] = o_p(1), \\ M_{1b} &= \phi_n O_p(n^{1/2}) = o_p(1) \quad \text{and} \quad M_{1c} = \phi_n O_p(n^{1/2}) = o_p(1), \end{aligned} \tag{E.13}$$

because $\phi_n = O(n^{-1})$. As in case (ii), we can apply the CLT to M_{1d} to obtain the Gaussian limit for M_1.

Next, we consider multiplying the first difference $\Delta c_{t+1} - \rho \Delta c_t$ in (E.1) by $\Delta c_t = \phi_n x_{t-1} + \sigma_c \epsilon_{c,t}$. We obtain

$$M_{2,t} \equiv \Delta c_t \left(\Delta c_{t+1} - \rho \Delta c_t \right) = M_{2a,t} + M_{2b,t} + M_{2c,t} + M_{2d,t} + \mu_2, \tag{E.14}$$

where

$$\begin{aligned} \mu_2 &\equiv \mathbb{E}\left[\Delta c_t \left(\Delta c_{t+1} - \rho \Delta c_t \right) \right] = \phi_n^2 (\rho_n - \rho) \mathbb{E}[x_{t-1}^2] - \rho \sigma_c^2, \\ M_{2a,t} &\equiv \phi_n^2 (\rho_n - \rho)(x_{t-1}^2 - \mathbb{E}[x_{t-1}^2]), \\ M_{2b,t} &\equiv \phi_n (\rho_n - \rho) \sigma_c x_{t-1} \epsilon_{c,t}, \\ M_{2c,t} &\equiv \phi_n x_{t-1} \left(\phi_n \epsilon_{x,t} + \sigma_c \epsilon_{c,t+1} - \rho \sigma_c \epsilon_{c,t} \right), \\ M_{2d,t} &\equiv \sigma_c \epsilon_{c,t} \left(\phi_n \epsilon_{x,t} + \sigma_c \epsilon_{c,t+1} - \rho \sigma_c \epsilon_{c,t} \right). \end{aligned} \tag{E.15}$$

As above, we consider three cases. In case (i) where ρ is bounded away from 1, we can apply the CLT directly. In case (ii),

$$M_{2a} = n^{-1/2} \sum_{t=1}^{n} M_{2a,t} = \phi_n^2 (\rho_n - \rho) n^{-1/2} \sum_{t=1}^{n} \left(x_{t-1}^2 - \mathbb{E}[x_{t-1}^2] \right) = o_p(1) \tag{E.16}$$

following the same arguments for M_{1a}. Similarly, we can show that M_{2b} and M_{2c}, defined similarly to M_{1b} and M_{1c} respectively, are both $o_p(1)$ following the arguments for M_{1b} and M_{1c}. Finally, the CLT always applies to M_{2d}, the counterpart of M_{1d}. In case (iii), same arguments for M_{1a} give

$$M_{2a} = \phi_n^2 \left[O_p(n^{3/2}) - O(n^{1/2})(1-\rho_n^2)^{-1} \right] = o_p(1). \tag{E.17}$$

Arguments for M_{1b}, M_{1c} and M_{1d} can be used to show the same results hold for their counterparts M_{2b}, M_{2c} and M_{2d} respectively.

Finally, the asset pricing moment condition always has a Gaussian limit because it is a location model with an i.i.d. error.

F Derivations for the Empirical Application

We first verify the baseline moment conditions since they do not depend on the model solution. Then, we solve the model and derive the equilibrium relations. Lastly, we verify the asset pricing moment conditions.

Verifying the Baseline Moment Conditions. We now verify the baseline moment conditions. The first moment condition is:

$$\begin{aligned}\mathbb{E}\left[(\Delta c_t - g_c) + p\mu_1(\alpha)\right] &= \mathbb{E}\left[\sigma_c \varepsilon_{c,t+1} - \zeta_{t+1}\right] + p\mu_1(\alpha) \\ &= -\mathbb{E}\left[x_{t+1}(\underline{v} + J_{t+1})\right] + p\mu_1(\alpha) \\ &= -\mathbb{E}\left[p_t\right]\mu_1(\alpha) + p\mu_1(\alpha) = 0. \end{aligned} \tag{F.1}$$

The second moment condition can be derived similarly.

The third moment condition can also be easily verified as follows:

$$\begin{aligned}\mathbb{E}\left[(\Delta c_t - g_c)^2 - \sigma_c^2 - p\mu_2(\alpha)\right] &= \mathbb{E}\left[(\sigma_c \varepsilon_{c,t+1} - \zeta_{t+1})^2 - \sigma_c^2 - p\mu_2(\alpha)\right] \\ &= \mathbb{E}\left[\sigma_c^2 \varepsilon_{c,t+1}^2 + x_{t+1}(\underline{v} + J_{t+1})^2\right] - \sigma_c^2 - p\mu_2(\alpha) \\ &= \mathbb{E}\left[p_t\right]\mu_2(\alpha) - p\mu_2(\alpha) = 0. \end{aligned} \tag{F.2}$$

The fourth moment condition can be verified in the same way.

The fifth moment condition is verified below.

$$\begin{aligned}&\mathbb{E}\left[\Delta c_{t-1}\left[\Delta c_{t+1} - \rho \Delta c_t + (1-\rho)(p\mu_1(\alpha) - g_c)\right]\right] \\ &= \mathbb{E}\left[\Delta c_{t-1}\left[(\sigma_c \varepsilon_{c,t+1} - \zeta_{t+1}) - \rho(\sigma_c \varepsilon_{c,t} - \zeta_t) + (1-\rho)p\mu_1(\alpha)\right]\right]. \end{aligned} \tag{F.3}$$

The law of iterated expectation further leads to

$$\begin{aligned}&\mathbb{E}\left[\Delta c_{t-1}\left[(\sigma_c \varepsilon_{c,t+1} - \zeta_{t+1}) - \rho(\sigma_c \varepsilon_{c,t} - \zeta_t) + (1-\rho)p\mu_1(\alpha)\right]\right] \\ &= -\mathbb{E}\left[\Delta c_{t-1}\mathbb{E}_{t-1}\left[\zeta_{t+1} - \rho\zeta_t - (1-\rho)p\mu_1(\alpha)\right]\right] \\ &= -\mathbb{E}\left[\Delta c_{t-1}\mathbb{E}_{t-1}\left[x_{t+1}(\underline{v} + J_{t+1}) - \rho x_t(\underline{v} + J_t) - (1-\rho)p\mu_1(\alpha)\right]\right]. \end{aligned} \tag{F.4}$$

Applying again the law of iterated expectation leads to

$$\begin{aligned}
&\mathbb{E}\left[\Delta c_{t-1}\mathbb{E}_{t-1}\left[x_{t+1}(\underline{v}+J_{t+1})-\rho x_t(\underline{v}+J_t)-(1-\rho)p\mu_1(\alpha)\right]\right]\\
&=\mathbb{E}\left[\Delta c_{t-1}\mathbb{E}_{t-1}\left[\mathbb{E}_t\left[x_{t+1}(\underline{v}+J_{t+1})\right]-\rho p_{t-1}\mu_1(\alpha)-(1-\rho)p\mu_1(\alpha)\right]\right]\\
&=\mathbb{E}\left[\Delta c_{t-1}\mathbb{E}_{t-1}\left[p_t-\rho p_{t-1}-(1-\rho)p\right]\right]\mu_1(\alpha)\\
&=\mathbb{E}\left[\Delta c_{t-1}\mathbb{E}_{t-1}\left[\sigma_p p\varepsilon_{p,t}\right]\right]\mu_1(\alpha)=0. \qquad \text{(F.5)}
\end{aligned}$$

The sixth moment condition can be verified in the same way.

Model Solution. We now derive the equilibrium of the model. Because the EIS coefficient is one, the first-order condition of optimal consumption leads to $C_t=(1-\delta)W_t$. Because of the homotheticity of the preference, it is natural to conjecture that

$$V_t=\mathcal{I}(p_t)C_t, \qquad \text{(F.6)}$$

where $\mathcal{I}(p_t)$ is a deterministic function of p_t, capturing the marginal value of net worth. The specification of the dynamics is consistent with the exponential-affine models, and thus, we further conjecture that

$$\mathcal{I}(p_t)=e^{I_0+I_1p_t}, \qquad \text{(F.7)}$$

with constants I_0 and I_1 to be determined by equilibrium conditions.

The constants I_0 and I_1 can be solved by plugging (F.6) and (F.7) into the recursive value function relation. Specifically, it holds that

$$I_0+I_1p_t+\ln C_t=(1-\delta)\ln C_t+(1-\gamma)^{-1}\delta\ln\mathbb{E}_t\left[e^{(1-\gamma)(I_0+I_1p_{t+1})}C_{t+1}^{1-\gamma}\right]. \qquad \text{(F.8)}$$

The relation above can be rewritten as

$$\begin{aligned}
I_0+I_1p_t=\;&\delta\left[I_0+I_1(1-\rho)p+I_1\rho p_t+g_c\right]+\frac{1}{2}\delta(1-\gamma)\left(I_1^2\sigma_p^2p^2+\sigma_c^2\right)\\
&+(1-\gamma)^{-1}\delta\left[\mathbf{1}_{\{p_t\geq 0\}}p_t\ell(\alpha,\gamma-1)-\mathbf{1}_{\{p_t<0\}}p_t\ell(\alpha,1-\gamma)\right], \qquad \text{(F.9)}
\end{aligned}$$

with $\ell(\alpha,x)\equiv e^{x\underline{v}}\dfrac{\alpha}{\alpha-x}-1$.

Because $\mathbf{1}_{\{p_t<0\}}p_t\approx 0$ under the relevant calibrations, we can rewrite (F.9) as

$$\begin{aligned}
I_0+I_1p_t\approx\;&\delta\left[I_0+I_1(1-\rho)p+I_1\rho p_t+g_c\right]+\frac{1}{2}\delta(1-\gamma)\left(I_1^2\sigma_p^2p^2+\sigma_c^2\right)\\
&+(1-\gamma)^{-1}\delta\mathbf{1}_{\{p_t\geq 0\}}p_t\ell(\alpha,\gamma-1). \qquad \text{(F.10)}
\end{aligned}$$

By matching the constant term and p_t term, we obtain that

$$I_1 \approx \delta I_1 \rho + (1-\gamma)^{-1} \delta \ell(\alpha, \gamma - 1) \tag{F.11}$$

$$I_0 \approx \delta I_0 + \delta I_1 (1-\rho) p + \delta g_c + \frac{1}{2} \delta (1-\gamma) \left(I_1^2 \sigma_p^2 p^2 + \sigma_c^2 \right). \tag{F.12}$$

Equation (F.11) has one root, which leads to the solution for I_1 in equilibrium:

$$I_1 = \frac{\ell(\alpha, \gamma - 1)}{(1-\gamma)(\delta^{-1} - \rho)}. \tag{F.13}$$

Thus, after plugging (F.13) into (F.12), we can obtain the solution for I_0 in equilibrium:

$$I_0 = \frac{\delta}{1-\delta} \left[I_1 (1-\rho) p + g_c + \frac{1}{2} (1-\gamma)(I_1^2 \sigma_p^2 p^2 + \sigma_c^2) \right]. \tag{F.14}$$

The equilibrium stochastic discount factor (SDF) is

$$M_{t+1} = \delta \left(\frac{C_{t+1}}{C_t} \right)^{-1} \frac{V_{t+1}^{1-\gamma}}{\mathbb{E}_t \left[V_{t+1}^{1-\gamma} \right]}. \tag{F.15}$$

Thus, by combining (F.6), (F.7), (F.13), and (F.14), the SDF expression in (F.15) can be rewritten as follows:

$$\begin{aligned} M_{t+1} &= \delta \left(\frac{C_{t+1}}{C_t} \right)^{-\gamma} \frac{e^{(1-\gamma)(I_0 + I_1 p_{t+1})}}{\mathbb{E}_t \left[e^{(1-\gamma)(I_0 + I_1 p_{t+1})} (C_{t+1}/C_t)^{1-\gamma} \right]} \\ &= \delta \left(\frac{C_{t+1}}{C_t} \right)^{-\gamma} \frac{e^{(1-\gamma)[I_0 + I_1(1-\rho)p + I_1 \rho p_t + I_1 \sigma_p p \varepsilon_{p,t+1}]}}{\mathbb{E}_t \left[e^{(1-\gamma)(I_0 + I_1 p_{t+1})} (C_{t+1}/C_t)^{1-\gamma} \right]}. \end{aligned} \tag{F.16}$$

Here, the expected value function with power $1-\gamma$ can be computed as follows:

$$\begin{aligned} &\mathbb{E}_t \left[e^{(1-\gamma)(I_0 + I_1 p_{t+1})} (C_{t+1}/C_t)^{1-\gamma} \right] \\ &\quad = \mathbb{E}_t \left[e^{(1-\gamma)(I_0 + I_1(1-\rho)p + I_1 \rho p_t + I_1 \sigma_p p \varepsilon_{p,t+1} + g_c + \sigma_c \varepsilon_{c,t+1} - \zeta_{t+1})} \right] \\ &\quad = e^{(1-\gamma)(I_0 + I_1(1-\rho)p + I_1 \rho p_t + g_c) + \frac{1}{2}(1-\gamma)^2 (I_1^2 \sigma_p^2 p^2 + \sigma_c^2)} \mathbb{E}_t \left[e^{-(1-\gamma)\zeta_{t+1}} \right] \\ &\quad = e^{(1-\gamma)(I_0 + I_1(1-\rho)p + I_1 \rho p_t + g_c) + \frac{1}{2}(1-\gamma)^2 (I_1^2 \sigma_p^2 p^2 + \sigma_c^2)} \left[1 + p_t \ell(\alpha, \gamma - 1) \right]. \end{aligned} \tag{F.17}$$

After plugging in the equilibrium value function and rearranging the terms, we get the log SDF, denoted by $m_{t+1} \equiv \ln M_{t+1}$ as follows:

$$\begin{aligned} m_{t+1} = \ln \delta - \gamma(g_c + \sigma_c \varepsilon_{c,t+1} - \zeta_{t+1}) + (1-\gamma) I_1 \sigma_p p \varepsilon_{p,t+1} \\ - (1-\gamma) g_c - \frac{1}{2}(1-\gamma)^2 (I_1^2 \sigma_p^2 p^2 + \sigma_c^2) - p_t \ell(\alpha, \gamma - 1). \end{aligned} \tag{F.18}$$

Re-arranging terms leads to

$$m_{t+1} = \Gamma_0 + \Gamma_1 p_t - \lambda_c \sigma_c \varepsilon_{c,t+1} - \lambda_p \sigma_p p \varepsilon_{p,t+1} + \lambda_\zeta \zeta_{t+1}, \tag{F.19}$$

where the predictive coefficients are

$$\begin{aligned} \Gamma_0 &= \ln \delta - g_c - \frac{1}{2}(1-\gamma)^2 \left(I_1^2 \sigma_p^2 p^2 + \sigma_c^2 \right) \\ \Gamma_1 &= -\ell(\alpha, \gamma - 1), \end{aligned} \tag{F.20}$$

and the loading coefficients are

$$\lambda_c = \gamma, \quad \lambda_p = (\gamma - 1) I_1, \quad \text{and } \lambda_\zeta = \gamma. \tag{F.21}$$

The log risk-free rate, denoted by $r_{f,t} = -\ln \mathbb{E}_t [M_{t+1}]$, is

$$\begin{aligned} r_{f,t} &= -\ln \mathbb{E}_t \left[e^{m_{t+1}} \right] \\ &= -\Gamma_0 - \frac{1}{2} \left(\lambda_c^2 \sigma_c^2 + \lambda_p^2 \sigma_p^2 p^2 \right) - \left[\Gamma_1 + \ell(\alpha, \lambda_\zeta) \right] p_t && \text{(F.22)} \\ &= -\ln \delta + g_c - \frac{1}{2}(2\gamma - 1)\sigma_c^2 - \left[\ell(\alpha, \gamma) - \ell(\alpha, \gamma - 1) \right] p_t. && \text{(F.23)} \end{aligned}$$

Using the Campbell-Shiller decomposition and log-linearization, we can represent the log return in terms of log price-dividend ratio and log dividend growth:

$$r_{m,t+1} = \kappa_{m,0} + \kappa_{m,1} z_{m,t+1} + \Delta d_{t+1} - z_{m,t}, \tag{F.24}$$

where

$$\kappa_{m,0} = \ln(1 + e^{\overline{z}_m}) - \kappa_{m,1} \overline{z}_m \tag{F.25}$$

and

$$\kappa_{m,1} = \frac{e^{\overline{z}_m}}{1 + e^{\overline{z}_m}} \tag{F.26}$$

and $\overline{z}_m$ is long-run mean of market log price-dividend ratio.

Using the log-linearization approximation, we search the equilibrium characterized by

$$z_{m,t} = A_{m,0} + A_{m,1} p_t, \tag{F.27}$$

where the constants $A_{m,0}$ and $A_{m,1}$ can be computed recursively as follows.

Define the period-t price of the dividend strip paid at the period $t+n$ as $H(D_t, p_t, n) = \mathbb{E}_t\left[M_{t,t+n} D_{t+n}\right]$ where $M_{t,t+n} \equiv e^{\sum_{i=1}^{n} m_{t+i}}$. The price function $H(D_t, p_t, n)$ satisfies the following recursive relations:

$$H(D_t, p_t, n) = \mathbb{E}_t\left[e^{m_{t+1}} H(D_{t+1}, p_{t+1}, n-1)\right] \tag{F.28}$$

$$H(D_t, p_t, 0) = D_t, \tag{F.29}$$

for arbitrary t and $n \geq 1$.

We conjecture that $H(D_t, p_t, n) = D_t e^{A_n + B_n p_t}$. Then, the recursive relations in (F.28) and (F.29) can be rewritten as follows:

$$\begin{aligned} e^{A_n + B_n p_t} &= \mathbb{E}_t\left[e^{\Delta d_{t+1} + m_{t+1} + A_{n-1} + B_{n-1} p_{t+1}}\right] \\ &= \mathbb{E}_t\left[e^{(g_d + \phi\sigma_c \varepsilon_{c,t+1} - \phi\zeta_{t+1}) + (\Gamma_0 + \Gamma_1 p_t - \lambda_c \sigma_c \varepsilon_{c,t+1} - \lambda_p \sigma_p p \varepsilon_{p,t+1} + \lambda_\zeta \zeta_{t+1}) + (A_{n-1} + B_{n-1} p_{t+1})}\right] \\ &= e^{\tilde{A}_n + \tilde{B}_n p_t} \mathbb{E}_t\left[e^{(\phi - \lambda_c)\sigma_c \varepsilon_{c,t+1} + (B_{n-1} - \lambda_p)\sigma_p p \varepsilon_{p,t+1} + (\lambda_\zeta - \phi)\zeta_{t+1}}\right] \\ &= e^{\tilde{A}_n + \tilde{B}_n p_t + \frac{1}{2}(\phi - \gamma)^2 \sigma_c^2 + \frac{1}{2}[B_{n-1} - (\gamma - 1) I_1]^2 \sigma_p^2 p^2} \mathbb{E}_t\left[e^{(\gamma - \phi)\zeta_{t+1}}\right], \end{aligned} \tag{F.30}$$

where $\tilde{A}_n = g_d + \Gamma_0 + A_{n-1} + B_{n-1}(1-\rho)p$, and $\tilde{B}_n = \Gamma_1 + B_{n-1}\rho$.

The moment generating function of ζ_{t+1} gives

$$\ln \mathbb{E}_t\left[e^{(\gamma - \phi)\zeta_{t+1}}\right] \approx p_t \ell(\alpha, \gamma - \phi). \tag{F.31}$$

Thus, A_n has the following recursive relation:

$$\begin{aligned} A_n &= \tilde{A}_n + \frac{1}{2}(\phi - \gamma)^2 \sigma_c^2 + \frac{1}{2}\left[B_{n-1} - (\gamma - 1) I_1\right]^2 \sigma_p^2 p^2 \\ &= g_d + \Gamma_0 + A_{n-1} + B_{n-1}(1-\rho)p + \frac{1}{2}(\phi - \gamma)^2 \sigma_c^2 + \frac{1}{2}\left[B_{n-1} - (\gamma - 1) I_1\right]^2 \sigma_p^2 p^2 \\ &= \ln \delta + (g_d - g_c) - \frac{1}{2}(1-\gamma)^2 \left(I_1^2 \sigma_p^2 p^2 + \sigma_c^2\right) + \frac{1}{2}(\phi - \gamma)^2 \sigma_c^2 \\ &\quad + A_{n-1} + B_{n-1}(1-\rho)p + \frac{1}{2}\left[B_{n-1} - (\gamma - 1) I_1\right]^2 \sigma_p^2 p^2, \end{aligned} \tag{F.32}$$

and B_n has the following recursive relation:

$$\begin{aligned} B_n &= \tilde{B}_n + \ell(\alpha, \gamma - \phi) \\ &= \rho B_{n-1} + \ell(\alpha, \gamma - \phi) - \ell(\alpha, \gamma - 1), \end{aligned} \tag{F.33}$$

with initial values $A_0 = B_0 = 0$.

Therefore, it holds that

$$B_n = \frac{1 - \rho^n}{1 - \rho} \left[\ell(\alpha, \gamma - \phi) - \ell(\alpha, \gamma - 1) \right]. \tag{F.34}$$

Because $\sigma_p^2 p \approx 0$, equation (F.36) can be rewritten as

$$\begin{aligned} A_n - A_{n-1} \approx \ln \delta + (g_d - g_c) + \frac{1}{2}(\phi - 1)(\phi + 1 - 2\gamma)\sigma_c^2 \\ + (1 - \rho^n) \left[\ell(\alpha, \gamma - \phi) - \ell(\alpha, \gamma - 1) \right] p. \end{aligned} \tag{F.35}$$

Thus, it holds that

$$A_n = n \ln \bar{\delta} - \frac{1 - \rho^n}{1 - \rho} \left[\ell(\alpha, \gamma - \phi) - \ell(\alpha, \gamma - 1) \right] p, \tag{F.36}$$

where $\bar{\delta}$ is referred to as the effective time preference coefficient (Barro, 2009), and the log of the effective time preference coefficient is equal to

$$\ln \bar{\delta} \equiv \ln \delta + (g_d - g_c) + \frac{1}{2}(\phi - 1)(\phi + 1 - 2\gamma)\sigma_c^2 + \left[\ell(\alpha, \gamma - \phi) - \ell(\alpha, \gamma - 1) \right] p. \tag{F.37}$$

Therefore, the log price-dividend ratio is

$$z_{m,t} = \ln \left[\sum_{n=1}^{+\infty} e^{A_n + B_n p_t} \right]. \tag{F.38}$$

According to Taylor expansion in terms of p_t around p, it follows that

$$A_{m,0} = \ln \left[\sum_{n=1}^{+\infty} e^{A_n + B_n p} \right] - A_{m,1} p \quad \text{and} \quad A_{m,1} = \frac{\sum_{n=1}^{+\infty} B_n e^{A_n + B_n p}}{\sum_{n=1}^{+\infty} e^{A_n + B_n p}}. \tag{F.39}$$

And thus, it holds that

$$
\begin{aligned}
A_{m,1} &= \frac{1}{1-\rho}\left[\ell(\alpha,\gamma-\phi)-\ell(\alpha,\gamma-1)\right] - \frac{\sum_{n=1}^{+\infty} e^{n(\ln\bar{\delta}+\ln\rho)}}{\sum_{n=1}^{+\infty} e^{n\ln\bar{\delta}}}\frac{1}{1-\rho}\left[\ell(\alpha,\gamma-\phi)-\ell(\alpha,\gamma-1)\right] \\
&= \left[\frac{1}{1-\rho} - \frac{\rho}{1-\rho}\frac{1-\bar{\delta}}{1-\rho\bar{\delta}}\right]\left[\ell(\alpha,\gamma-\phi)-\ell(\alpha,\gamma-1)\right] \\
&= \frac{1-\rho\bar{\delta}-\rho(1-\bar{\delta})}{(1-\rho)(1-\rho\bar{\delta})}\left[\ell(\alpha,\gamma-\phi)-\ell(\alpha,\gamma-1)\right] \\
&= \frac{1-\rho}{(1-\rho)(1-\rho\bar{\delta})}\left[\ell(\alpha,\gamma-\phi)-\ell(\alpha,\gamma-1)\right] \\
&= \frac{1}{1-\rho\bar{\delta}}\left[\ell(\alpha,\gamma-\phi)-\ell(\alpha,\gamma-1)\right]. \qquad \text{(F.40)}
\end{aligned}
$$

In principle, the persistence parameter ρ can be infinitely close to 1, and thus, we require the effective time preference coefficient to be less than one (i.e., $\bar{\delta} < 1$).

According to (F.34) and (F.36), it follows that

$$
\begin{aligned}
A_{m,0} &= \ln\left[\sum_{n=1}^{\infty} e^{n\ln\bar{\delta}}\right] - A_{m,1}p \\
&= \ln\left[\frac{\bar{\delta}}{1-\bar{\delta}}\right] - A_{m,1}p, \qquad \text{(F.41)}
\end{aligned}
$$

and the long-run average log price-dividend ratio is

$$\bar{z}_m = A_{m,0} + A_{m,1}p = \ln\left[\frac{\bar{\delta}}{1-\bar{\delta}}\right]. \qquad \text{(F.42)}$$

Plugging back into (F.26), we obtain that

$$\kappa_{m,1} = \bar{\delta} = \delta e^{(g_d-g_c)+\frac{1}{2}(\phi-1)(\phi+1-2\gamma)\sigma_c^2+[\ell(\alpha,\gamma-\phi)-\ell(\alpha,\gamma-1)]p}. \qquad \text{(F.43)}$$

Because $\sigma_c^2 \approx 0$ and $p \approx 0$, the following approximation works well for relevant parameter regions:

$$\kappa_{m,1} = \bar{\delta} \approx \delta e^{(g_d-g_c)}. \qquad \text{(F.44)}$$

According to (F.24) and (F.27), the log market return can be rewritten as

$$r_{m,t+1} - \mathbb{E}_t\left[r_{m,t+1}\right] = \beta_c\sigma_c\varepsilon_{c,t+1} + \beta_p\sigma_p p\varepsilon_{p,t+1} - \beta_\zeta\left[\zeta_{t+1} - p_t\mu_1(\alpha)\right], \qquad \text{(F.45)}$$

where $\beta_c = \phi$, $\beta_p = \kappa_{m,1}A_{m,1}$, and $\beta_\zeta = \phi$. The Euler equation for the log market return is

$$1 = \mathbb{E}_t\left[e^{r_{m,t+1}+m_{t+1}}\right], \qquad \text{(F.46)}$$

which leads to

$$\mathbb{E}_t\left[e^{m_{t+1}}\right] = \mathbb{E}_t\left[e^{r_{m,t+1}-r_{f,t}+m_{t+1}}\right]. \tag{F.47}$$

It further leads to

$$\begin{aligned} & e^{\mathbb{E}_t[m_{t+1}]+\frac{1}{2}\left[\gamma^2\sigma_c^2+\lambda_p^2\sigma_p^2p^2\right]-\gamma p_t\mu_1(\alpha)}\mathbb{E}_t\left[e^{\gamma\zeta_{t+1}}\right] \\ & \quad = e^{\mathbb{E}_t[m_{t+1}]+\mathbb{E}_t[r_{m,t+1}]-r_{f,t}+\frac{1}{2}\left[(\phi-\gamma)^2\sigma_c^2+(\beta_p-\lambda_p)^2\sigma_p^2p^2\right]-(\gamma-\phi)p_t\mu_1(\alpha)}\mathbb{E}_t\left[e^{(\gamma-\phi)\zeta_{t+1}}\right]. \end{aligned} \tag{F.48}$$

Finally, rearranging terms leads to

$$\mathbb{E}_t\left[r_{m,t+1}\right] - r_{f,t} = \phi\gamma\sigma_c^2 + \beta_p\lambda_p\sigma_p^2p^2 + \left[\ell(\alpha,\gamma) - \ell(\alpha,\gamma-\phi) - \phi\mu_1(\alpha)\right]p_t - \frac{1}{2}\left(\phi^2\sigma_c^2 + \beta_p^2\sigma_p^2p^2\right).$$

Now, we derive the yield of the defaultable government bond, denoted by $y_{b,t}$, and we express the log government bond return as follows:

$$r_{b,t+1} = y_{b,t} - x_{b,t+1}(\underline{v} + J_{t+1}). \tag{F.49}$$

A default on the government bond occurs with probability q conditional on the occurrence of a disaster. Thus, by definition, it holds that

$$\mathbb{E}_t\left[r_{b,t+1}\right] = y_{b,t} - p_t q\mu_1(\alpha). \tag{F.50}$$

According to the Euler equation of the defaultable government bond and the risk-free bond, it holds that

$$\mathbb{E}_t\left[e^{m_{t+1}+r_{b,t+1}-r_{f,t}}\right] = \mathbb{E}_t\left[e^{m_{t+1}}\right]. \tag{F.51}$$

Some calculations show that the following relation approximately holds:

$$\begin{aligned} & \ln\mathbb{E}_t\left[e^{m_{t+1}+r_{b,t+1}-r_{f,t}}\right] \\ & \quad = \Gamma_0 + \Gamma_1 p_t + y_{b,t} - r_{f,t} + \frac{1}{2}\gamma^2\sigma_c^2 + \frac{1}{2}\lambda_p^2\sigma_p^2p^2 + \ln\mathbb{E}_t\left[e^{\gamma x_{t+1}v_{t+1} - x_{b,t+1}v_{t+1}}\right] && \text{(F.52)} \\ & \quad = \Gamma_0 + \Gamma_1 p_t + y_{b,t} - r_{f,t} + \frac{1}{2}\gamma^2\sigma_c^2 + \frac{1}{2}\lambda_p^2\sigma_p^2p^2 + \left[(1-q)\ell(\alpha,\gamma) + q\ell(\alpha,\gamma-1)\right]p_t. && \text{(F.53)} \end{aligned}$$

Combining (F.23), (F.51), and (F.53), it follows that

$$y_{b,t} - r_{f,t} = q\left[\ell(\alpha,\gamma) - \ell(\alpha,\gamma-1)\right]p_t. \tag{F.54}$$

Further, putting (F.50) and (F.54) together, we obtain the following relation:

$$\mathbb{E}_t\left[r_{b,t+1}\right] - r_{f,t} = q\left[\ell(\alpha,\gamma) - \ell(\alpha,\gamma-1) - \mu_1(\alpha)\right]p_t. \tag{F.55}$$

Therefore, the conditional mean of excess log returns of the market portfolio relative to the defaultable government bill is

$$\begin{aligned}\mathbb{E}_t\left[r_{m,t+1}-r_{b,t+1}\right]=&\ \phi\gamma\sigma_c^2+\beta_p\lambda_p\sigma_p^2p^2+\left[\ell(\alpha,\gamma)-\ell(\alpha,\gamma-\phi)-\phi\mu_1(\alpha)\right]p_t\\&-\frac{1}{2}\left(\phi^2\sigma_c^2+\beta_p^2\sigma_p^2p^2\right)-q\left[\ell(\alpha,\gamma)-\ell(\alpha,\gamma-1)-\mu_1(\alpha)\right]p_t,\end{aligned}\tag{F.56}$$

and the conditional variance of excess returns of the market portfolio relative to the defaultable government bill is

$$\begin{aligned}\mathbb{V}ar_t\left[r_{m,t+1}-r_{b,t+1}\right]&=\mathbb{V}ar_t\left[\phi\sigma_c\varepsilon_{c,t+1}+\beta_p\sigma_p p\varepsilon_{p,t+1}-(\phi x_{t+1}-x_{b,t+1})(\underline{v}+J_{t+1})\right]\\&=\phi^2\sigma_c^2+\beta_p^2\sigma_p^2p^2+\left(\phi^2-2\phi q+q\right)p_t\mu_2(\alpha)-(\phi-q)^2p_t^2\mu_1(\alpha)^2.\end{aligned}\tag{F.57}$$

Verifying the Asset Pricing Moment Conditions. We now verify the asset pricing moment conditions. The first moment condition is the expected log return of defaultable government bills:

$$\mathbb{E}\left[r_{b,t}\right]-\omega_1(\vartheta)=\left(\mathbb{E}\left[r_{b,t}\right]-r_{f,t-1}\right)+r_{f,t-1}-\omega_1(\vartheta).\tag{F.58}$$

Plugging (F.23) and (F.55) into the equation above, it follows that

$$\begin{aligned}\mathbb{E}\left[r_{b,t}\right]-\omega_1(\vartheta)&=q\left[\ell(\alpha,\gamma)-\ell(\alpha,\gamma-1)-\mu_1(\alpha)\right]\mathbb{E}\left[p_{t-1}\right]\\&\quad-\ln\delta+g_c-\frac{1}{2}(2\gamma-1)\sigma_c^2-\left[\ell(\alpha,\gamma)-\ell(\alpha,\gamma-1)\right]\mathbb{E}\left[p_{t-1}\right]-\omega_1(\vartheta)\\&=-\ln\delta+g_c-\frac{1}{2}(2\gamma-1)\sigma_c^2-(1-q)p\left[\ell(\alpha,\gamma)-\ell(\alpha,\gamma-1)\right]-qp\mu_1(\alpha)-\omega_1(\vartheta)\\&=-\ln\delta+g_c-\frac{1}{2}(2\gamma-1)\sigma_c^2-qp\mu_1(\alpha)-(1-q)h_1(\vartheta)\frac{p}{\alpha-\gamma}-\omega_1(\vartheta)\\&=0,\end{aligned}\tag{F.59}$$

where $h_1(\vartheta)\equiv\alpha\left[e^{\underline{v}\gamma}-e^{\underline{v}(\gamma-1)}\frac{\alpha-\gamma}{\alpha-\gamma+1}\right]$.

The second asset pricing moment condition is the unconditional variance of the log government bill return:

$$\mathbb{E}\left[r_{b,t}-\omega_1(\vartheta)\right]^2-\omega_2(\vartheta)=\mathbb{V}ar\left[\mathbb{E}_{t-1}(r_{b,t})\right]+\mathbb{E}\left[\mathbb{V}ar_{t-1}(r_{b,t})\right]-\omega_2(\vartheta)\tag{F.60}$$

Plugging in the equilibrium expressions for $\mathbb{V}ar\left[\mathbb{E}_{t-1}(r_{b,t})\right]$ and $\mathbb{E}\left[\mathbb{V}ar_{t-1}(r_{b,t})\right]$, the equation above can further lead to

$$\begin{aligned}
&\mathbb{E}\left[r_{b,t}-\omega_1(\vartheta)\right]^2-\omega_2(\vartheta)\\
&=\left[(1-q)(\ell(\alpha,\gamma)-\ell(\alpha,\gamma-1))+q\mu_1(\alpha)\right]^2\frac{\sigma_p^2p^2}{1-\rho^2}\\
&\quad+qp\mu_2(\alpha)-q^2p^2\mu_1(\alpha)^2\left(1+\frac{\sigma_p^2}{1-\rho^2}\right)-\omega_2(\vartheta).
\end{aligned}\tag{F.61}$$

According to the definition of the function $\omega_2(\vartheta)$ and the equation above, it follows that

$$\mathbb{E}\left[r_{b,t}-\omega_1(\vartheta)\right]^2-\omega_2(\vartheta)=0.\tag{F.62}$$

The third asset pricing moment condition is the unconditional mean of the excess log market return relative to the defaultable government bills. The excess log market return is

$$\begin{aligned}
r_{m,t}^e=&\ \mathbb{E}_{t-1}\left[r_{m,t}^e\right]+\phi\sigma_c\varepsilon_{c,t}+\beta_p\sigma_p\sqrt{p}\varepsilon_{p,t}\\
&-\phi\left[x_t(\underline{v}+J_t)-p_{t-1}\mu_1(\alpha)\right]+\left[x_{b,t}(\underline{v}+J_t)-qp_{t-1}\mu_1(\alpha)\right],
\end{aligned}\tag{F.63}$$

with the conditional expected excess log market return to be

$$\begin{aligned}
\mathbb{E}_{t-1}\left[r_{m,t}^e\right]=&\ \phi\gamma\sigma_c^2+\beta_p\lambda_p\sigma_p^2p^2+\left[\ell(\alpha,\gamma)-\ell(\alpha,\gamma-\phi)-\phi\mu_1(\alpha)\right]p_{t-1}\\
&-\frac{1}{2}\left(\phi^2\sigma_c^2+\beta_p^2\sigma_p^2p^2\right)-q\left[\ell(\alpha,\gamma)-\ell(\alpha,\gamma-1)-\mu_1(\alpha)\right]p_{t-1}.
\end{aligned}\tag{F.64}$$

Therefore, the unconditional mean of excess log market returns is

$$\mathbb{E}\left[r_{m,t}^e\right]=h_0(\vartheta)+h_3(\vartheta)\frac{p}{\alpha-\gamma},\tag{F.65}$$

$$\begin{aligned}
\text{where}\quad h_0(\vartheta)&=\phi\gamma\sigma_c^2+\beta_p\lambda_p\sigma_p^2p^2-\frac{1}{2}\left(\phi^2\sigma_c^2+\beta_p^2\sigma_p^2p^2\right),\\
h_3(\vartheta)&=\alpha\left[(1-q)e^{\underline{v}\gamma}-e^{\underline{v}(\gamma-\phi)}\frac{\alpha-\gamma}{\alpha-\gamma+\phi}+qe^{\underline{v}(\gamma-1)}\frac{\alpha-\gamma}{\alpha-\gamma+1}\right]-(\alpha-\gamma)(\phi-q)\mu_1(\alpha).
\end{aligned}$$

The fourth asset pricing moment condition is the unconditional variance of the excess log market return relative to the defaultable government bills. The unconditional variance has the following decomposition:

$$\mathbb{V}ar\left[r_{m,t+1}^e\right]=\mathbb{E}\left[\mathbb{V}ar_t(r_{m,t+1}^e)\right]+\mathbb{V}ar\left[\mathbb{E}_t\left(r_{m,t+1}^e\right)\right].\tag{F.66}$$

The unconditional mean of the conditional variance is

$$\mathbb{E}\left[\mathbb{V}ar_t\left(r^e_{m,t+1}\right)\right] = \mathbb{E}\left[\phi^2\sigma_c^2 + \beta_p^2\sigma_p^2 p^2 + p_t(q - 2\phi q + \phi^2)\mu_2(\alpha) - p_t^2(q-\phi)^2\mu_1(\alpha)^2\right] \tag{F.67}$$
$$= \phi^2\sigma_c^2 + \beta_p^2\sigma_p^2 p^2 + (q - 2\phi q + \phi^2)\mu_2(\alpha)p - (q-\phi)^2\mu_1(\alpha)^2\left(\frac{\sigma_p^2 p^2}{1-\rho^2} + p^2\right).$$

The unconditional variance of the conditional mean is

$$\mathbb{V}ar\left[\mathbb{E}_t\left(r^e_{m,t+1}\right)\right] = \mathbb{V}ar\left[\left((1-q)\ell(\alpha,\gamma) - \ell(\alpha,\gamma-\phi) + q\ell(\alpha,\gamma-1) - (\phi-q)\mu_1(\alpha)\right)p_t\right]$$
$$= \left[(1-q)\ell(\alpha,\gamma) - \ell(\alpha,\gamma-\phi) + q\ell(\alpha,\gamma-1) - (\phi-q)\mu_1(\alpha)\right]^2\frac{\sigma_p^2 p^2}{1-\rho^2}$$
$$= h_3(\vartheta)^2\frac{1}{(\alpha-\gamma)^2}\frac{\sigma_p^2 p^2}{1-\rho^2}. \tag{F.68}$$

The sixth asset pricing moment condition can be derived as follows, and the fifth asset pricing moment can be derived similarly. The excess log market return in period $t+1$ is

$$r^e_{m,t+1} = h_0(\vartheta) + h_3(\vartheta)\frac{p_t}{\alpha-\gamma} + e_{t+1}, \tag{F.69}$$

where e_{t+1} is a random variable such that $\mathbb{E}_t\left[e_{t+1}\right] = 0$. Thus, the excess log market return in period $t+1$ can further expressed in terms of p_{t-1}:

$$r^e_{m,t+1} = h_0(\vartheta) + h_3(\vartheta)(\alpha-\gamma)^{-1}\left[(1-\rho)p + \rho p_{t-1} + \sigma_p p\varepsilon_{p,t}\right] + e_{t+1}$$
$$= h_0(\vartheta) + h_3(\vartheta)\frac{p}{\alpha-\gamma} + h_3(\vartheta)(\alpha-\gamma)^{-1}\rho(p_{t-1}-p) + \tilde{e}_t, \tag{F.70}$$

where $\mathbb{E}_{t-1}\left[\tilde{e}_t\right] = 0$ with $\tilde{e}_t \equiv h_3(\vartheta)(\alpha-\gamma)\sigma_p p\varepsilon_{p,t} + e_{t+1}$.

The log price-dividend ratio is

$$z_{m,t} - \overline{z}_m = \frac{1}{1-\rho\overline{\delta}}h_2(\vartheta)(p_t - p). \tag{F.71}$$

Plugging (F.71) into (F.70), it follows that

$$r^e_{m,t+1} = \omega_3(\vartheta) + \frac{\rho(1-\rho\overline{\delta})}{\alpha-\gamma}h_2(\vartheta)^{-1}h_3(\vartheta)(z_{m,t-1} - \overline{z}_m) + \tilde{e}_t. \tag{F.72}$$

Therefore, according to the definition of $\omega_6(\vartheta)$ and the equation above, the asset pricing moment condition follows,

$$\mathbb{E}_{t-1}\left[r^e_{m,t+1} - \omega_6(\vartheta)(z_{m,t-1} - \overline{z}_m) - \omega_3(\vartheta)\right] = 0. \tag{F.73}$$

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