This note concerns Lemma 9.10 in the Supplemental Material of Andrews and Cheng (2012). This is an argmax theorem that shows if (i) a stochastic process M_n weakly converges to another stochastic process M, and (ii) a set A_n converges to another set A_0 in the Hausdorff metric, then the maximizer of M_n over the parameter space A_n converges in distribution to the maximizer of M over A_0, under the standard regularities conditions given in the Lemma. Cox (2022) points out that although the conclusion is correct as long as the parameter space is separable, the proof requires modification following Theorem 1 of Cox (2022). Specifically, equation (9.98) in the proof assumes that $F \cap A_n$ converges to $F \cap A_0$ for every closed set F before applying the extended continuous mapping theorem (CMT) from van der Vaart and Wellner (1996) to obtain $\sup_{h \in F \cap A_n} M_n(h) \to_d \sup_{h \in F \cap A_0} M(h)$. This argument does not hold for some closed sets F. For example, suppose $A_n = [1 + 1/n, 2]$, $A_0 = [1, 2]$, and $F = [0, 1] \cup \{2\}$. In this case, $A_n \cap F = \{2\} \subset A_0 \cap F = \{1, 2\}$. To get around such issues in the proof, Cox (2022) provides a strategy. Roughly speaking, the requirement $A_n \cap F$ converging to $A_0 \cap F$ is replaced by the requirement $A_n \cap F$ converging to a *subset* of $A_0 \cap F$ along a subsequence under a compact metric space, before one applies the extended CMT. The rest of the proof follows in a similar fashion. Cox (2022) provides the proof and additional applications of this argmax theorem.

References

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