



An eigenvalue problem for a fully nonlinear elliptic equation with gradient constraint

Ryan Hynd¹

Received: 13 November 2015 / Accepted: 9 January 2017
© Springer-Verlag Berlin Heidelberg 2017

Abstract We consider the problem of finding $\lambda \in \mathbb{R}$ and a function $u : \mathbb{R}^n \rightarrow \mathbb{R}$ that satisfy the PDE

$$\max \{ \lambda + F(D^2u) - f(x), H(Du) \} = 0, \quad x \in \mathbb{R}^n.$$

Here F is elliptic, positively homogeneous and superadditive, f is convex and superlinear, and H is typically assumed to be convex. Examples of this type of PDE arise in the theory of singular ergodic control. We show that there is a unique λ^* for which the above equation has a solution u with appropriate growth as $|x| \rightarrow \infty$. Moreover, associated to λ^* is a convex solution u^* that has essentially bounded second derivatives, provided F is uniformly elliptic and H is uniformly convex. It is unknown whether or not u^* is unique up to an additive constant; however, we verify that this is the case when $n = 1$ or when F, f, H are “rotational.”

Mathematics Subject Classification 35R35 · 35P30 · 49J20

1 Introduction

In this note, we will analyze the following eigenvalue problem: find $\lambda \in \mathbb{R}$ and $u : \mathbb{R}^n \rightarrow \mathbb{R}$ satisfying the PDE

$$\max \{ \lambda + F(D^2u) - f(x), H(Du) \} = 0, \quad x \in \mathbb{R}^n. \quad (1.1)$$

Communicated by N.Trudinger.

The author was partially supported by NSF Grant DMS-1301628.

✉ Ryan Hynd
rhynd@math.upenn.edu

¹ Department of Mathematics, University of Pennsylvania, Philadelphia, PA, USA

Here $Du = (u_{x_i})$ and $D^2u = (u_{x_i x_j})$. A standing assumption will be that $F : S_n(\mathbb{R}) \rightarrow \mathbb{R}$ is elliptic, positively homogeneous, and superadditive:

$$\begin{cases} -\Theta \operatorname{tr} N \leq F(M + N) - F(M) \leq -\theta \operatorname{tr} N, & (N \geq 0) \\ F(tM) = tF(M) \\ F(M) + F(N) \leq F(M + N) \end{cases} \tag{1.2}$$

for each $M, N \in S_n(\mathbb{R})$, $t \geq 0$ and some $\theta, \Theta \geq 0$. If $\theta > 0$, we say F is uniformly elliptic, and $S_n(\mathbb{R})$ denotes the collection of real, symmetric $n \times n$ matrices. A prototypical example of F is

$$F(M) = \min_{1 \leq k \leq N} \{-\operatorname{tr}(A_k M)\},$$

where the eigenvalues of each $A_1, \dots, A_N \in S_n(\mathbb{R})$ belong to the interval $[\theta, \Theta]$.

We will also assume throughout that the gradient constraint function $H \in C(\mathbb{R}^n)$ satisfies

$$\begin{cases} H(0) < 0 \\ \{p \in \mathbb{R}^n : H(p) \leq 0\} \text{ is compact and strictly convex.} \end{cases} \tag{1.3}$$

Moreover, we will always suppose that f is convex and superlinear

$$\lim_{|x| \rightarrow \infty} \frac{f(x)}{|x|} = \infty.$$

An equation that motivated this study is

$$\max \{\lambda - \Delta u - f(x), |Du| - 1\} = 0, \quad x \in \mathbb{R}^n. \tag{1.4}$$

In previous work [14], we showed there is a unique eigenvalue $\lambda^* \in \mathbb{R}$ such that the PDE (1.4) admits a viscosity solution u satisfying the growth condition

$$\lim_{|x| \rightarrow \infty} \frac{u(x)}{|x|} = 1.$$

The eigenvalue λ^* is also known to have an ergodic control theoretic interpretation. We refer the reader to [3, 12, 14, 20, 21, 24] for more information on how this PDE arises in stochastic control and financial applications.

In (1.4), $H(p) = |p| - 1$. And from our remarks above, it is natural to study solutions of (1.1) subject to a suitable growth condition. To this end, we define the function

$$\ell(v) := \max\{p \cdot v : H(p) \leq 0\}, \quad v \in \mathbb{R}^n$$

which is also known as the support function of the convex set $\{p \in \mathbb{R}^n : H(p) \leq 0\}$. The main result of this paper is as follows.

Theorem 1.1 *Assume (1.2), (1.3), and that f is convex and superlinear.*

(i) *There is a unique $\lambda^* \in \mathbb{R}$ such that (1.1) has a viscosity solution $u \in C(\mathbb{R}^n)$ satisfying the growth condition*

$$\lim_{|x| \rightarrow \infty} \frac{u(x)}{\ell(x)} = 1. \tag{1.5}$$

Associated to λ^ is a convex viscosity solution u^* that satisfies (1.5).*

(ii) Suppose that F is uniformly elliptic, H is convex and that there are $\sigma, \Sigma > 0$ such that

$$\sigma |\xi|^2 \leq D^2H(p)\xi \cdot \xi \leq \Sigma |\xi|^2, \quad \xi \in \mathbb{R}^n \tag{1.6}$$

for Lebesgue almost every $p \in \mathbb{R}^n$. Then we may choose u^* to satisfy $D^2u^* \in L^\infty(\mathbb{R}^n; S_n(\mathbb{R}))$.

The assumptions (1.2) on F were first made to be able to verify part (i) of the above theorem. We did not expect that these same assumptions would guarantee the regularity of solutions of (1.4) as detailed in part (ii); especially because we have recently shown that equations such as (1.4) have solutions that are locally $W^{2,\infty}$ provided F is uniformly elliptic and convex [15]. While in this paper, (1.2) requires F to be uniformly elliptic but concave. Also note that when F is concave, the PDE (1.1) is not uniformly elliptic and neither convex nor concave, which are usually needed to obtain bounds on the second derivatives of solutions.

Fortunately, we found a way to exploit the degree one homogeneity of F to produce a solution of (1.1) that has uniformly bounded second derivatives on \mathbb{R}^n . Our approach is to apply Bernstein’s method to a penalized version of (1.1). In deriving estimates on the solution of this penalized equation, we exploit the identity $F(M) = F_{M_{ij}}(M)M_{ij}$, which F satisfies if it is differentiable at M . In general, we use an approximate version of this identity (Lemma 4.2) at a critical juncture of our argument (Lemma 4.4) which ultimately results in a one sided second derivative estimate. The full second derivative estimate then follows from the convexity of solutions and that any solution of (1.1) is a subsolution of a uniformly elliptic PDE. Consequently, the hypotheses (1.2) are natural for our eigenvalue problem.

When $\lambda = \lambda^*$ in (1.1), we will call solutions that satisfy the growth condition (1.5) *eigenfunctions*. It is unknown if eigenfunctions are unique up to an additive constant. However, we establish below that when $n = 1$ any two convex eigenfunctions differ by a constant; see Proposition 5.1. We also show that if F, f and H are “rotational,” then u^* can be chosen radial and twice continuously differentiable. This generalizes Theorem 2.3 of [18] and Theorem 1.3 of [14].

Theorem 1.2 *Suppose*

$$\begin{cases} f(Ox) = f(x) \\ H(O^t p) = H(p) \\ F(OMO^t) = F(M) \end{cases} \tag{1.7}$$

for each $x, p \in \mathbb{R}^n, M \in S_n(\mathbb{R})$ and orthogonal $n \times n$ matrix O . If F is uniformly elliptic and H satisfies (1.6), then there is a radial eigenfunction $u^* \in C^2(\mathbb{R}^n)$.

In Proposition 5.2 below, we assume (1.7) and show any two convex, radial eigenfunctions differ by an additive constant. Unfortunately, we do not know if this symmetry assumption ensures that every eigenfunction is radial. Finally, we verify a certain minmax formula for λ^* .

Theorem 1.3 *Define*

$$\lambda_+ := \inf \left\{ \sup_{H(D\psi(x)) < 0} \{-F(D^2\psi(x)) + f(x)\} : \psi \in C^2(\mathbb{R}^n), \liminf_{|x| \rightarrow \infty} \frac{\psi(x)}{\ell(x)} \geq 1 \right\}$$

and

$$\lambda_- := \sup \left\{ \inf_{x \in \mathbb{R}^n} \{-F(D^2\phi(x)) + f(x)\} : \phi \in C^2(\mathbb{R}^n), H(D\phi) \leq 0 \right\}$$

Then

$$\lambda_- \leq \lambda^* \leq \lambda_+.$$

If there is an eigenfunction u^* that satisfies $D^2u^* \in L^\infty(\mathbb{R}^n; S_n(\mathbb{R}))$, then $\lambda^* = \lambda_+$.

The organization of this paper is as follows. In Sect. 2, we verify the uniqueness of eigenvalues as detailed in Theorem 1.1 part (i). Then we establish the existence of an eigenvalue λ^* in Section 3. Next, we verify Theorem 1.1 part (ii) in Sect. 4 and prove Theorem 1.2 in Sect. 5. Section 6 of this paper is dedicated to the proof of Theorem 1.3. Lastly, we would like to acknowledge the hospitality of the University of Pennsylvania’s Center of Race & Equity in Education where this paper was written.

2 Comparison principle

In this section, we show there can be at most one eigenvalue as detailed in Theorem 1.1. As equation (1.1) is a fully nonlinear elliptic equation for a scalar function u , we will employ the theory of viscosity solutions [2, 6, 7, 12]. In particular, we will use results and notation from the “user guide” [7]. Moreover, going forward we typically will omit the modifier “viscosity” when we refer to sub- and supersolutions.

We begin our discussion with a basic proposition about subsolutions of the first order PDE $H(Du) = 0$. It is now appropriate to mention a few minor, yet important facts. First, we can replace H in (1.1) with the explicit convex gradient constraint

$$H_0(p) := \max_{|v|=1} \{p \cdot v - \ell(v)\}$$

since $H(p) \leq 0$ if and only if $H_0(p) \leq 0$ (Theorem 8.24 in [22]). Next, the assumptions (1.3) imply there are positive constants c_0, c_1 such that

$$c_0|v| \leq \ell(v) \leq c_1|v|, \quad v \in \mathbb{R}^n. \tag{2.1}$$

It also then follows that

$$|p| - c_1 \leq H_0(p) \leq |p| - c_0, \quad p \in \mathbb{R}^n. \tag{2.2}$$

Lemma 2.1 *A function $u \in C(\mathbb{R}^n)$ satisfies*

$$H(Du(x)) \leq 0, \quad x \in \mathbb{R}^n \tag{2.3}$$

if and only if

$$u(x) - u(y) \leq \ell(x - y), \quad x, y \in \mathbb{R}^n. \tag{2.4}$$

Proof Assume (2.3). Then u is Lipschitz by (2.2), and $H(Du(x)) \leq 0$ for almost every $x \in \mathbb{R}^n$. Let $u^\epsilon := \eta^\epsilon * u$ be a standard mollification of u . That is, $\eta \in C_c^\infty(\mathbb{R}^n)$ is a nonnegative, radial function supported in $B_1(0)$ that satisfies $\int_{\mathbb{R}^n} \eta(z) dz = 1$ and $\eta^\epsilon := \epsilon^{-n} \eta(\cdot/\epsilon)$. It is readily verified that $u^\epsilon \in C^\infty(\mathbb{R}^n)$ and u^ϵ converges to u uniformly as ϵ tends to 0; see Appendix C.5 of [10] for more on mollification. As H_0 is convex, we have by Jensen’s inequality

$$H_0(Du^\epsilon) = H_0(D(\eta^\epsilon * u)) = H_0(\eta^\epsilon * Du) \leq \eta^\epsilon * H_0(Du) \leq 0.$$

It follows that for any $x, y \in \mathbb{R}^n$

$$u^\epsilon(x) - u^\epsilon(y) = \int_0^1 Du^\epsilon(y + t(x - y)) \cdot (x - y)dt \leq \ell(x - y).$$

Sending $\epsilon \rightarrow 0^+$ gives (2.4).

For the converse, suppose there is $p \in \mathbb{R}^n$ such that

$$u(x) \leq u(x_0) + p \cdot (x - x_0) + o(|x - x_0|)$$

as $x \rightarrow x_0$. Substituting $x = x_0 - tv$ for $t > 0$ and $|v| = 1$ above gives

$$u(x_0) - t\ell(v) \leq u(x_0 - tv) \leq u(x_0) - tp \cdot v + o(t).$$

As a result $p \cdot v \leq \ell(v)$. As v was arbitrary, $H(p) \leq 0$. □

Corollary 2.2 *The function ℓ satisfies (2.3). Moreover, at any $x \in \mathbb{R}^n$ for which ℓ is differentiable*

$$\ell(x) = D\ell(x) \cdot x \text{ and } H(D\ell(x)) = 0.$$

Proof As ℓ is convex and positively homogeneous, it is subadditive. Therefore, $\ell(x) \leq \ell(y) + \ell(x - y)$ for each $x, y \in \mathbb{R}^n$. By the previous lemma, ℓ satisfies (2.3). Now suppose that ℓ is differentiable at x , and choose ξ such that $H(\xi) \leq 0$ and $\ell(x) = \xi \cdot x$. Then, as $y \rightarrow x$

$$\begin{aligned} \xi \cdot y &\leq \ell(y) \\ &= \ell(x) + D\ell(x) \cdot (y - x) + o(|y - x|) \\ &= \xi \cdot x + D\ell(x) \cdot (y - x) + o(|y - x|). \end{aligned}$$

Choosing $y = x + tv$, for $t > 0$ and $v \in \mathbb{R}^n$ gives $\xi \cdot v \leq D\ell(x) \cdot v + o(1)$ as $t \rightarrow 0^+$. Thus, $\xi = D\ell(x)$ and $H(D\ell(x)) \leq 0$. If $x \neq 0$,

$$H_0(D\ell(x)) \geq D\ell(x) \cdot \frac{x}{|x|} - \ell\left(\frac{x}{|x|}\right) = \frac{D\ell(x) \cdot x - \ell(x)}{|x|} = 0$$

and so $H(D\ell(x)) = 0$. Conversely, if $x = 0$, then ℓ is linear since it is positively homogeneous. However, this would contradict (2.1). □

The following assertion is a comparison principle for eigenvalues that makes use of the growth condition (1.5).

Proposition 2.3 *Assume $u \in USC(\mathbb{R}^n)$ is a subsolution of (1.1) with eigenvalue λ and $v \in LSC(\mathbb{R}^n)$ is a supersolution of (1.1) with eigenvalue μ . If*

$$\limsup_{|x| \rightarrow \infty} \frac{u(x)}{\ell(x)} \leq 1 \leq \liminf_{|x| \rightarrow \infty} \frac{v(x)}{\ell(x)}, \tag{2.5}$$

the $\lambda \leq \mu$.

Remark 2.4 Any subsolution u of (1.1) satisfies $H(Du) \leq 0$. By Lemma 2.1, u then satisfies (2.4) and therefore the first inequality in (2.5) automatically holds. We have included both inequalities in (2.5) simply for aesthetic purposes, and we continue this practice throughout this paper.

Proof For $\tau \in (0, 1)$ and $\eta > 0$, set

$$w^\tau(x, y) := \tau u(x) - v(y), \quad \varphi^\eta(x, y) := \frac{1}{2\eta}|x - y|^2$$

$x, y \in \mathbb{R}^n$. Observe

$$\begin{aligned} (w^\tau - \varphi^\eta)(x, y) &= \tau(u(x) - u(y)) + \tau u(y) - v(y) - \frac{1}{2\eta}|x - y|^2 \\ &\leq \tau \ell(x - y) + \tau u(y) - v(y) - \frac{1}{2\eta}|x - y|^2 \\ &\leq \tau c_1|x - y| + \tau u(y) - v(y) - \frac{1}{2\eta}|x - y|^2 \\ &\leq \eta \tau^2 c_1^2 + \tau u(y) - v(y) - \frac{1}{4\eta}|x - y|^2. \end{aligned} \tag{2.6}$$

In view of (2.5), $\lim_{|y| \rightarrow \infty} (\tau u(y) - v(y)) = -\infty$ and so

$$\lim_{|x|+|y| \rightarrow \infty} (w^\tau - \varphi^\eta)(x, y) = -\infty.$$

As a result, there is (x_η, y_η) maximizing $w^\tau - \varphi^\eta$.

By Theorem 3.2 in [7], for each $\rho > 0$ there are $X, Y \in S_n(\mathbb{R})$ with $X \leq Y$ such that

$$\left(\frac{x_\eta - y_\eta}{\eta}, X\right) \in \overline{J}^{2,+}(\tau u)(x_\eta)$$

and

$$\left(\frac{x_\eta - y_\eta}{\eta}, Y\right) \in \overline{J}^{2,-}v(y_\eta).$$

Note that

$$\begin{aligned} H_0\left(\frac{x_\eta - y_\eta}{\eta}\right) &= H_0\left(\tau \frac{x_\eta - y_\eta}{\tau \eta} + (1 - \tau)0\right) \\ &\leq \tau H_0\left(\frac{x_\eta - y_\eta}{\tau \eta}\right) + (1 - \tau)H_0(0) \\ &\leq (1 - \tau)H_0(0) \\ &< 0. \end{aligned}$$

As v is a supersolution of (1.1),

$$\mu + F(Y) - f(y_\eta) \geq 0.$$

Since F is elliptic and positively homogeneous,

$$\begin{aligned} \tau \lambda - \mu &\leq -\tau F\left(\frac{X}{\tau}\right) + F(Y) + \tau f(x_\eta) - f(y_\eta) \\ &= -F(X) + F(Y) + \tau f(x_\eta) - f(y_\eta) \\ &\leq \tau f(x_\eta) - f(y_\eta) \\ &= f(x_\eta) - f(y_\eta) + (\tau - 1)f(x_\eta) \\ &\leq f(x_\eta) - f(y_\eta) + (\tau - 1)\inf_{\mathbb{R}^n} f. \end{aligned} \tag{2.7}$$

We now claim that $(y_\eta)_{\eta>0} \subset \mathbb{R}^n$ is bounded. To see this, recall inequality (2.6). If there is a sequence $\eta_k \rightarrow 0$ as $k \rightarrow \infty$ for which $|y_{\eta_k}|$ is unbounded, then $(w^\tau - \varphi^{\eta_k})(x_{\eta_k}, y_{\eta_k})$ tends to $-\infty$ as $k \rightarrow \infty$. However,

$$\begin{aligned} (w^\tau - \varphi^{\eta_k})(x_{\eta_k}, y_{\eta_k}) &= \sup_{\mathbb{R}^n \times \mathbb{R}^n} (w^\tau - \varphi^{\eta_k}) \\ &\geq (w^\tau - \varphi^\eta)(0, 0) \\ &= \tau u(0) - v(0). \end{aligned}$$

Thus, $(y_\eta)_{\eta>0}$ and similarly $(x_\eta)_{\eta>0}$ is bounded. It then follows from Lemma 3.1 in [7] that

$$\lim_{\eta \rightarrow 0^+} \frac{|x_\eta - y_\eta|^2}{2\eta} = 0$$

and $(x_\eta, y_\eta)_{\eta>0} \subset \mathbb{R}^n \times \mathbb{R}^n$ has a cluster point (x_τ, x_τ) . Passing to the limit along an appropriate sequence η tending to 0 in (2.7) then gives

$$\tau\lambda - \mu \leq (\tau - 1) \inf_{\mathbb{R}^n} f. \tag{2.8}$$

We conclude after sending $\tau \rightarrow 1^-$. □

Corollary 2.5 *There can be at most one $\lambda \in \mathbb{R}$ for which (1.1) has a solution u satisfying (1.5).*

We are uncertain whether or not eigenfunctions u are uniquely defined up to an additive constant. However, we do know that if F is not uniformly elliptic and f is not strictly convex, eigenfunctions are not necessarily unique. For instance when $F \equiv 0$ and $H(p) = |p| - 1$, equation (1.1) reduces to

$$\max\{\lambda - f, |Du| - 1\} = 0, \quad \mathbb{R}^n. \tag{2.9}$$

It is easily verified that $\lambda^* = \inf_{\mathbb{R}^n} f$ and $u(x) = |x - x_0|$ is a solution of (2.9) for each x_0 such that $\inf_{\mathbb{R}^n} f = f(x_0)$. Notice that if there is another point $y_0 \neq x_0$ where f attains its minimum, then $u(x) = |x - y_0|$ is another solution.

We will give some conditions in Proposition 5.1 below that guarantee uniqueness when $n = 1$. However, we postpone this discussion until after we have considered the regularity of solutions of (1.1). We conclude this section by giving a few examples with explicit solutions.

Example 2.6 Assume $n = 1$, and consider the eigenvalue problem

$$\begin{cases} \max\{\lambda - u'' - x^2, |u'| - 1\} = 0, & x \in \mathbb{R} \\ \lim_{|x| \rightarrow \infty} \frac{u(x)}{|x|} = 1 \end{cases}.$$

Direct computation gives the explicit eigenvalue

$$\lambda^* = (2/3)^{2/3}$$

with a corresponding eigenfunction

$$\begin{aligned} u^*(x) &= \inf_{|y| < (\lambda^*)^{1/2}} \left\{ \frac{\lambda^*}{2} y^2 - \frac{1}{12} y^4 + |x - y| \right\} \\ &= \begin{cases} \frac{\lambda^*}{2} x^2 - \frac{1}{12} x^4, & |x| < (\lambda^*)^{1/2} \\ \frac{\lambda^*}{2} [(\lambda^*)^{1/2}]^2 - \frac{1}{12} [(\lambda^*)^{1/2}]^4 + (x - (\lambda^*)^{1/2}), & x \geq (\lambda^*)^{1/2} \\ \frac{\lambda^*}{2} [(\lambda^*)^{1/2}]^2 - \frac{1}{12} [(\lambda^*)^{1/2}]^4 - (x + (\lambda^*)^{1/2}), & x \leq -(\lambda^*)^{1/2} \end{cases}. \end{aligned}$$

One checks additionally that $u^* \in C^2(\mathbb{R})$. In fact, searching for a solution that is twice continuously differentiable lead us to the particular value of λ^* .

Example 2.7 The problem in the previous example can be generalized to any dimension $n \in \mathbb{N}$

$$\begin{cases} \max \{ \lambda - \Delta u - |x|^2, \max_{1 \leq i \leq n} |u_{x_i}| - 1 \} = 0, & x \in \mathbb{R}^n \\ \lim_{|x| \rightarrow \infty} u(x) / \sum_{i=1}^n |x_i| = 1 \end{cases} \quad (2.10)$$

Note that this problem corresponds to (1.1) when $F(M) = -\text{tr } M$, $f(x) = |x|^2$ and $H(p) = \max_{1 \leq i \leq n} |p_i| - 1$. In this case, $\ell(v) = \sum_{i=1}^n |v_i|$. Now assume (λ_1, u_1) is a solution of the eigenvalue problem in the previous example. Then $\lambda^* = n\lambda_1$ and

$$u^*(x) = \sum_{i=1}^n u_1(x_i)$$

is a solution of the eigenvalue problem (2.10) with $\lambda = \lambda^*$. Moreover, $u^* \in C^2(\mathbb{R}^n)$.

3 Existence of an eigenvalue

In order to prove the existence of an eigenvalue, we will study solutions of the following PDE for $\delta > 0$.

$$\max \{ \delta u + F(D^2u) - f(x), H(Du) \} = 0, \quad x \in \mathbb{R}^n. \quad (3.1)$$

This approach was inspired by the work of J. Menaldi, M. Robin and M. Taksar [20]. Employing the same techniques used to verify Proposition 2.3 above, we can establish the following assertion.

Proposition 3.1 *Assume $\delta > 0$, $u \in USC(\mathbb{R}^n)$ is a subsolution of (3.1) and $v \in LSC(\mathbb{R}^n)$ is a supersolution of (3.1). If u and v satisfy (2.5), then $u \leq v$.*

It is now immediate that there can be at most one solution of (3.1) that satisfies the growth condition (1.5). We will call this solution u_δ . To verify that u_δ exists, we can appeal to Perron’s method once we have appropriate sub and supersolutions. To this end, we first characterize the largest function v that is less than a given function g and satisfies $H(Dv) \leq 0$.

Lemma 3.2 *Assume $g \in C(\mathbb{R}^n)$ is superlinear. The unique solution of the PDE*

$$\max\{v - g, H(Dv)\} = 0, \quad x \in \mathbb{R}^n \quad (3.2)$$

that satisfies the growth condition (1.5) is given by the inf-convolution of g and ℓ

$$v(x) := \inf_{y \in \mathbb{R}^n} \{g(y) + \ell(x - y)\} \quad (3.3)$$

Proof The uniqueness follows from Proposition 3.1. In particular, this equation corresponds to (3.1) with $F \equiv 0$ and $\delta = 1$. Therefore, we only verify that v given in (3.3) is a solution that satisfies the growth condition (1.5). Choosing $y = x$ gives, $v(x) \leq g(x)$. Also note $x \mapsto g(y) + \ell(x - y)$ satisfies (2.4), which implies that v does as well. Hence, v is a subsolution of (3.2). In particular, $\limsup_{|x| \rightarrow \infty} v(x)/\ell(x) \leq 1$. Using $\ell(x - y) \geq \ell(x) - \ell(y)$,

$$v(x) \geq \inf_{y \in \mathbb{R}^n} \{g(y) - \ell(y)\} + \ell(x).$$

As g is assumed superlinear, $\inf_{\mathbb{R}^n} \{g(y) - \ell(y)\}$ is finite. Thus, $\liminf_{|x| \rightarrow \infty} v(x)/\ell(x) \geq 1$.

Finally, if ψ is another subsolution of (3.2)

$$\begin{aligned} v(x) &= \inf_{y \in \mathbb{R}^n} \{g(y) + \ell(x - y)\} \\ &\geq \inf_{y \in \mathbb{R}^n} \{\psi(y) + \ell(x - y)\} \\ &\geq \psi(x). \end{aligned}$$

By Lemma 4.4 of [7], v must be a supersolution of (3.2). □

The solution of (3.2) when $g(x) = \frac{1}{2}|x|^2$ will be of particular interest to us and will help us construct a useful supersolution of PDE (3.1).

Lemma 3.3 *Let $g(x) := \frac{1}{2}|x|^2$ and v the solution of (3.2) subject to the growth condition (1.5). Then*

$$v(x) = \frac{1}{2}|x|^2$$

when $H(x) \leq 0$, and

$$H(Dv) = 0$$

in $\{x \in \mathbb{R}^n : H(x) > 0\}$.

Proof Recall that $H(x) \leq 0$ implies $\ell(v) \geq x \cdot v$ for all $v \in \mathbb{R}^n$. Thus

$$\begin{aligned} v(x) &= \inf_{y \in \mathbb{R}^n} \left\{ \frac{1}{2}|y|^2 + \ell(x - y) \right\} \\ &\geq \inf_{y \in \mathbb{R}^n} \left\{ \frac{1}{2}|y|^2 + x \cdot (x - y) \right\} \\ &= \inf_{y \in \mathbb{R}^n} \left\{ \frac{1}{2}|y - x|^2 + \frac{1}{2}|x|^2 \right\} \\ &= \frac{1}{2}|x|^2. \end{aligned}$$

As $v(x) \leq \frac{1}{2}|x|^2$ for all x , the first claim follows.

Now suppose that $H(x) > 0$. Then there is a $v_0 \in \mathbb{R}^n$ with $|v_0| = 1$ such that $\ell(v_0) < x \cdot v_0$. Fix $\epsilon > 0$ so small that $\ell(v_0) < x \cdot v_0 - \epsilon$. Then

$$\begin{aligned} v(x) &= \inf_{y \in \mathbb{R}^n} \left\{ \frac{1}{2}|y - x|^2 + \ell(y) \right\} \\ &\leq \frac{1}{2}|(\epsilon v_0) - x|^2 + \ell(\epsilon v_0) \\ &= \frac{1}{2}|x|^2 + \frac{\epsilon^2}{2}|v_0|^2 - (\epsilon v_0) \cdot x + \ell(\epsilon v_0) \\ &= \frac{1}{2}|x|^2 + \frac{\epsilon^2}{2} + \epsilon[-v_0 \cdot x + \ell(v_0)] \\ &\leq \frac{1}{2}|x|^2 + \frac{\epsilon^2}{2} - \epsilon^2 \\ &< \frac{1}{2}|x|^2. \end{aligned}$$

Since v satisfies (3.2), the PDE $H(Du) = 0$ holds on the open set $\{x \in \mathbb{R}^n : H(x) > 0\}$. \square

We are now ready to exhibit sub and supersolutions of (3.1) that are comparable to $\ell(x)$ for large values of $|x|$.

Lemma 3.4 *Let $\delta \in (0, 1)$. There are constants $K_1, K_2 \geq 0$ such that*

$$\bar{u}(x) = \frac{K_1}{\delta} + \inf_{y \in \mathbb{R}^n} \left\{ \frac{1}{2}|y|^2 + \ell(x - y) \right\} \tag{3.4}$$

is a supersolution of (3.1) satisfying (1.5) and

$$\underline{u}(x) = (\ell(x) - K_2)^+ + \inf_{\mathbb{R}^n} f \tag{3.5}$$

is a subsolution of (3.1) satisfying (1.5).

Proof 1. Choose

$$K_1 := -F(I_n) + \sup_{H(x) \leq 0} f(x).$$

Lemma 3.3 implies $\bar{u}(x) = \frac{K_1}{\delta} + \frac{1}{2}|x|^2$ when $H(x) \leq 0$. Thus,

$$\delta \bar{u} + F(D^2 \bar{u}) - f \geq K_1 + F(I_n) - f \geq 0$$

on $\{x \in \mathbb{R}^n : H(x) < 0\}$.

We also have by Lemma 3.3 that $H(D\bar{u}) = 0$ on $\{x \in \mathbb{R}^n : H(x) > 0\}$. We will now verify that $H(D\bar{u}(x_0)) = 0$ when $H(x_0) = 0$. To this end, suppose that

$$\bar{u}(x_0) + p \cdot (x - x_0) + o(|x - x_0|) \leq \bar{u}(x)$$

as $x \rightarrow x_0$. Using $\bar{u}(x_0) = \frac{K_1}{\delta} + \frac{1}{2}|x_0|^2$ and $\bar{u}(x) \leq \frac{K_1}{\delta} + \frac{1}{2}|x|^2$ with the above inequality gives

$$\frac{1}{2}|x_0|^2 + p \cdot (x - x_0) + o(|x - x_0|) \leq \frac{1}{2}|x|^2,$$

as $x \rightarrow x_0$. It follows that $p = x_0$, and so $H(p) = H(x_0) = 0$.

2. Choose $K_2 \geq 0$ so large that

$$(\ell(x) - K_2)^+ \leq f(x) - \inf_{\mathbb{R}^n} f, \quad x \in \mathbb{R}^n.$$

Such a K_2 exists by the assumption that f is superlinear and (2.1). Observe \underline{u} defined in (3.5) satisfies (2.4); thus $H(D\underline{u}) \leq 0$. And as ℓ is convex, \underline{u} is convex. Therefore, $F(D^2 \underline{u}) \leq 0$ and

$$\delta \underline{u} + F(D^2 \underline{u}) - f \leq \delta \underline{u} - f \leq (\ell - K_2)^+ + \inf_{\mathbb{R}^n} f - f \leq 0, \quad x \in \mathbb{R}^n$$

for $\delta \leq 1$. \square

A key property of u_δ is that it is a convex function. This is critical to the arguments to follow. We also remark that our proof of this fact below was inspired by Korevaar’s work [17] and is an adaption of Lemma 3.7 in [14]. The new feature we verify here is that the assumption that F is superadditive still produces a convex solution of (3.1).

Proposition 3.5 *The function u_δ is convex.*

Proof For $\tau \in (0, 1)$ and $\eta > 0$, we define

$$w^\tau(x, y, z) := \tau u(z) - \frac{u(x) + u(y)}{2}$$

and

$$\varphi^\eta(x, y, z) = \frac{1}{2\eta} \left| \frac{x+y}{2} - z \right|^2$$

for $x, y, z \in \mathbb{R}^n$. Notice that

$$\begin{aligned} (w^\tau - \varphi_\eta)(x, y, z) &= \tau \left\{ u(z) - u\left(\frac{x+y}{2}\right) \right\} - \frac{1}{2\eta} \left| \frac{x+y}{2} - z \right|^2 \\ &\quad + \tau u\left(\frac{x+y}{2}\right) - \frac{u(x) + u(y)}{2} \\ &\leq \left(\tau \ell \left(\frac{x+y}{2} - z \right) - \frac{1}{2\eta} \left| \frac{x+y}{2} - z \right|^2 \right) \\ &\quad + \tau u\left(\frac{x+y}{2}\right) - \frac{u(x) + u(y)}{2}. \end{aligned}$$

By the growth condition (1.5), it follows that

$$\lim_{|x|+|y| \rightarrow \infty} \left\{ \tau u\left(\frac{x+y}{2}\right) - \frac{u(x) + u(y)}{2} \right\} = -\infty$$

and therefore

$$\lim_{|x|+|y|+|z| \rightarrow \infty} (w^\tau - \varphi_\eta)(x, y, z) = -\infty.$$

In particular, there is $(x_\eta, y_\eta, z_\eta) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$ maximizing $w^\tau - \varphi_\eta$. By Theorem 3.2 in [7], there are $X, Y, Z \in \mathcal{S}(n)$ such that

$$\begin{cases} (-2D_x \varphi_\eta(x_\eta, y_\eta, z_\eta), X) \in \bar{J}^{2,-} u(x_\eta) \\ (-2D_y \varphi_\eta(x_\eta, y_\eta, z_\eta), Y) \in \bar{J}^{2,-} u(y_\eta) \\ (\frac{1}{\tau} D_z \varphi_\eta(x_\eta, y_\eta, z_\eta), Z) \in \bar{J}^{2,+} u(z_\eta) \end{cases} \tag{3.6}$$

and

$$\tau Z \leq \frac{1}{2}(X + Y). \tag{3.7}$$

Now set

$$\begin{aligned} p_\eta &:= -2D_x \varphi_\eta(x_\eta, y_\eta, z_\eta) = -2D_y \varphi_\eta(x_\eta, y_\eta, z_\eta) = D_z \varphi_\eta(x_\eta, y_\eta, z_\eta) \\ &= \frac{1}{\eta} \left(z_\eta - \frac{x_\eta + y_\eta}{2} \right). \end{aligned}$$

By the bottom inclusion in (3.6),

$$\max\{\delta u(z_\eta) + F(Z) - f(z_\eta), H(p_\eta/\tau)\} \leq 0.$$

It follows that

$$H(p_\eta) = H\left(\tau \frac{p_\eta}{\tau} + (1 - \tau)0\right) < 0$$

and by the top two inclusions in (3.6),

$$\begin{cases} \delta u(x_\eta) + F(X) - f(x_\eta) \geq 0 \\ \delta u(y_\eta) + F(Y) - f(y_\eta) \geq 0 \end{cases}.$$

Combining these inequalities with (3.7) gives

$$\begin{aligned} \delta w^\tau(x, y, z) &\leq \delta w(x_\eta, y_\eta, z_\eta) \\ &= \tau \delta u(z_\eta) - \frac{\delta u(x_\eta) + \delta u(y_\eta)}{2} \\ &\leq \tau(-F(Z) + f(z_\eta)) - \frac{(-F(X) + f(x_\eta)) + (-F(Y) + f(y_\eta))}{2} \\ &= \left[-F(\tau Z) + \frac{F(X) + F(Y)}{2} \right] + \tau f(z_\eta) - \frac{f(x_\eta) + f(y_\eta)}{2} \\ &\leq \left[-F\left(\frac{X + Y}{2}\right) + \frac{F(X) + F(Y)}{2} \right] + \tau f(z_\eta) - \frac{f(x_\eta) + f(y_\eta)}{2} \\ &\leq f(z_\eta) - \frac{f(x_\eta) + f(y_\eta)}{2} + (\tau - 1)f(z_\eta) \\ &\leq f(z_\eta) - \frac{f(x_\eta) + f(y_\eta)}{2} + (\tau - 1) \inf_{\mathbb{R}^n} f \end{aligned} \tag{3.8}$$

for each $(x, y, z) \in \mathbb{R}^n$.

Another basic estimate for $w^\tau - \varphi_\eta$ that stems from (3.6) and (2.1) is

$$(w^\tau - \varphi_\eta)(x, y, z) \leq \tau u\left(\frac{x + y}{2}\right) - \frac{u(x) + u(y)}{2} + \tau^2 c_1^2 \eta.$$

This inequality gives that $(x_\eta, y_\eta)_{\eta>0} \subset \mathbb{R}^n \times \mathbb{R}^n$ is bounded. For were this not the case, $(w^\tau - \varphi_\eta)(x_\eta, y_\eta, z_\eta)$ tends to $-\infty$ yet

$$\begin{aligned} (w^\tau - \varphi_\eta)(x_\eta, y_\eta, z_\eta) &= \max_{x,y,z} (w^\tau - \varphi_\eta)(x, y, z) \\ &\geq (w^\tau - \varphi_\eta)(0, 0, 0) \\ &= (\tau - 1)u(0) \\ &> -\infty, \end{aligned}$$

for each $\eta > 0$. Similarly, $(z_\eta)_{\eta>0} \subset \mathbb{R}^n$ is bounded.

Again we appeal to Lemma 3.1 in [7], which asserts the existence of a cluster point $(x_\tau, y_\tau, (x_\tau + y_\tau)/2)$ of $((x_\eta, y_\eta, z_\eta))_{\eta>0}$ that maximizes

$$(x, y) \mapsto \tau u\left(\frac{x + y}{2}\right) - \frac{u(x) + u(y)}{2}.$$

Thus, we may pass to the limit through an appropriate sequence of η tending to 0 in (3.8) to find for any $x, y \in \mathbb{R}^n$

$$\begin{aligned} \tau u\left(\frac{x + y}{2}\right) - \frac{u(x) + u(y)}{2} &\leq f\left(\frac{x_\tau + y_\tau}{2}\right) \\ &\quad - \frac{f(x_\tau) + f(y_\tau)}{2} + (\tau - 1) \inf_{\mathbb{R}^n} f \leq (\tau - 1) \inf_{\mathbb{R}^n} f. \end{aligned}$$

Here we have used the convexity of f . Finally, we conclude upon sending $\tau \rightarrow 1^-$. □

By Aleksandrov’s theorem (section 6.4 of [11]), u_δ is twice differentiable at Lebesgue almost every $x \in \mathbb{R}^n$. At any such x , if $H(Du(x)) < 0$, then x must be uniformly bounded for

$$f(x) = \delta u_\delta(x) + F(D^2u_\delta) \leq \delta u_\delta(x).$$

Recall that f is superlinear and u_δ grows at most linearly. As precise statement is as follows.

Corollary 3.6 *There is a constant R , independent of $\delta \in (0, 1)$, such that if $p \in J^{1,-}u_\delta(x)$ and $H(p) < 0$, then $|x| \leq R$.*

Proof As u_δ is convex, $J^{1,-}u_\delta(x) = \partial u(x)$; see proposition 4.7 in [2]. It then follows that $(p, O_n) \in J^{2,-}u_\delta(x)$. Thus,

$$\max\{\delta u_\delta(x) - f(x), H(p)\} \geq 0.$$

As $H(p) < 0$, it must be that $\delta u_\delta(x) - f(x) \geq 0$. As a result,

$$f(x) \leq \delta u_\delta(x) \leq K_1 + \ell(x) \leq K_1 + c_1|x|.$$

Thus, $|x| \leq R$ for some R that is independent of $\delta \in (0, 1)$. □

Another important corollary is the following “extension formula” for solutions. We interpret this formula informally as: once the values of $u_\delta(x)$ are known for each x satisfying $H(Du_\delta(x)) < 0$, u_δ is determined on all of \mathbb{R}^n .

Corollary 3.7 *Let*

$$\Omega_\delta := \mathbb{R}^n \setminus \{x \in \mathbb{R}^n : H(Du_\delta(x)) \geq 0 \text{ in the viscosity sense } \}. \tag{3.9}$$

Then

$$u_\delta(x) = \inf \{u_\delta(y) + \ell(x - y) : y \in \Omega_\delta\}, \quad x \in \mathbb{R}^n. \tag{3.10}$$

Moreover, the infimum in (3.10) can be taken over $\partial\Omega_\delta$ when $x \notin \Omega_\delta$.

Proof Set $u = u_\delta$ and define v to be the right hand side of (3.10). Since $u(x) \leq u(y) + \ell(x - y)$ for each $x, y \in \mathbb{R}^n$, $u \leq v$. If $x \in \Omega_\delta$, there is a sequence $(x_k)_{k \in \mathbb{N}} \subset \Omega_\delta$ converging to x as $k \rightarrow \infty$. Clearly, $v(x) \leq u(x_k) + \ell(x - x_k)$ and sending $k \rightarrow \infty$ gives $v(x) \leq u(x)$. Thus, $u(x) = v(x)$ for $x \in \Omega_\delta$.

Observe that $v(x) - v(y) \leq \ell(x - y)$ for all $x, y \in \mathbb{R}^n$. Therefore, v satisfies the PDE $H(Dv) \leq 0$ on \mathbb{R}^n . In particular,

$$\begin{cases} H(Dv) \leq 0 \leq H(Du), & x \in \mathbb{R}^n \setminus \overline{\Omega_\delta} \\ v = u, & x \in \partial\Omega_\delta \end{cases}$$

while

$$\limsup_{|x| \rightarrow \infty} \frac{v(x)}{\ell(x)} \leq 1 \leq \liminf_{|x| \rightarrow \infty} \frac{u(x)}{\ell(x)}.$$

It follows from an argument similar to one given in Proposition 2.3 used to derive (2.8), that

$$\tau v - u \leq (\tau - 1) \inf_{\mathbb{R}^n} f$$

for each $\tau \in (0, 1)$. In particular, $v \leq u$ on $\mathbb{R}^n \setminus \overline{\Omega_\delta}$. So we are able to conclude (3.10).

Now suppose $x \notin \Omega_\delta$ and choose $y \in \Omega_\delta$ such that $u(x) = u(y) + \ell(x - y)$. There is a $t \in [0, 1]$ such that

$$z = ty + (1 - t)x \in \partial\Omega_\delta.$$

Observe that since u is convex and ℓ is positively homogeneous

$$\begin{aligned} u(z) + \ell(x - z) &= u(ty + (1 - t)x) + \ell(t(x - y)) \\ &\leq t(u(y) + \ell(x - y)) + (1 - t)u(x) \\ &= tu(x) + (1 - t)u(x) \\ &\leq u(x). \end{aligned}$$

Thus, the minimum in (3.10) occurs on the boundary of $\partial\Omega_\delta$ when $x \notin \Omega_\delta$. □

We will now verify the existence of an eigenvalue. Let $\delta \in (0, 1)$ and set

$$\begin{cases} \lambda_\delta := \delta u_\delta(0) \\ v_\delta(x) := u_\delta(x) - u_\delta(0), \quad x \in \mathbb{R}^n. \end{cases}$$

In view of (3.4) and (3.5),

$$-\left(\inf_{\mathbb{R}^n} f\right)^- \leq \lambda_\delta \leq K_1; \tag{3.11}$$

and by (2.1),

$$\begin{cases} |v_\delta(x)| \leq c_1|x| \\ |v_\delta(x) - v_\delta(y)| \leq c_1|x - y| \end{cases} \tag{3.12}$$

for $x, y \in \mathbb{R}^n$ and $0 < \delta < 1$.

Proof (part (i) of Theorem 1.1) By (3.11) and (3.12), there is a sequence of positive numbers $(\delta_k)_{k \in \mathbb{N}}$ tending to 0, $\lambda^* \in \mathbb{R}$ and $u^* \in C(\mathbb{R}^n)$ such that $\lambda_{\delta_k} \rightarrow \lambda^*$ and $v_{\delta_k} \rightarrow u^*$ locally uniformly on \mathbb{R}^n . By the stability of viscosity solutions under locally uniform convergence (Lemma 6.1 in [7]), u^* satisfies (1.1) with $\lambda = \lambda^*$.

In view of the extension formula (3.10), (2.1), and Corollary 3.6,

$$\begin{aligned} v_{\delta_k}(x) &= u_{\delta_k}(x) - u_{\delta_k}(0) \\ &= \inf_{y \in \Omega_{\delta_k}} \{u_{\delta_k}(y) - u_{\delta_k}(0) + \ell(x - y)\} \\ &\geq \inf_{y \in \Omega_{\delta_k}} \{-c_1|y| + \ell(x - y)\} \\ &\geq \inf_{y \in \Omega_{\delta_k}} \{-c_1|y| + \ell(x) - \ell(y)\} \\ &= \ell(x) - 2c_1 \sup_{y \in \Omega_{\delta_k}} |y| \\ &\geq \ell(x) - 2c_1R. \end{aligned}$$

Thus, $u^*(x) \geq \ell(x) - 2c_1R$ and in particular, u^* satisfies the growth condition (1.5). It now follows that λ^* is the desired eigenvalue. □

We also have the following characterization of the eigenvalue λ^* . See also [1] for a similar characterization of eigenvalues of operators that are uniformly elliptic, fully nonlinear, and positively homogeneous.

Corollary 3.8 *Let λ^* be as described in part (i) of Theorem 1.1. Then*

$$\lambda^* = \sup\{\lambda \in \mathbb{R} : \text{there is a subsolution } u \text{ of (1.1) with eigenvalue } \lambda \text{ satisfying } \limsup_{|x| \rightarrow \infty} \frac{u(x)}{\ell(x)} \leq 1\} \tag{3.13}$$

and

$$\lambda^* = \inf\{\mu \in \mathbb{R} : \text{there is a supersolution } v \text{ of (1.1) with eigenvalue } \mu \text{ satisfying } \liminf_{|x| \rightarrow \infty} \frac{v(x)}{\ell(x)} \geq 1\}. \tag{3.14}$$

In particular, choosing $\lambda = \inf_{\mathbb{R}^n} f$ and $u \equiv 0$ in (3.13) gives $\lambda^* \geq \inf_{\mathbb{R}^n} f$. And selecting $\mu = -F(I_n) + \sup_{H(x) \leq 0} f(x)$ and $v(x) = \inf_{\mathbb{R}^n} \{|y|^2/2 + \ell(x - y)\}$ in (3.14) gives $\lambda^* \leq -F(I_n) + \sup_{H(x) \leq 0} f(x)$. In summary, we have the bounds on λ^*

$$\inf_{x \in \mathbb{R}^n} f(x) \leq \lambda^* \leq -F(I_n) + \sup_{H(x) \leq 0} f(x).$$

4 Regularity of solutions

Our goal in this section is to prove part (ii) of Theorem 1.1. To this end, we will assume that F is uniformly elliptic, assume H satisfies (1.6) and derive a uniform upper bound on D^2u_δ . Recall u_δ is the unique solution of (3.1) that satisfies (1.5). We will first use an easy semiconcavity argument to bound $D^2u_\delta(x)$ for all large values of $|x|$. Then we will pursue second derivatives bounds on u_δ for smaller values of $|x|$. To this end, we will employ to the so-called ‘‘penalty method’’ introduced by L. C. Evans [8]. For other related work, consult also [15, 16, 23, 26].

4.1 Preliminaries

An important identity for us will be

$$\ell(v) = \inf_{\lambda > 0} \lambda H^*\left(\frac{v}{\lambda}\right), \quad v \in \mathbb{R}^n \setminus \{0\} \tag{4.1}$$

where $H^*(w) = \sup_{p \in \mathbb{R}^n} \{p \cdot w - H(p)\}$ is the Legendre transform of H ; see exercise 11.6 of [22]. This formula is crucial to our method for deriving second derivatives estimates on u_δ for large values of $|x|$.

Lemma 4.1 *Define Ω_δ as in (3.9). There is a constant C such that*

$$D^2u_\delta(x) \leq \frac{C}{\text{dist}(x, \Omega_\delta)} I_n$$

for Lebesgue almost every $x \in \mathbb{R}^n \setminus \overline{\Omega_\delta}$.

Proof We will employ formula (4.1). We will also use that

$$H^*(0) > 0 \tag{4.2}$$

and

$$\frac{1}{\Sigma} |\xi|^2 \leq D^2H^*(w)\xi \cdot \xi \leq \frac{1}{\sigma} |\xi|^2, \quad \xi \in \mathbb{R}^n \tag{4.3}$$

for almost every $w \in \mathbb{R}^n$. Let $v \in \mathbb{R}^n \setminus \{0\}$ and $\lambda > 0$. Note (4.3) implies

$$\begin{aligned} \lambda H^*(0) + DH^*(0) \cdot v + \frac{1}{2\Sigma\lambda} |v|^2 &\leq \lambda H^*\left(\frac{v}{\lambda}\right) \leq \lambda H^*(0) \\ &\quad + DH^*(0) \cdot v + \frac{1}{2\sigma\lambda} |v|^2. \end{aligned} \tag{4.4}$$

Thus, $\lim_{\lambda \rightarrow 0^+} \lambda H^*(v/\lambda) = +\infty$. And with (4.2), we also conclude that $\lim_{\lambda \rightarrow \infty} \lambda H^*(v/\lambda) = +\infty$. As $\lambda \mapsto \lambda H^*(v/\lambda)$ is strictly convex, there is a unique $\lambda = \lambda(v) > 0$ for which $\ell(v) = \lambda(v)H^*(v/\lambda(v))$.

Using the positive homogeneity of ℓ , for $t > 0$

$$\begin{aligned} \lambda(tv)H^*\left(\frac{tv}{\lambda(tv)}\right) &= \ell(tv) \\ &= t\ell(v) \\ &= t\lambda(v)H^*\left(\frac{v}{\lambda(v)}\right) \\ &= t\lambda(v)H^*\left(\frac{tv}{t\lambda(v)}\right). \end{aligned}$$

Thus, $\lambda(tv) = t\lambda(v)$. It also follows from (4.4) that

$$\gamma := \inf_{|v|=1} \lambda(v) > 0.$$

In particular, $\lambda(v) \geq \gamma|v|$, for each $v \neq 0$.

Again let $v \neq 0$, and choose $h \in \mathbb{R}^n$ so small in norm that $v \pm h \neq 0$. Then for $\lambda = \lambda(v)$

$$\begin{aligned} \ell(v+h) - 2\ell(v) + \ell(v-h) &\leq \lambda H^*\left(\frac{v+h}{\lambda}\right) - 2\lambda H^*\left(\frac{v}{\lambda}\right) + \lambda H^*\left(\frac{v-h}{\lambda}\right) \\ &= \lambda \left[H^*\left(\frac{v}{\lambda} + \frac{h}{\lambda}\right) - 2H^*\left(\frac{v}{\lambda}\right) + H^*\left(\frac{v}{\lambda} - \frac{h}{\lambda}\right) \right] \\ &\leq \lambda \frac{1}{\sigma} \left| \frac{h}{\lambda} \right|^2 \\ &= \frac{1}{\sigma\lambda} |h|^2 \\ &\leq \frac{1}{\gamma\sigma|v|} |h|^2. \end{aligned}$$

Now we can employ the extension formula (3.10). Let $x \in \mathbb{R}^n \setminus \overline{\Omega}_\delta$ and choose h so small that $x \pm h \in \mathbb{R}^n \setminus \overline{\Omega}_\delta$. Selecting $y \in \partial\Omega_\delta$ so that $u_\delta(x) = u_\delta(y) + \ell(x - y)$ gives

$$\begin{aligned} u_\delta(x+h) - 2u_\delta(x) + u_\delta(x-h) &\leq \ell(x-y+h) - 2\ell(x-y) + \ell(x-y-h) \\ &\leq \frac{1}{\gamma\sigma|x-y|} |h|^2 \\ &\leq \frac{C}{\text{dist}(x, \partial\Omega_\delta)} |h|^2. \end{aligned}$$

The claim follows as u_δ is differentiable Lebesgue almost everywhere. □

In order to complete the proof of part (ii) of Theorem 1.1, we must bound the second derivatives on u_δ on some subset of \mathbb{R}^n that includes $\overline{\Omega}_\delta$. Before we detail our approach, it

will be necessary for us to differentiate (a smoothing) of F . To this end, we extend F to the space $M_n(\mathbb{R})$ of all $n \times n$ real matrices as follows

$$\overline{F}(M) := F\left(\frac{1}{2}(M + M^t)\right), \quad M \in M_n(\mathbb{R}).$$

We can then treat $\overline{F}(M)$ as a function of the n^2 real entries of the matrix $M \in M_n(\mathbb{R})$. It is readily checked that \overline{F} is uniformly elliptic, positively homogeneous and superadditive on $M_n(\mathbb{R})$. In particular, \overline{F} satisfies (1.2) for each $M, N \in M_n(\mathbb{R})$ and $t \geq 0$. This allows us to identify F with \overline{F} and we shall do this for the remainder of this section.

We now define F^ϱ as the standard mollification of F

$$F^\varrho(M) := \int_{M_n(\mathbb{R})} \eta^\varrho(N)F(M - N)dN, \quad M \in M_n(\mathbb{R}).$$

The integral above is over the n^2 real variables $N = (N_{ij}) \in M_n(\mathbb{R})$, and as in Lemma 2.1, $\eta \in C_c^\infty(M_n(\mathbb{R}))$ is a nonnegative function that is supported in $\{M \in M_n(\mathbb{R}) : |M| \leq 1\}$ and $\eta(M)$ only depends on $|M|$. Moreover, η satisfies $\int_{\mathbb{R}^n} \eta(Z)dZ = 1$ and we have defined $\eta^\varrho := \varrho^{-n^2}\eta(\cdot/\varrho)$. See also section 4 of [15] or Proposition 9.8 in [5] for more details on mollifying functions of matrices.

It is readily verified that $F^\varrho \in C^\infty(M_n(\mathbb{R}))$ and, with the help of (1.2), F^ϱ is uniformly elliptic, concave and satisfies

$$F^\varrho(M) \leq F(M) \leq F^\varrho(M) + \sqrt{n}\Theta\varrho, \quad M \in M_n(\mathbb{R}). \tag{4.5}$$

However, F^ϱ is not in general positively homogeneous. Nevertheless, F^ϱ inherits a certain almost homogeneity property.

Lemma 4.2 *For every $M \in M_n(\mathbb{R})$,*

$$F^\varrho(M) = F_{M_{ij}}^\varrho(M)M_{ij} - \int_{M_n(\mathbb{R})} \eta^\varrho(N)F_{M_{ij}}(M - N)N_{ij}dN.$$

In particular,

$$|F^\varrho(M) - F_{M_{ij}}^\varrho(M)M_{ij}| \leq \sqrt{n}\Theta\varrho, \quad M \in M_n(\mathbb{R}). \tag{4.6}$$

Proof By the ellipticity assumption (1.2), F is Lipschitz continuous. Rademacher’s Theorem then implies that F is differentiable for Lebesgue almost every $M \in M_n(\mathbb{R})$, which we identify with \mathbb{R}^{n^2} . Therefore,

$$F_{M_{ij}}^\varrho(M) = \int_{M_n(\mathbb{R})} \eta^\varrho(N)F_{M_{ij}}(M - N)dN.$$

See Theorem 1 of section 5.3 in [10] for an easy verification of this equality. Since F is positively homogenous of degree one,

$$F(M) = F_{M_{ij}}(M)M_{ij}$$

for Lebesgue almost every $M \in M_n(\mathbb{R})$. And therefore,

$$\begin{aligned}
 F_{M_{ij}}^\varrho(M)M_{ij} &= \int_{M_n(\mathbb{R})} \eta^\varrho(N)F_{M_{ij}}(M - N)M_{ij}dN \\
 &= \int_{M_n(\mathbb{R})} \eta^\varrho(N)F_{M_{ij}}(M - N)(M_{ij} - N_{ij})dN \\
 &\quad + \int_{M_n(\mathbb{R})} \eta^\varrho(N)F_{M_{ij}}(M - N)N_{ij}dN \\
 &= \int_{M_n(\mathbb{R})} \eta^\varrho(N)F(M - N)dN + \int_{M_n(\mathbb{R})} \eta^\varrho(N)F_{M_{ij}}(M - N)N_{ij}dN \\
 &= F^\varrho(M) + \int_{M_n(\mathbb{R})} \eta^\varrho(N)F_{M_{ij}}(M - N)N_{ij}dN.
 \end{aligned}$$

The ellipticity assumption (1.2) also implies

$$-\Theta|\xi|^2 \leq F_{M_{ij}}(M)\xi_i\xi_j \leq -\theta|\xi|^2, \quad \xi \in \mathbb{R}^n$$

for almost every $M \in M_n(\mathbb{R})$. Therefore,

$$\begin{aligned}
 \left| \int_{M_n(\mathbb{R})} \eta^\varrho(N)F_{M_{ij}}(M - N)N_{ij}dN \right| &\leq \int_{M_n(\mathbb{R})} \eta^\varrho(N) |F_{M_{ij}}(M - N)N_{ij}| dN \\
 &\leq \int_{M_n(\mathbb{R})} \eta^\varrho(N) \sqrt{\sum_{ij=1}^n (F_{M_{ij}}(M - N))^2} |N|dN \\
 &\leq \sqrt{n}\Theta \int_{M_n(\mathbb{R})} \eta^\varrho(N)|N|dN \\
 &= \sqrt{n}\Theta\varrho \int_{|Z|\leq 1} \eta(Z)|Z|dZ \quad (Z = N/\varrho) \\
 &\leq \sqrt{n}\Theta\varrho \int_{|Z|\leq 1} \eta(Z)dZ \\
 &= \sqrt{n}\Theta\varrho.
 \end{aligned}$$

□

We will additionally need to smooth out H and f , and we will do so by using the mollifications $H^\varrho = \eta^\varrho * H$ and $f^\varrho = \eta^\varrho * f$. Here η is a standard mollifier on \mathbb{R}^n . Let us select ϱ_1 so small that

$$H^\varrho(0) < 0, \quad \varrho \in (0, \varrho_1). \tag{4.7}$$

We will also employ $u_{\delta,\varrho}$, the solution of (3.1) with H^ϱ and f^ϱ replacing H and f that is subject to the growth condition (1.5) with $\ell^\varrho(v) = \sup\{p \cdot v : H^\varrho(p) \leq 0\}$ replacing ℓ . Some useful pointwise estimates on $u_{\delta,\varrho}$ are as follows.

Lemma 4.3 *Assume $\delta \in (0, 1)$ and $\varrho \in (0, \varrho_1)$. Then*

$$\inf_{\mathbb{R}^n} f \leq u_{\delta,\varrho} \leq \frac{1}{\delta} \left(n\Theta + \sup_{H \leq 0} f^\varrho \right) + \ell. \tag{4.8}$$

Proof Using test functions as in (3.4) and (3.5) that correspond to (3.1) with F , H^ϱ , and f^ϱ , we find

$$\inf_{\mathbb{R}^n} f^\varrho \leq u_{\delta,\varrho} \leq \frac{1}{\delta} \left(-F(I_n) + \sup_{H^\varrho \leq 0} f^\varrho \right) + \ell^\varrho. \tag{4.9}$$

By the convexity of H and f , Jensen’s inequality gives $H \leq H^\varrho$ and $f \leq f^\varrho$. It then follows that $\ell^\varrho \leq \ell$. By the uniform ellipticity of F , we also have $-F(I_n) \leq n\Theta$. These observations together with (4.9) give (4.8). \square

4.2 The penalty method

Now we fix $\delta \in (0, 1)$, $\rho \in (0, \rho_1)$ and choose a ball $B = B_R(0) \subset \mathbb{R}^n$ so large that

$$H(p) \leq 0 \implies |p| \leq R. \tag{4.10}$$

For $\epsilon > 0$, we will study solutions $u = u^\epsilon$ of the PDE

$$\delta u + F^\varrho(D^2u) + \beta_\epsilon(H^\varrho(Du)) = f^\varrho - \delta u_{\delta,\varrho}(0), \quad x \in B \tag{4.11}$$

subject to the boundary condition

$$u(x) = u_{\delta,\varrho}(x) - u_{\delta,\varrho}(0), \quad x \in \partial B. \tag{4.12}$$

Recall that $u_{\delta,\varrho}$ is the solution of (3.1) with F , H^ϱ , and f^ϱ subject to the growth condition (1.5) with $\ell^\varrho(v) = \sup\{p \cdot v : H^\varrho(p) \leq 0\}$ instead of ℓ .

In (4.11), F^ϱ is a standard mollification of F and the family $\{\beta_\epsilon\}_{\epsilon>0}$ of functions each satisfy

$$\begin{cases} \beta_\epsilon \in C^\infty(\mathbb{R}) \\ \beta_\epsilon(z) = 0, & z \leq 0 \\ \beta_\epsilon(z) > 0, & z > 0 \\ \beta'_\epsilon \geq 0, \\ \beta''_\epsilon \geq 0, \\ \beta_\epsilon(z) = (z - \epsilon)/\epsilon, & z \geq 2\epsilon. \end{cases} \tag{4.13}$$

Our intuition is that β_ϵ is a smoothing of the Lipschitz function $z \mapsto (z/\epsilon)^+$; and therefore, solutions of (4.11) should be close to solutions of

$$\max\{\delta u + F^\varrho(D^2u) - (f^\varrho - \delta u_{\delta,\varrho}(0)), H^\varrho(Du)\} = 0, \quad x \in B$$

that satisfy (4.12). These solutions will in turn be very close to $u_{\delta,\varrho} - u_{\delta,\varrho}(0)|_B$ for ϱ small; this is verified in Proposition 4.8 below.

By a theorem of N. Trudinger (Theorem 8.2 in [25]) there is a unique classical solution $u^\epsilon \in C^\infty(B) \cap C(\bar{B})$ solving (4.11) and satisfying the boundary condition (4.12). This result relies on the Evans–Krylov a priori estimates for solutions of concave, fully nonlinear elliptic equations and the continuity method [9, 19]. Along with the concavity of F , the main structural condition that allows us to apply this theorem is that $p \mapsto \beta_\epsilon(H^\varrho(p))$ grows at most quadratically for each $\epsilon > 0$. We remark that u^ϵ naturally depends on the other parameters $\delta \in (0, 1)$ and $\varrho \in (0, \varrho_1)$; we have chosen not to indicate this dependence for ease of notation.

Since $u_{\delta,\varrho}$ solves (3.1) with $F, H^\varrho,$ and $f^\varrho,$ we have from (4.13) and (4.5) that

$$\begin{aligned} \delta(u_{\delta,\varrho} - u_{\delta,\varrho}(0)) + F^\varrho(D^2u_{\delta,\varrho}) + \beta_\epsilon(H^\varrho(Du_{\delta,\varrho})) &= \delta u_{\delta,\varrho} + F^\varrho(D^2u_{\delta,\varrho}) - \delta u_{\delta,\varrho}(0) \\ &\leq \delta u_{\delta,\varrho} + F(D^2u_{\delta,\varrho}) - \delta u_{\delta,\varrho}(0) \\ &\leq f^\varrho - \delta u_{\delta,\varrho}(0). \end{aligned}$$

In view of (4.11) and (4.12), $u_{\delta,\varrho} - u_{\delta,\varrho}(0) \leq u^\epsilon$ by a routine maximum principle argument. Also note

$$F^\varrho(D^2u^\epsilon) \leq f^\varrho - \delta u_{\delta,\varrho}(0) - \delta u^\epsilon \leq f^\varrho - \delta u_{\delta,\varrho}.$$

The Aleksandrov–Bakelman–Pucci estimate (Theorem 3.6 in [5], Theorem 17.3 in [13]) then implies

$$\sup_B u^\epsilon \leq C \left(\sup_{\partial B} |u_{\delta,\varrho} - u_{\delta,\varrho}(0)| + \sup_B |f^\varrho - \delta u_{\delta,\varrho}| \right)$$

for some constant $C = C(\text{diam}(B), n, \theta, \Theta).$ Since $H(Du_{\delta,\varrho}) \leq H^\varrho(Du_{\delta,\varrho}) \leq 0,$ (4.10) gives

$$|u_{\delta,\varrho}(x) - u_{\delta,\varrho}(0)| \leq c_1 R, \quad x \in B.$$

By combining this estimate with (4.8), we have in summary

$$|u^\epsilon|_{L^\infty(B)} \leq C \left\{ c_1 R + \left(\inf_{\mathbb{R}^n} f \right)^- + \sup_B \ell + n\Theta + \sup_B |f^\varrho| \right\}. \tag{4.14}$$

We will use this estimate to obtain bounds on the higher derivatives of u^ϵ that will be independent of all $\epsilon > 0$ and sufficiently small.

We are now in a position to derive uniform estimates on the derivatives of $u^\epsilon.$ We will borrow from the recent work by the author and H. Mawi on fully nonlinear elliptic equations with convex gradient constraints [15]. Note however, one of the main assumptions in [15] is that the nonlinearity is uniformly elliptic and *convex*; note the class of nonlinearities we study in this paper satisfy (1.2) and are *concave*. We will make use of Lemma 4.2 instead of a convexity assumption on $F.$

We will also employ the uniform convexity assumption (1.6), which implies

$$\begin{cases} H^\varrho(p) \geq H^\varrho(0) + DH^\varrho(0) \cdot p + \frac{\sigma}{2}|p|^2 \\ DH^\varrho(p) \cdot p - H^\varrho(p) \geq -H^\varrho(0) + \frac{\sigma}{2}|p|^2 \\ |DH^\varrho(p)| \leq |DH^\varrho(0)| + \sqrt{n}\Sigma|p| \end{cases} \quad (p \in \mathbb{R}^n). \tag{4.15}$$

And we choose $\varrho_1 > 0$ sufficiently smaller if necessary so that (4.7) holds and

$$\begin{cases} |H^\varrho(0)| \leq |H(0)| + 1 \\ |DH^\varrho(0)| \leq |DH(0)| + 1 \\ |f^\varrho|_{W^{1,\infty}(B)} \leq |f|_{W^{1,\infty}(B_{R+1}(0))} \quad (B = B_R(0)) \end{cases}$$

for $0 < \varrho < \varrho_1.$ In stating our uniform estimates below, it will be convenient for us to label the following list

$$\Pi := \left(\sigma, \Sigma, \theta, \Theta, n, R, c_1, H(0), |DH(0)|, |f|_{W^{1,\infty}(B_{R+1}(0))}, \inf_{\mathbb{R}^n} f, \sup_B \ell, \varrho_1 \right).$$

Lemma 4.4 *Let $\delta \in (0, 1)$, $\varrho \in (0, \varrho_1)$, $\epsilon \in (0, 1)$ and suppose $\zeta \in C_c^\infty(B)$ is nonnegative. There is a constant C depending only on the list Π and $|\zeta|_{W^{2,\infty}(B)}$ such that*

$$\zeta(x)|Du^\epsilon(x)| \leq C, \quad x \in B.$$

Proof 1. Set

$$M_\epsilon := \sup_{x \in B} |\zeta(x)Du^\epsilon(x)|$$

and define

$$v^\epsilon(x) := \frac{1}{2}\zeta^2(x)|Du^\epsilon(x)|^2 - \alpha_\epsilon u^\epsilon(x).$$

Here α_ϵ is a positive constant that will be chosen below. We will first obtain a bound on v^ϵ from above and then use the resulting estimate to bound M_ϵ . We emphasize that each constant below will only depend on the list Π and $|\zeta|_{W^{2,\infty}(B)}$; in particular, the constants will not depend on ϵ and α_ϵ .

2. We first differentiate equation (4.11) with respect to x_k ($k = 1 \dots, n$) to get

$$\delta u_{x_k}^\epsilon + F_{M_{ij}}^\varrho(D^2u^\epsilon)u_{x_i x_j x_k}^\epsilon + \beta'_\epsilon(H^\varrho(Du^\epsilon))DH^\varrho(Du^\epsilon) \cdot Du_{x_k}^\epsilon = f_{x_k}^\varrho. \tag{4.16}$$

We suppress ϵ, ϱ dependence and function arguments and use (4.16) to compute

$$\begin{aligned} F_{M_{ij}}v_{x_i x_j} + \beta' H_{p_k}v_{x_k} &= (F_{M_{ij}}\zeta_{x_i}\zeta_{x_j} + \zeta F_{M_{ij}}\zeta_{x_i x_j})|Du|^2 \\ &\quad + 4F_{M_{ij}}\zeta\zeta_{x_i}Du \cdot Du_{x_j} + \zeta^2 F_{M_{ij}}Du_{x_i} \cdot Du_{x_j} \\ &\quad - \beta' H_{p_k}(\alpha u_{x_k} - \zeta\zeta_{x_k}|Du|^2) \\ &\quad + \zeta^2 u_{x_k}(f_{x_k} - \delta u_{x_k}) - \alpha F_{M_{ij}}u_{x_i x_j}. \end{aligned} \tag{4.17}$$

We reiterate that in (4.17), we have written u for u^ϵ , v for v^ϵ , F for $F^\varrho(D^2u^\epsilon)$, β for $\beta_\epsilon(H^\varrho(Du^\epsilon))$, H for $H^\varrho(Du^\epsilon)$ and f for f^ϱ . We will continue this convention for the remainder of this proof.

3. Now we recall Lemma 4.2. In particular, the inequality (4.6) along with the convexity of $\beta = \beta_\epsilon$ implies

$$\begin{aligned} -F_{M_{ij}}u_{x_i x_j} &:= -F_{M_{ij}}(D^2u)(D^2u)_{ij} \\ &\leq -F(D^2u) + \sqrt{n}\Theta_{\varrho_1} \\ &= \beta(H(Du)) + \delta u - (f - \delta u_{\delta, \varrho}(0)) + \sqrt{n}\Theta_{\varrho_1} \\ &\leq H(Du)\beta'(H(Du)) + \delta u - (f - \delta u_{\delta, \varrho}(0)) + \sqrt{n}\Theta_{\varrho_1}. \end{aligned}$$

Combining with (4.17) gives

$$\begin{aligned} F_{M_{ij}}v_{x_i x_j} + \beta' H_{p_k}v_{x_k} &\leq (F_{M_{ij}}\zeta_{x_i}\zeta_{x_j} + \zeta F_{M_{ij}}\zeta_{x_i x_j})|Du|^2 \\ &\quad + 4F_{M_{ij}}\zeta\zeta_{x_i}Du \cdot Du_{x_j} + \zeta^2 F_{M_{ij}}Du_{x_i} \cdot Du_{x_j} \\ &\quad - \beta'(\alpha(H_{p_k}u_{x_k} - H) - \zeta H_{p_k}\zeta_{x_k}|Du|^2) \\ &\quad + \zeta^2 u_{x_k}(f_{x_k} - \delta u_{x_k}) + \alpha(\delta u - (f - \delta u_{\delta, \varrho}(0)) + \sqrt{n}\Theta_{\varrho_1}). \end{aligned} \tag{4.18}$$

3. Assume $x_0 \in \bar{B}$ is a maximizing point for v . If $x_0 \in \partial B$, then $v \leq -\alpha(u_{\delta, \varrho}(x_0) - u_{\delta, \varrho}(0)) \leq \alpha c_1 R$. Therefore,

$$v \leq C(\alpha + 1). \tag{4.19}$$

Alternatively, suppose $x_0 \in B$. If $\beta' = \beta'(H(Du(x_0))) \leq 1 < 1/\epsilon$, then $H(Du(x_0)) \leq 2\epsilon \leq 2$. By (4.15), $|Du(x_0)|$ is bounded from above independently of ϵ . Hence, the inequality (4.19) holds for an appropriate constant C . The final situation to consider is when $\beta' = \beta'(H(Du(x_0))) > 1$, which we will now assume.

Recall the uniform ellipticity assumption gives

$$\eta^2 F_{M_{ij}} Du_{x_i} \cdot Du_{x_j} \leq -\zeta^2 \theta |D^2u|^2.$$

And employing necessary conditions $Dv(x_0) = 0$ and $D^2v(x_0) \leq 0$ and the Cauchy-Schwarz inequality to the term $4F_{M_{ij}} \eta \eta_{x_i} Du \cdot Du_{x_j} \leq (\zeta |D^2u|)(C|D\zeta||Du|)$, we evaluate (4.18) at the point x_0 to get

$$\begin{aligned} 0 &\leq C(|Du|^2 + 1 + \alpha) - \beta'(\alpha(H_{p_k} u_{x_k} - H) - \zeta H_{p_k} \zeta_{x_k} |Du|^2) \\ &\leq C(|Du|^2 + 1 + \alpha) - \beta'(\sigma\alpha|Du|^2 - C_0(1 + \zeta|Du|)|Du|^2) \\ &\leq C\beta' \{ |Du|^2 + 1 + \alpha - \sigma\alpha|Du|^2 + C_0(1 + \zeta|Du|)|Du|^2 \}. \end{aligned}$$

After multiplying through by $\zeta = \zeta(x_0)^2$ we have

$$0 \leq C\beta' \{ (\zeta|Du|)^2 + 1 + \alpha - \sigma\alpha(\zeta|Du|)^2 + C_0(1 + \zeta|Du|)(\zeta|Du|)^2 \} \tag{4.20}$$

which of course holds at x_0 .

We now choose

$$\alpha = \alpha_\epsilon := \frac{2C_0}{\sigma} M_\epsilon.$$

Note $\sigma\alpha \geq 2C_0\zeta(x_0)|Du(x_0)|$ and so (4.20) gives

$$0 \leq C\beta' \{ (\zeta|Du|)^2 + 1 + \alpha - 2C_0(\zeta|Du|)^3 + C_0(1 + \zeta|Du|)(\zeta|Du|)^2 \}.$$

As $\beta' > 1$, the expression in the parentheses is necessarily nonnegative. It follows that there is a constant C such that

$$\zeta(x_0)|Du(x_0)| \leq C(1 + \alpha)^{1/3}.$$

As a result, (4.19) holds for another appropriately chosen constant C .

4. Therefore,

$$M_\epsilon^2 = \sup_B |\zeta Du^\epsilon|^2 = 2 \sup_B (v^\epsilon + \alpha_\epsilon u^\epsilon) \leq C(\alpha_\epsilon + 1) \leq C \left(\frac{2C_0}{\sigma} M_\epsilon + 1 \right).$$

Consequently, M_ϵ is bounded above independently of $\epsilon \in (0, 1)$. □

Next we assert that $\beta_\epsilon(H^\varrho(Du^\epsilon))$ is locally bounded, independently of all ϵ sufficiently small.

Lemma 4.5 *Let $\delta \in (0, 1)$, $\varrho \in (0, \varrho_1)$, $\epsilon \in (0, 1)$ and suppose $\zeta \in C_c^\infty(B)$ is nonnegative. There is a constant C depending only on the list Π and $|\zeta|_{W^{2,\infty}(B)}$ such that*

$$\zeta(x)\beta_\epsilon(H^\varrho(Du^\epsilon(x))) \leq C, \quad x \in B.$$

We omit a proof of Lemma 4.5 as the proof of Lemma 3.3 in our recent work [15] immediately applies here. We also note that

$$F^\varrho(D^2u^\epsilon) = -\beta_\epsilon(H^\varrho(Du^\epsilon)) + f^\varrho - \delta u_{\delta,\varrho}(0) - \delta u^\epsilon$$

is locally bounded, independently of $\epsilon \in (0, 1]$. By the $W_{loc}^{2,p}$ estimates for fully nonlinear elliptic equations due to L. Caffarelli (Theorem 1 in [4], Theorem 7.1 in [5]), we have the following.

Lemma 4.6 *Let $\delta \in (0, 1)$, $\varrho \in (0, \varrho_1)$, $\epsilon \in (0, 1)$, $p \in (n, \infty)$, and assume $G \subset B$ is open with $\bar{G} \subset B$. There is a constant C depending on p , the list Π , and G such that*

$$|D^2u^\epsilon|_{L^p(G)} \leq C \{ |u^\epsilon|_{L^\infty(B)} + 1 \}.$$

Proof Assume $B_r(x_0) \subset B$ is nonempty, and choose $\zeta \in C_c^\infty(B_r(x_0))$ such that $0 \leq \zeta \leq 1$, $\zeta \equiv 1$ on $B_{r/2}(x_0)$ and

$$|D\zeta|_{L^\infty(B_{r/2}(x_0))} \leq \frac{C}{r}, \quad |D^2\zeta|_{L^\infty(B_{r/2}(x_0))} \leq \frac{C}{r^2}.$$

From Lemma 4.5, $\beta_\epsilon(H^\varrho(Du^\epsilon(x))) \leq C_1$ for $x \in B_{r/2}(x_0)$ for some C_1 depending only on the list Π and r . In view of (4.8), we also have that $\delta u_{\delta,\varrho}(0)$ is bounded independently of $\delta \in (0, 1)$. By the assumption that F is uniformly elliptic and concave, Theorem 7.1 in [5] implies there is a universal constant c_0 such that

$$\begin{aligned} r^2 |D^2u^\epsilon|_{L^p(B_{r/4}(x_0))} &\leq c_0 \{ |u^\epsilon|_{L^\infty(B_{r/2}(x_0))} + |f^\varrho - \delta u_{\delta,\varrho}(0) \\ &\quad - \delta u^\epsilon - \beta_\epsilon(H^\varrho(Du^\epsilon))|_{L^\infty(B_{r/2}(x_0))} \} \\ &\leq c_0 \{ |u^\epsilon|_{L^\infty(B_{r/2}(x_0))} + |f^\varrho - \delta u_{\delta,\varrho}(0) - \delta u^\epsilon|_{L^\infty(B_{r/2}(x_0))} + C_1 \} \\ &\leq C_0 \{ |u^\epsilon|_{L^\infty(B)} + |f^\varrho|_{L^\infty(B)} + 1 \}. \end{aligned}$$

Here C_0 only depends only on the list Π .

Now select $r = \frac{1}{2} \text{dist}(\partial G, B)$ and cover \bar{G} with finitely many balls $B_{r/4}(x_1), \dots, B_{r/4}(x_m)$, with each $x_1, \dots, x_m \in G$. Then

$$\begin{aligned} \int_G |D^2u^\epsilon(x)|^p dx &\leq \int_{\bigcup_{i=1}^m B_{r/4}(x_i)} |D^2u^\epsilon(x)|^p dx \\ &\leq \sum_{i=1}^m \int_{B_{r/4}(x_i)} |D^2u^\epsilon(x)|^p dx \\ &\leq m C_0^p (|u^\epsilon|_{L^\infty(B)} + |f^\varrho|_{L^\infty(B)} + 1)^p. \end{aligned}$$

□

In view of our uniform estimates, we are now ready to send $\epsilon \rightarrow 0^+$ in the Eq.(4.11).

Proposition 4.7 *Let $\delta \in (0, 1)$, $\varrho \in (0, \varrho_1)$, $p \in (n, \infty)$ and assume $G \subset B$ is open with $\bar{G} \subset B$.*

- (i) *There is $v_{\delta,\varrho} \in C(\bar{B}) \cap W_{loc}^{2,p}(B)$ such that $u^\epsilon \rightarrow v_{\delta,\varrho}$, as $\epsilon \rightarrow 0^+$, uniformly in \bar{B} and weakly in $W^{2,p}(G)$.*
- (ii) *Moreover, $v_{\delta,\varrho}$ is the unique solution of the boundary value problem*

$$\begin{cases} \max\{\delta v + F^\varrho(D^2v) - (f^\varrho - \delta u_{\delta,\varrho}(0)), H^\varrho(Dv)\} = 0 & x \in B \\ v = u_{\delta,\varrho} - u_{\delta,\varrho}(0) & x \in \partial B \end{cases} \quad (4.21)$$

- (iii) *There is a constant C depending on p , the list Π , and G such that*

$$|D^2v_{\delta,\varrho}|_{L^p(G)} \leq C \{ |v_{\delta,\varrho}|_{L^\infty(B)} + 1 \} \quad (4.22)$$

and

$$-C \leq F^\varrho(D^2v_{\delta,\varrho}(x)) \tag{4.23}$$

for Lebesgue almost every $x \in G$.

Proof (i) – (ii) The convergence to v satisfying (4.21) is proved very similar to Proposition 4.1 in [14] and part (ii) of Theorem 1.1 in [15], so we omit the details. In both arguments, the uniqueness of solutions of a related boundary value problem of the type (4.21) is crucial. A uniqueness proof can be adapted from Proposition 2.3 (see also Proposition 2.2 of [15]); we leave the details to the reader.

(iii) The bound (4.22) follows from part (i) and Lemma 4.6. Let us now verify (4.23). Assume that E is open and $\bar{E} \subset B$. Recall that F^ϱ is concave. As u^ϵ converges to $v_{\delta,\varrho}$ weakly $W_{loc}^{2,p}(B)$,

$$\limsup_{\epsilon \rightarrow 0^+} \int_E F^\varrho(D^2u^\epsilon(x))dx \leq \int_E F^\varrho(D^2v_{\delta,\varrho}(x))dx. \tag{4.24}$$

By Lemma 4.5, there is a constant C depending only the list Π and E such that $F^\varrho(D^2u^\epsilon) \geq -C$ on E . Inequality (4.24) then gives

$$-C \leq F^\varrho(D^2v_{\delta,\varrho}(x))$$

for almost every $x \in E$. As E is arbitrary, the conclusion follows. □

Proposition 4.8 *Let $\delta \in (0, 1)$, $p \in (n, \infty)$ and assume $G \subset B$ is open with $\bar{G} \subset B$.*

- (i) *Then $v_{\delta,\varrho} \rightarrow u_\delta - u_\delta(0)$, as $\varrho \rightarrow 0^+$, uniformly on \bar{B} and weakly in $W^{2,p}(G)$.*
- (ii) *There is a constant C depending on p , the list Π , and G such that*

$$-C \leq F(D^2u_\delta(x))$$

for almost every $x \in G$.

Proof (i) By (4.14) and part (i) of the previous proposition,

$$|v_{\delta,\varrho}|_{L^\infty(B)} \leq C \left\{ c_1 R + \left(\inf_{\mathbb{R}^n} f \right)^- + \sup_B \ell + n\Theta + \sup_B |f^\varrho| \right\}$$

for each $\varrho \in (0, \varrho_1)$. We also have $H(Dv_{\delta,\varrho}) \leq H^\varrho(Dv_{\delta,\varrho}) \leq 0$, so that $|Dv_{\delta,\varrho}| \leq c_1$. Therefore, there is a sequence $(\varrho_k)_{k \in \mathbb{N}}$ tending to 0 and $(v_{\delta,\varrho_k})_{k \in \mathbb{N}}$ converging uniformly to a solution $w \in C(\bar{B})$ of the PDE

$$\begin{cases} \max\{\delta w + F(D^2w) - (f - \delta u_\delta(0)), H(Dw)\} = 0 & x \in B \\ w = u_\delta - u_\delta(0) & x \in \partial B \end{cases}$$

As in part (i) of the previous proposition, $w = u_\delta - u_\delta(0)$ is the unique solution of this PDE. Therefore, it turns out that $v_{\delta,\varrho} \rightarrow u_\delta - u_\delta(0)$ uniformly on \bar{B} . The weak convergence assertion follows easily from this uniform convergence and the estimate (4.22).

(ii) Let $U \subset G$ be measurable and recall that $F^\varrho \leq F$ and that F is concave. By (4.23), we have there is a constant C depending on p , the list Π , $1/\text{dist}(\partial G, B)$ and G such that

$$\begin{aligned} -C|U| &\leq \limsup_{\varrho \rightarrow 0^+} \int_U F^\varrho(D^2v_{\delta,\varrho}(x))dx \\ &\leq \limsup_{\varrho \rightarrow 0^+} \int_U F(D^2v_{\delta,\varrho}(x))dx \\ &\leq \int_U F(D^2(u_\delta(x) - u_\delta(0)))dx \\ &= \int_U F(D^2u_\delta(x))dx. \end{aligned}$$

□

Corollary 4.9 For each $\delta \in (0, 1)$, $D^2u_\delta \in L^\infty(\mathbb{R}^n; S_n(\mathbb{R}))$. Moreover, there is a constant C depending only on the list Π for which

$$|D^2u_\delta|_{L^\infty(\mathbb{R}^n; S_n(\mathbb{R}))} \leq C.$$

for each $\delta \in (0, 1)$.

Proof Choose $R_1 > 0$ so that $\Omega_\delta \subset B_{R_1}(0)$ for all $\delta \in (0, 1)$; such an R_1 exists by Corollary 3.6. Lemma 4.1 gives that there is a universal constant C such that

$$D^2u_\delta(x) \leq \frac{C}{R_1} I_n$$

for almost every $|x| \geq 2R_1$.

Now select $R > 2R_1$ so large that (4.10) is also satisfied. Part (ii) of Proposition 4.8, with $G = B_{2R_1}(0)$ and $B = B_R(0)$, gives a constant C_1 depending on R_1 and the list Π such that

$$-C_1 \leq F(D^2u_\delta(x)) \tag{4.25}$$

for almost every $|x| \leq 2R_1$. Since u_δ is convex (Proposition 3.5), the uniform ellipticity assumption on F implies

$$F(D^2u_\delta(x)) \leq -\theta \Delta u_\delta(x) \tag{4.26}$$

for almost every $x \in \mathbb{R}^n$. Therefore, we can again appeal to the convexity of u_δ and employ (4.25) and (4.26) to get

$$D^2u_\delta(x) \leq \Delta u_\delta(x) I_n \leq \frac{C_1}{\theta} I_n$$

for almost every $|x| \leq 2R_1$. □

Proof (part (ii) of Theorem 1.1) By the convexity of u_δ and Corollary 4.9, there is a constant C independent of $\delta \in (0, 1)$ for which

$$0 \leq u_\delta(x + h) - 2u_\delta(x) + u_\delta(x - h) \leq C|h|^2$$

for every $x, h \in \mathbb{R}^n$. The assertion now follows from passing to the limit along an appropriate sequence of δ tending to 0 as was done in the proof of part (i) of Theorem 1.1. □

Remark 4.10 By part (ii) of Theorem 1.1, Du_δ exists everywhere and is continuous. By Corollary 3.6

$$\Omega_\delta = \{x \in \mathbb{R}^n : H(Du_\delta(x)) < 0\}$$

is open and bounded.

5 1D and rotationally symmetric problems

Now we will discuss a few results for solutions of the eigenvalue problem (1.1) when the dimension $n = 1$ and when F, f, H satisfy the symmetry hypothesis (1.7):

$$\begin{cases} f(Ox) = f(x) \\ H(O^t p) = H(p) \\ F(OMO^t) = F(M) \end{cases}$$

for each $x, p \in \mathbb{R}^n, M \in S_n(\mathbb{R})$ and orthogonal $n \times n$ matrix O . First, we prove Theorem 1.2 which involves the regularity of symmetric eigenfunctions. Then we consider the uniqueness of eigenfunctions of (1.1) that satisfy the growth condition (1.5).

Proof (Theorem 1.2) The assumption (1.7) implies that u_δ is radial; this follows from the uniqueness assertion Proposition 3.1. In particular, u^* constructed in the proof of part (i) of Theorem 1.1 will also be radial. Consequently, there is a function $\phi : [0, \infty) \rightarrow \mathbb{R}$ such that $u^*(x) = \phi(|x|)$. As u^* is convex, ϕ is nondecreasing and convex. Moreover, for almost every $x \in \mathbb{R}^n$

$$\begin{cases} Du^*(x) = \phi'(|x|) \frac{x}{|x|} \\ D^2u^*(x) = \phi''(|x|) \frac{x \otimes x}{|x|^2} + \frac{\phi'(|x|)}{|x|} \left(I_n - \frac{x \otimes x}{|x|^2} \right) \end{cases} .$$

Similar arguments imply $f(x) = f_0(|x|)$ for a nondecreasing, convex function f_0 . Likewise $H(p)$ only depends on $|p|$ and so $\{p \in \mathbb{R}^n : H(p) \leq 0\}$ is a ball centered at the origin. Thus, $\ell(v) = a|v|$ for some $a > 0$, and as a result $H_0(p) = |p| - a$. The assumption (1.7) also implies $F = F(M)$ only depends on the eigenvalues of M . In particular, the symmetric function $G(\mu_1, \dots, \mu_n) := F(\text{diag}(\mu_1, \dots, \mu_n))$ completely determines F . And as F is uniformly elliptic,

$$G(\mu_1 + h, \dots, \mu_n) - G(\mu_1, \dots, \mu_n) \leq -\theta h.$$

for $h \geq 0$.

From our comments above, ϕ satisfies

$$\max \left\{ \lambda^* + G \left(\frac{\phi'}{r}, \dots, \frac{\phi'}{r}, \phi'' \right) - f_0(r), \phi' - a \right\} = 0, \quad r > 0. \tag{5.1}$$

And since ϕ' is nondecreasing,

$$\{r > 0 : \phi'(r) < a\} = (0, r_0)$$

for some $r_0 > 0$; this is another way of expressing $\Omega := \{x \in \mathbb{R}^n : H(Du^*(x)) < 0\} = B_{r_0}(0)$. Thus, $\phi' = a$ for $r \geq r_0$ and $\phi \in C^2(\mathbb{R} \setminus \{r_0\}) \cap C_{loc}^{1,1}(\mathbb{R})$. Furthermore, as $\phi''(r_0+) = 0$, we just need to show $\phi''(x_0-) = 0$. This limit would imply $\phi \in C^2(\mathbb{R})$ and consequently $u^* \in C^2(\mathbb{R}^n)$.

Recall the left hand limit $\phi''(x_0-)$ exists and is nonnegative since ϕ is convex. By (5.1),

$$\lambda^* + G\left(\frac{\phi'}{r}, \dots, \frac{\phi'}{r}, \phi''\right) - f_0(r) \leq 0, \quad r > 0.$$

Sending $r \rightarrow r_0^+$ gives

$$\lambda^* + G\left(\frac{a}{r_0}, \dots, \frac{a}{r_0}, 0\right) - f_0(r_0) \leq 0. \tag{5.2}$$

Now,

$$\lambda^* + G\left(\frac{\phi'}{r}, \dots, \frac{\phi'}{r}, \phi''\right) - f_0(r) = 0, \quad r \in (0, r_0)$$

and sending $r \rightarrow r_0^-$ gives

$$\lambda^* + G\left(\frac{a}{r_0}, \dots, \frac{a}{r_0}, \phi''(x_0-)\right) - f_0(r_0) = 0. \tag{5.3}$$

Combining (5.2) and (5.3) gives

$$G\left(\frac{a}{r_0}, \dots, \frac{a}{r_0}, 0\right) \leq f_0(r_0) - \lambda^* = G\left(\frac{a}{r_0}, \dots, \frac{a}{r_0}, \phi''(x_0-)\right).$$

By the monotonicity of G in each of its arguments, $\phi''(x_0-) \leq 0$. Thus $\phi''(x_0) = 0$. \square

Proposition 5.1 *Assume $n = 1$. Any two convex solutions of (1.1) that satisfy (1.5) differ by an additive constant.*

Proof Assume u_1, u_2 are convex and satisfy

$$\begin{cases} \max\{\lambda^* + F(u'') - f, H(u')\} = 0, & x \in \mathbb{R} \\ \lim_{|x| \rightarrow \infty} \frac{u(x)}{\ell(x)} = 1 \end{cases}.$$

As in the proof of Theorem 1.2, we may deduce that necessarily $u_1, u_2 \in C^2(\mathbb{R})$. Also observe

$$H_0(p) = \max_{v \pm 1} \{pv - \ell(v)\} = \max\{p - \ell(1), -p - \ell(-1)\}.$$

In particular,

$$\{p \in \mathbb{R} : H(p) \leq 0\} = [-\ell(-1), \ell(1)].$$

It then follows from the convexity of u_1 and u_2 that

$$I_1 := \{x \in \mathbb{R} : H(u'_1(x)) < 0\} \quad \text{and} \quad I_2 := \{x \in \mathbb{R} : H(u'_2(x)) < 0\}$$

are bounded, open intervals.

Let us first assume $I_1 = I_2 = (\alpha, \beta)$. Then

$$\lambda^* + F(u''_1) - f = 0 = \lambda^* + F(u''_2) - f, \quad x \in (\alpha, \beta).$$

As F is uniformly elliptic $u''_1 = u''_2 = F^{-1}(f - \lambda^*)$ for $x \in (\alpha, \beta)$. Hence, $u'_1 - u'_2$ is constant. The above characterization of $\{p \in \mathbb{R} : H(p) \leq 0\}$ also implies

$$\begin{cases} u'_1(x) = u'_2(x) = -\ell(-1), & x \in (-\infty, \alpha] \\ u'_1(x) = u'_2(x) = \ell(1), & x \in [\beta, \infty) \end{cases}$$

It now follows that necessarily $u'_1 = u'_2$ and so $u_1 - u_2$ is constant.

Now we are left to prove that $I_1 = I_2$; for definiteness, we shall assume $I_1 = (\alpha_1, \beta_1)$ and $I_2 = (\alpha_2, \beta_2)$. First suppose that $I_1 \cap I_2 = \emptyset$ and without loss of generality $\beta_1 < \alpha_2$. Then on I_1 , $\lambda^* + F(u'_1) - f = 0$ and $u'_2 = -\ell(-1)$. We always have $\lambda^* + F(u'_2) - f \leq 0$ which implies $\lambda^* - f \leq 0$ on I_1 since $u''_2 = 0$. It then follows that $F(u'_1) = f - \lambda^* \geq 0$ and thus $u'_1 \leq 0$. As u is convex, $u'_1 = 0$ in I_1 . However, u'_1 is constant and it would then be impossible for $u'_1(\alpha_1) = -\ell(-1) < 0$ and $u'_1(\beta_1) = \ell(1) > 0$. Therefore, $I_1 \cap I_2 \neq \emptyset$.

Without any loss of generality, we may assume $\alpha_1 < \alpha_2 \leq \beta_1$. Repeating our argument above, we find $u''_1 = 0$ on $[\alpha_1, \alpha_2]$. It must be that u'_1 is constant and thus equal to $-\ell(-1)$ on $[\alpha_1, \alpha_2]$. But then $H(u'_1) = 0$ on (α_1, α_2) , which contradicts the definition of I_1 . Hence, $I_1 = I_2$ and the assertion follows. \square

Proposition 5.2 *Assume the symmetry condition (1.7) and that F is uniformly elliptic. Then any two convex, rotationally symmetric solutions of (1.1) that satisfy (1.5) differ by an additive constant.*

Proof As remarked in the above proof of Theorem 1.2, the symmetry assumption on H results in $H_0(p) = |p| - a$ for some $a > 0$. Now assume u_1, u_2 are convex, rotationally symmetric solutions of (1.1) that satisfy (1.5). Then it follows

$$\{x \in \mathbb{R}^n : H(Du_i(x)) < 0\} = B_{r_i}(0)$$

for $i = 1, 2$ and some $r_1, r_2 > 0$. Thus,

$$u_i(x) = a|x| + b_i, \quad |x| \geq r_i \tag{5.4}$$

for some constants b_i . If $r_1 = r_2 =: r$, then

$$\begin{cases} F(D^2u_1) = f(x) - \lambda^* = F(D^2u_2), & x \in B_r(0) \\ u_1 = ar + b_1, \quad u_2 = ar + b_2, & x \in \partial B_r(0). \end{cases}$$

As F is uniformly elliptic, $u_1 \equiv u_2 + b_1 - b_2$ on $\overline{B_r(0)}$ and thus on \mathbb{R}^n .

Now suppose $r_1 < r_2$, and set $v := u_2 + b_1 - b_2$. By (5.4), $u_1 \equiv v$ for $|x| \geq r_2$. Since,

$$F(D^2u_1) \leq f(x) - \lambda^* = F(D^2v), \quad x \in B_{r_2}(0)$$

the Aleksandrov-Bakelman-Pucci estimate (Theorem 3.6 in [5], Theorem 17.3 in [13]) then gives $u_1 \equiv v$ in $B_{r_2}(0)$, from which we conclude the proof, or $u_1 < v$ in $B_{r_2}(0)$. However if $u_1 < v$ in $B_{r_2}(0)$, Hopf's Lemma (see the appendix of [1]) implies

$$\frac{\partial v}{\partial \nu}(x_0) < \frac{\partial u_1}{\partial \nu}(x_0) \tag{5.5}$$

for each $x_0 \in \partial B_{r_2}(0)$. Here $\nu = x_0/|x_0|$. As u_1 and v are rotational and convex, (5.5) gives

$$|Dv(x_0)| < |Du_1(x_0)| \leq a.$$

However, $|Dv| = a$ on $\partial B_{r_2}(0)$. This contradicts the hypothesis that $r_1 < r_2$. \square

6 Minmax formulae

This final section is devoted entirely to the proof of Theorem 1.3. In particular, we will make use of the characterizations of λ^* given in (3.13) and (3.14). We will also use that the functions H and H_0 have the same sign.

Let $\phi \in C^2(\mathbb{R}^n)$ and suppose that $H(D\phi) \leq 0$. If

$$\lambda_\phi := \inf_{\mathbb{R}^n} \{-F(D^2\phi(x)) + f(x)\} > -\infty,$$

then ϕ is a subsolution of (1.1) with eigenvalue λ_ϕ . By (3.13), $\lambda_\phi \leq \lambda^*$. Hence, $\lambda_- = \sup_\phi \lambda_\phi \leq \lambda^*$. Now let $\psi \in C^2(\mathbb{R}^n)$ satisfy

$$\liminf_{|x| \rightarrow \infty} \frac{\psi(x)}{\ell(x)} \geq 1.$$

If

$$\mu_\psi := \sup_{H(D\psi) < 0} \{-F(D^2\psi(x)) + f(x)\} < \infty,$$

then ψ is a supersolution of (1.1) with eigenvalue μ_ψ . It follows from (3.14) that $\lambda_\psi \geq \lambda^*$. As a result, $\lambda_+ = \inf_\psi \mu_\psi \geq \lambda^*$.

Now let u^* be an eigenfunction associated with λ^* that satisfies $D^2u^* \in L^\infty(\mathbb{R}^n; S_n(\mathbb{R}))$. As in Remark 4.10,

$$\Omega_0 := \{x \in \mathbb{R}^n : H(Du^*(x)) < 0\}$$

is open and bounded. For $\epsilon > 0$ and $\tau > 1$, set

$$u^{\epsilon, \tau} = \tau u^\epsilon = \tau(\eta^\epsilon * u^*).$$

Here $u^\epsilon = \eta^\epsilon * u^*$ is a standard mollification of u^* . Observe that H_0 is Lipschitz continuous with Lipschitz constant no more than one; in view of the basic estimate $|Du^* - Du^\epsilon|_{L^\infty(\mathbb{R}^n)} \leq \epsilon |D^2u^*|_{L^\infty(\mathbb{R}^n)}$,

$$H_0(Du^*(x)) \leq H_0(Du^\epsilon(x)) + \epsilon |D^2u^*|_{L^\infty(\mathbb{R}^n)}, \quad x \in \mathbb{R}^n. \tag{6.1}$$

Also note for any $x \in \mathbb{R}^n$ where $H(Du^{\epsilon, \tau}(x)) < 0$,

$$H_0(Du^\epsilon(x)) = H_0\left(\frac{1}{\tau} Du^{\epsilon, \tau}(x) + \frac{\tau-1}{\tau} 0\right) < \frac{\tau-1}{\tau} H_0(0) < 0. \tag{6.2}$$

In view of (6.1) and (6.2), we can choose $\epsilon_1 = \epsilon_1(\tau) > 0$ such that

$$\varrho := -\left(\frac{\tau-1}{\tau} H_0(0) + \epsilon_1 |D^2u^*|_{L^\infty(\mathbb{R}^n)}\right) > 0$$

and thus

$$\{x \in \mathbb{R}^n : H(Du^{\epsilon, \tau}(x)) < 0\} \subset \{x \in \mathbb{R}^n : H_0(Du^*(x)) < -\varrho\}$$

for $\epsilon \in (0, \epsilon_1)$. Since $\{x \in \mathbb{R}^n : H_0(Du^*(x)) < -\varrho\}$ is a proper open subset of Ω_0 we can further select $\epsilon_2 = \epsilon_2(\tau) > 0$ so that

$$\{x \in \mathbb{R}^n : H_0(Du^*(x)) < -\varrho\} \subset \Omega^\epsilon := \{x \in \mathbb{R}^n : \text{dist}(x, \partial\Omega_0) > \epsilon\} \tag{6.3}$$

for $\epsilon \in (0, \min\{\epsilon_1, \epsilon_2\})$.

By assumption, u^* satisfies $\lambda^* + F(D^2u^*) - f = 0$ for almost every $x \in \Omega_0$. Mollifying both sides of this equation gives $\lambda^* + F(D^2u^*)^\epsilon - f^\epsilon = 0$ in Ω^ϵ . Since F is concave

$$\begin{aligned} F(D^2u^\epsilon(x)) &= F\left(\int_{\mathbb{R}^n} \eta^\epsilon(y) D^2u^*(x-y) dy\right) \\ &\geq \int_{\mathbb{R}^n} \eta^\epsilon(y) F(D^2u^*(x-y)) dy = F(D^2u^*)^\epsilon(x). \end{aligned}$$

Consequently, $\lambda^* + F(D^2u^\epsilon) - f^\epsilon \geq 0$, in Ω^ϵ . And since Ω_0 is bounded, $|f^\epsilon - f|_{L^\infty(\Omega_0)} = o(1)$ as $\epsilon \rightarrow 0^+$. Therefore,

$$\lambda^* + F(D^2u^\epsilon) - f \geq o(1), \quad x \in \Omega^\epsilon. \quad (6.4)$$

as $\epsilon \rightarrow 0^+$.

We can now combine the inclusion (6.3) and the inequality (6.4). For $\epsilon \in (0, \min\{\epsilon_1, \epsilon_2\})$

$$\begin{aligned} \lambda^+ &\leq \sup_{H(Du^{\epsilon,\tau}) < 0} \{-F(D^2u^{\epsilon,\tau}(x)) + f(x)\} \\ &= \sup_{H(Du^{\epsilon,\tau}) < 0} \{-\tau F(D^2u^\epsilon(x)) + f(x)\} \\ &= \sup_{H(Du^{\epsilon,\tau}) < 0} \{-F(D^2u^\epsilon(x)) + f(x)\} + O(\tau - 1) \\ &\leq \sup_{\Omega^\epsilon} \{-F(D^2u^\epsilon(x)) + f(x)\} + O(\tau - 1) \\ &\leq \lambda^* + o(1) + O(\tau - 1). \end{aligned}$$

We conclude by first ending $\epsilon \rightarrow 0^+$ and then $\tau \rightarrow 1^+$.

References

1. Armstrong, S.N.: Principal eigenvalues and an anti-maximum principle for homogeneous fully nonlinear elliptic equations. *J. Differ. Equ.* **246**(7), 2958–2987 (2009)
2. Bardi, M., Capuzzo-Dolcetta, I.: Optimal Control and Viscosity Solutions of Hamilton–Jacobi–Bellman equations. With appendices by Maurizio Falcone and Pierpaolo Soravia. Systems and Control: Foundations and Applications. Birkhäuser Boston, Inc., Boston (1997)
3. Borkar, V., Budhiraja, A.: Ergodic control for constrained diffusions: characterization using HJB equations. *SIAM J. Control Optim.* **43**(4), 1467–1492 (2004/05)
4. Caffarelli, Luis A.: Interior a priori estimates for solutions of fully nonlinear equations. *Ann. Math. (2)* **130**(1), 189–213 (1989)
5. Caffarelli, L.A., Cabré, X.: Fully Nonlinear Elliptic Equations. American Mathematical Society Colloquium Publications, vol. 43. American Mathematical Society, Providence (1995)
6. Crandall, M.: Viscosity Solutions: A Primer. Viscosity Solutions and Applications. Lecture Notes in Math., vol. 1660. Springer, Berlin (1997)
7. Crandall, M., Ishii, H., Lions, P.L.: User’s guide to viscosity solutions of second order partial differential equations. *Bull. Am. Math. Soc. (N.S.)* **27**(1), 1–67 (1992)
8. Evans, L.C.: A second-order elliptic equation with gradient constraint. *Commun. Partial Differ. Equ.* **4**(5), 555–572 (1979)
9. Evans, L.C.: Classical solutions of fully nonlinear, convex, second-order elliptic equations. *Commun. Pure Appl. Math.* **35**(3), 333–363 (1982)
10. Evans, L.C.: Partial Differential Equations. Graduate Studies in Mathematics, vol. 19, 2nd edn. American Mathematical Society, Providence, Providence (2010)
11. Evans, L.C., Gariepy, R.: Measure Theory and Fine Properties of Functions. Studies in Advanced Mathematics. CRC Press, Boca Raton (1992)
12. Fleming, H., Soner, H.: Controlled Markov Processes and Viscosity Solutions. Stochastic Modeling and Applied Probability, vol. 25, 2nd edn. Springer, New York (2006)
13. Gilbarg, D., Trudinger, N.: Elliptic Partial Differential Equations of Second Order. Springer, Berlin (1998)
14. Hynd, R.: The eigenvalue problem of singular ergodic control. *Commun. Pure Appl. Math.* **65**(5), 649–682 (2012)
15. Hynd, R., Mawi, H.: On Hamilton–Jacobi–Bellman equations with convex gradient constraints. *Interfaces Free Bound.* **18**, 291–315 (2016)
16. Ishii, H., Koike, S.: Boundary regularity and uniqueness for an elliptic equation with gradient constraint. *Commun. Partial Differ. Equ.* **8**(4), 317–346 (1983)
17. Korevaar, N.J.: Convex solutions to nonlinear elliptic and parabolic boundary value problems. *Indiana Univ. Math. J.* **32**(4), 603–614 (1983)

18. Kruk, L.: Optimal policies for n -dimensional singular stochastic control problems. II. The radially symmetric case. *Ergodic control. SIAM J. Control Optim.* **39**(2), 635–659 (2000)
19. Krylov, N.V.: Boundedly inhomogeneous elliptic and parabolic equations in a domain. (Russian) *Izv. Akad. Nauk SSSR Ser. Mat.* **47**(1), 75–108 (1983)
20. Menaldi, J.-L., Robin, M., Taksar, M.: Singular ergodic control for multidimensional Gaussian processes. *Math. Control Signals Syst.* **5**(1), 93–114 (1992)
21. Øksendal, B., Sulem, A.: *Applied Stochastic Control of Jump Diffusions*, 2nd edn. Universitext. Springer, Berlin (2007)
22. Rockafellar, R.T., Wets, R.: *Variational Analysis. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*, vol. 317. Springer, Berlin (1998)
23. Soner, H.M., Shreve, S.: Regularity of the value function for a two-dimensional singular stochastic control problem. *SIAM J. Control Optim.* **27**(4), 876–907 (1989)
24. Soner, H.M., Touzi, N.: Homogenization and asymptotics for small transaction costs. *SIAM J. Control Optim.* **51**(4), 2893–2921 (2013)
25. Trudinger, N.: Fully nonlinear, uniformly elliptic equations under natural structure conditions. *Trans. Am. Math. Soc.* **278**(2), 751–769 (1983)
26. Wiegner, M.: The $C^{1,1}$ -character of solutions of second order elliptic equations with gradient constraint. *Commun. Partial Differ. Equ.* **6**(3), 361–371 (1981)