Journal of Functional Analysis 272 (2017) 4873–4918

Contents lists available at ScienceDirect

Journal of Functional Analysis

www.elsevier.com/locate/jfa

Approximation of the least Rayleigh quotient for degree p homogeneous functionals

Ryan Hynd [a,1], Erik Lindgren [b,*,2]

[a] Department of Mathematics, University of Pennsylvania, United States
[b] Department of Mathematics, KTH, Sweden

A R T I C L E I N F O

Article history:
Received 13 September 2016
Accepted 28 February 2017
Available online 6 March 2017
Communicated by E. Milman

MSC:
35A15
35B40
35K55
47J10

Keywords:
Nonlinear eigenvalue problem
Doubly nonlinear evolution
Inverse iteration
Large time behavior

A B S T R A C T

We present two novel methods for approximating minimizers of the abstract Rayleigh quotient $\Phi(u)/\|u\|^p$. Here Φ is a strictly convex functional on a Banach space with norm $\|\cdot\|$, and Φ is assumed to be positively homogeneous of degree $p \in (1, \infty)$. Minimizers are shown to satisfy $\partial\Phi(u) - \lambda\mathcal{J}_p(u) \ni 0$ for a certain $\lambda \in \mathbb{R}$, where $\mathcal{J}_p$ is the subdifferential of $\frac{1}{p}\|\cdot\|^p$. The first approximation scheme is based on inverse iteration for square matrices and involves sequences that satisfy

$$\partial\Phi(u_k) - \mathcal{J}_p(u_{k-1}) \ni 0 \quad (k \in \mathbb{N}).$$

The second method is based on the large time behavior of solutions of the doubly nonlinear evolution

$$\mathcal{J}_p(\dot{v}(t)) + \partial\Phi(v(t)) \ni 0 \quad (a.e.\ t > 0)$$

and more generally p-curves of maximal slope for Φ. We show that both schemes have the remarkable property that the Rayleigh quotient is nonincreasing along solutions and that properly scaled solutions converge to a minimizer of $\Phi(u)/\|u\|^p$. These results are new even for Hilbert spaces and their primary application is in the approximation of optimal

* Corresponding author.
E-mail address: eriklin@kth.se (E. Lindgren).

[1] Partially supported by NSF grant DMS-1301628.
[2] Supported by the Swedish Research Council, grant no. 2012-3124. Partially supported by the Royal Swedish Academy of Sciences.

http://dx.doi.org/10.1016/j.jfa.2017.02.024

constants and extremal functions for inequalities in Sobolev spaces.

1. Introduction

We begin with an elementary, motivating example. Let A be a real symmetric, positive definite $n \times n$ matrix. The smallest eigenvalue σ of A is given by the least value the Rayleigh quotient $(Av \cdot v)/\|v\|^2$ may assume. That is,

$$\sigma = \inf\left\{\frac{Av \cdot v}{\|v\|^2} : v \neq 0\right\}$$

where $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^n$. It is evident that for $w \neq 0$

$$Aw = \sigma w \qquad \text{if and only if} \qquad \sigma = \frac{Aw \cdot w}{\|w\|^2}. \tag{1.1}$$

We will recall two methods that are used to approximate σ and its corresponding eigenvectors. The first is inverse iteration:

$$Au_k = u_{k-1}, \quad k \in \mathbb{N} \tag{1.2}$$

for a given $u_0 \in \mathbb{R}^n$. It turns out that the limit $\lim_{k\to\infty} \sigma^k u_k$ exists and satisfies (1.1) when it is not equal to $0 \in \mathbb{R}^n$. In this case,

$$\lim_{k\to\infty} \frac{Au_k \cdot u_k}{\|u_k\|^2} = \sigma.$$

The second method is based on the large time limit of solutions of the ordinary differential equation

$$\dot{v}(t) + Av(t) = 0 \quad (t > 0). \tag{1.3}$$

It is straightforward to verify that $\lim_{t\to\infty} e^{\sigma t} v(t)$ exists and satisfies (1.1) when it is not equal to $0 \in \mathbb{R}^n$. In this case,

$$\lim_{t\to\infty} \frac{Av(t) \cdot v(t)}{\|v(t)\|^2} = \sigma.$$

The purpose of this paper is to generalize these convergence assertions to Rayleigh quotients that are defined on Banach spaces. In particular, we will show how these ideas provide new understanding of optimality conditions for functional inequalities in Sobolev spaces.

Let X be a Banach space over $\mathbb{R}$ with norm $\|\cdot\|$ and topological dual X^*. We will study functionals $\Phi : X \to [0,\infty]$ that are proper, convex, lower semicontinuous, and have compact sublevel sets. Moreover, we will assume that each Φ is strictly convex on its domain

$$\mathrm{Dom}(\Phi) := \{u \in X : \Phi(u) < \infty\},$$

and is positively homogeneous of degree $p \in (1,\infty)$. That is

$$\Phi(tu) = t^p\Phi(u)$$

for each $t \geq 0$ and $u \in X$. These properties will be assumed throughout this entire paper. Moreover, we will always assume $p \in (1,\infty)$ and write $q = p/(p-1)$ for the dual Hölder exponent to p.

In our motivating example described above, $X = \mathbb{R}^n$ equipped with the Euclidean norm and $\Phi(u) = \frac{1}{2}Au \cdot u$. In the spirit of that example, we consider finding $u \in X \setminus \{0\}$ that minimizes the abstract Rayleigh quotient $\Phi(u)/\|u\|^p$. To this end, we define

$$\lambda_p := \inf\left\{p\frac{\Phi(u)}{\|u\|^p} : u \neq 0\right\} \tag{1.4}$$

to be the least Rayleigh quotient associated with Φ. We will argue below that minimizers of $\Phi(u)/\|u\|^p$ exist and that for $w \neq 0$

$$\partial\Phi(w) - \lambda_p \mathcal{J}_p(w) \ni 0 \qquad \text{if and only if} \qquad \lambda_p = p\frac{\Phi(w)}{\|w\|^p}.$$

Here

$$\partial\Phi(u) := \{\xi \in X^* : \Phi(w) \geq \Phi(u) + \langle \xi, w-u\rangle \text{ for all } w \in X\}$$

is the subdifferential of Φ at u and $\mathcal{J}_p(u)$ is the subdifferential of $\frac{1}{p}\|\cdot\|^p$ at $u \in X$. We are using the notation $\langle \xi, u\rangle := \xi(u)$ and we will write $\|\xi\|_* := \sup\{|\langle \xi, u\rangle| : \|u\| \leq 1\}$ for the norm on X^*. It is straightforward to verify

$$\mathcal{J}_p(u) = \left\{\xi \in X^* : \langle \xi, u\rangle = \frac{1}{p}\|u\|^p + \frac{1}{q}\|\xi\|_*^q\right\} = \left\{\xi \in X^* : \|\xi\|_*^q = \|u\|^p\right\}. \tag{1.5}$$

See for instance equation (1.4.6) in [2]. We also remark that the Hahn–Banach Theorem implies $\mathcal{J}_p(u) \neq \emptyset$ for each $u \in X$.

For some functionals Φ, any two minimizers of $\Phi(u)/\|u\|^p$ are linearly dependent. In the motivating example above, where $X = \mathbb{R}^n$ equipped with the Euclidean norm and $\Phi(u) = \frac{1}{2}Au \cdot u$, this would amount to the eigenspace of the first eigenvalue of A being one dimensional. This observation leads to the following definition and terminology, which is central to the main assertions of this work.

Definition 1.1. λ_p defined in (1.4) is said to be *simple* if

$$\lambda_p = \frac{p\Phi(u)}{\|u\|^p} = \frac{p\Phi(v)}{\|v\|^p},$$

for $u, v \in X \setminus \{0\}$, implies that u and v are linearly dependent.

In analogy with (1.2), we will study the *inverse iteration* scheme: for $u_0 \in X$

$$\partial\Phi(u_k) - \mathcal{J}_p(u_{k-1}) \ni 0, \quad k \in \mathbb{N}. \tag{1.6}$$

We will see below that solutions of this scheme exist and satisfy

$$\frac{p\Phi(u_k)}{\|u_k\|^p} \leq \frac{p\Phi(u_{k-1})}{\|u_{k-1}\|^p} \quad \text{and} \quad \frac{\|u_k\|}{\|u_{k+1}\|} \leq \frac{\|u_{k-1}\|}{\|u_k\|}$$

for each $k \in \mathbb{N}$ provided $u_0 \in \text{Dom}(\Phi) \setminus \{0\}$. An important number that is related to this scheme and that will appear throughout this paper is

$$\mu_p := \lambda_p^{\frac{1}{p-1}}.$$

The following theorem asserts that if we scale u_k by appropriate powers of μ_p, the resulting sequence converges to a minimizer of $\Phi(u)/\|u\|^p$.

Theorem 1.2. *Assume that λ_p is simple and that $(u_k)_{k\in\mathbb{N}}$ satisfies (1.6) with $u_0 \in X$. Then the limit $w := \lim_{k\to\infty} \mu_p^k u_k$ exists and $w \in Dom(\Phi)$. Moreover,*

$$\Phi(w) = \lim_{k\to\infty} \Phi(\mu_p^k u_k).$$

If $w \neq 0$, w is a minimizer of $\Phi(u)/\|u\|^p$,

$$\lambda_p = \lim_{k\to\infty} \frac{p\Phi(u_k)}{\|u_k\|^p}, \quad \textit{and} \quad \mu_p = \lim_{k\to\infty} \frac{\|u_{k-1}\|}{\|u_k\|}.$$

Next we will present a convergence result for a flow analogous to the differential equation (1.3). To this end, we will study the large time behavior of solutions of the doubly nonlinear evolution

$$\mathcal{J}_p(\dot{v}(t)) + \partial\Phi(v(t)) \ni 0 \quad (a.e.\ t > 0). \tag{1.7}$$

The main reason we will study this flow is that the function

$$t \mapsto \frac{p\Phi(v(t))}{\|v(t)\|^p}$$

is nonincreasing on any interval of time for which it is defined.

However, instead of restricting our attention to paths $v : [0, \infty) \to X$ that satisfy (1.7) at almost every $t > 0$, we will study p-curves of maximal slope for Φ; see Definition 4.1 below. These are locally absolutely continuous paths such that $t \mapsto \Phi(v(t))$ decreases as much as possible in the sense of the chain rule. It turns out that p-curves of maximal slope for Φ satisfy (1.7) when they are differentiable and they have been shown to exist in general Banach spaces for any prescribed $v(0) \in \mathrm{Dom}(\Phi)$ (Chapters 1–3 of [2]). While most of the Banach spaces X we have in mind satisfy the Radon–Nikodym property, which guarantees the almost everywhere differentiability of absolutely continuous paths (Chapter VII, Section 6 of [18]), our proof does not rely on this assumption.

Theorem 1.3. *Assume that λ_p is simple and that v is a p-curve of maximal slope for Φ with $v(0) \in Dom(\Phi)$. Then the limit $w := \lim_{t\to\infty} e^{\mu_p t} v(t)$ exists and $w \in Dom(\Phi)$. Moreover,*

$$\Phi(w) = \lim_{t\to\infty} \Phi(e^{\mu_p t} v(t)).$$

If $w \neq 0$, then w is a minimizer of $\Phi(u)/\|u\|^p$ and

$$\lambda_p = \lim_{t\to\infty} \frac{p\Phi(v(t))}{\|v(t)\|^p}.$$

Remark 1.4. Without the assumption that λ_p is simple, weaker versions of Theorem 1.2 and Theorem 1.3 still hold. See Remarks 3.7 and 4.11.

Our primary motivation for this work was in approximating optimal constants and extremal functions in various Sobolev inequalities. See Table 1 for the examples we will apply our results to. In each case and throughout this paper, $\Omega \subset \mathbb{R}^n$ is bounded, open and connected with C^1 boundary $\partial\Omega$; the mapping $T : W^{1,p}(\Omega) \to L^p(\partial\Omega; \sigma)$ is the Sobolev trace operator and σ is $n - 1$ dimensional Hausdorff measure. The organization of this paper is as follows. In Section 2, we will present some preliminary information and discuss examples. Then we will prove Theorem 1.2 and Theorem 1.3 in Sections 3 and 4, respectively.

2. Preliminaries

In this section, we will show that the functional $\Phi(u)/\|u\|^p$ has a minimizer $w \in X\backslash\{0\}$ that satisfies

$$\partial\Phi(w) - \lambda_p \mathcal{J}_p(w) \ni 0. \tag{2.1}$$

Note that (2.1) holds if there are $\xi \in \partial\Phi(w)$, $\zeta \in \mathcal{J}_p(w)$ such that $\xi - \lambda_p \zeta = 0$ or equivalently if $\partial\Phi(w) \cap \lambda_p \mathcal{J}_p(w) \neq \emptyset$. Next we shall discuss the projection of elements of X onto rays in the direction of nonzero vectors. Finally, we will present some examples of

Table 1
Rayleigh quotients in Sobolev spaces.

Rayleigh quotient	Space	Functional inequality
$\dfrac{\int_\Omega \|Du\|^p dx}{\int_\Omega \|u\|^p dx}$	$W_0^{1,p}(\Omega)$	$\lambda_p \int_\Omega \|u\|^p dx \le \int_\Omega \|Du\|^p dx$
$\dfrac{\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\|u(x)-u(y)\|^p}{\|x-y\|^{n+ps}} dxdy}{\int_\Omega \|u\|^p dx}$	$W_0^{s,p}(\Omega)$	$\lambda_p \int_\Omega \|u\|^p dx \le \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\|u(x)-u(y)\|^p}{\|x-y\|^{n+ps}} dxdy$
$\dfrac{\int_\Omega \|Du\|^p dx + \beta \int_{\partial\Omega} \|Tu\|^p d\sigma}{\int_\Omega \|u\|^p dx}$	$W^{1,p}(\Omega)$	$\lambda_p \int_\Omega \|u\|^p dx \le \int_\Omega \|Du\|^p dx + \beta \int_{\partial\Omega} \|Tu\|^p d\sigma$
$\dfrac{\int_\Omega \|Du\|^p dx}{\inf_{c\in\mathbb{R}} \int_\Omega \|u+c\|^p dx}$	$W^{1,p}(\Omega)$	$\lambda_p \inf_{c\in\mathbb{R}} \int_\Omega \|u+c\|^p dx \le \int_\Omega \|Du\|^p dx$
$\dfrac{\int_\Omega \|Du\|^p dx}{\\|u\\|_\infty^p}$	$W_0^{1,p}(\Omega)$ $(p>n)$	$\lambda_p \\|u\\|_\infty^p \le \int_\Omega \|Du\|^p dx$
$\dfrac{\int_\Omega (\|Du\|^p + \|u\|^p) dx}{\int_{\partial\Omega} \|Tu\|^p d\sigma}$	$W^{1,p}(\Omega)$	$\lambda_p \int_{\partial\Omega} \|Tu\|^p d\sigma \le \int_\Omega (\|Du\|^p + \|u\|^p)\, dx$

homogeneous functionals on Hilbert and Sobolev spaces that will be revisited throughout this work.

Let us first begin with a basic fact about the degree p homogeneous convex functionals Φ we are considering. For each $u \in \text{Dom}(\Phi)$ and $\zeta \in \partial\Phi(u)$,

$$p\Phi(u) = \langle \zeta, u\rangle \tag{2.2}$$

(Lemma 3.9 in [3]). However, the following proposition, which is equivalent to (2.2), will be more useful to us.

Proposition 2.1. *Assume $u, v \in Dom(\Phi)$ and $\zeta \in \partial\Phi(u)$. Then*

$$\langle \zeta, v\rangle \le [p\Phi(u)]^{1-\frac{1}{p}} [p\Phi(v)]^{\frac{1}{p}}. \tag{2.3}$$

Equality holds in (2.3) if and only if u and v are linearly dependent.

Proof. If $u = 0$, $\Phi(u) = 0$ and so

$$t^p \Phi(v) = \Phi(tv) \geq \langle \zeta, tv \rangle = t\langle \zeta, v \rangle$$

for each $t > 0$. Dividing by t and sending $t \to 0^+$, gives $\langle \zeta, v \rangle \leq 0$. Therefore, the claim holds for $u = 0$ and it trivially holds for $v = 0$, so we assume otherwise.

Suppose that $p\Phi(u) = p\Phi(v) = 1$. Then by (2.2) and the convexity of Φ

$$\begin{aligned}\langle \zeta, v \rangle &= \langle \zeta, v - u + u \rangle \\ &= \langle \zeta, v - u \rangle + \langle \zeta, u \rangle \\ &= \langle \zeta, v - u \rangle + p\Phi(u) \\ &\leq \Phi(v) - \Phi(u) + p\Phi(u) \\ &= \frac{1}{p}(p\Phi(v)) + \left(1 - \frac{1}{p}\right)(p\Phi(u)) \\ &= 1.\end{aligned}$$

The inequality above is strict when $u \neq v$, as Φ is strictly convex on its domain.

In general, we observe that if $\zeta \in \partial\Phi(u)$, then the homogeneity of Φ implies that $\zeta/c^{p-1} \in \partial\Phi(u/c)$ for each $c > 0$. Therefore,

$$\left\langle \frac{\zeta}{[p\Phi(u)]^{1-\frac{1}{p}}}, \frac{v}{[p\Phi(v)]^{\frac{1}{p}}} \right\rangle \leq 1$$

by the above computation. Hence, we conclude the assertion (2.3). Equality occurs only when $u/[p\Phi(u)]^{1/p} = v/[p\Phi(v)]^{1/p}$; that is, when u and v are linearly dependent. □

Now we shall argue that at least one minimizer $w \neq 0$ of $\Phi(u)/\|u\|^p$ exists. It would then follow that $\lambda_p > 0$ and therefore

$$\lambda_p \|u\|^p \leq p\Phi(u), \quad u \in X. \tag{2.4}$$

Proposition 2.2. *There exists $w \in X \setminus \{0\}$ for which*

$$\lambda_p = p\frac{\Phi(w)}{\|w\|^p}. \tag{2.5}$$

Moreover, $\lambda_p > 0$ and thus (2.4) holds.

Proof. Suppose $(u^k)_{k\in\mathbb{N}}$ is a minimizing sequence for λ_p. Without any loss of generality, we may assume $u^k \in \mathrm{Dom}(\Phi)$, $u^k \neq 0$ for each $k \in \mathbb{N}$ and

$$\lambda_p = \lim_{k\to\infty} p\frac{\Phi(u^k)}{\|u^k\|^p}.$$

Set $v^k := u^k/\|u_k\|$, and notice $\|v^k\| = 1$ and $\sup_{k\in\mathbb{N}} \Phi(v^k) < \infty$. By the compactness of the sublevel sets of Φ, there is a subsequence $(v^{k_j})_{j\in\mathbb{N}}$ that converges to some $w \in X$ that satisfies $\|w\| = 1$. Since Φ is degree p homogeneous and lower semicontinuous,

$$\lambda_p = \lim_{j\to\infty} p\frac{\Phi(u^{k_j})}{\|u^{k_j}\|^p} = \lim_{j\to\infty} p\Phi\left(\frac{u^{k_j}}{\|u^{k_j}\|}\right) = \lim_{j\to\infty} p\Phi(v^{k_j}) = \liminf_{j\to\infty} p\Phi(v^{k_j}) \geq p\Phi(w).$$

Thus, w satisfies (2.5). Since $w \neq 0$ and Φ is strictly convex with $\Phi(0) = 0$, $\Phi(w) > 0$. Thus, $\lambda_p > 0$. $\square$

Proposition 2.3. *An element $w \in X \setminus \{0\}$ satisfies (2.5) if and only if w satisfies (2.1).*

Proof. Suppose w satisfies (2.5) and $\xi \in \mathcal{J}_p(w)$. Then for each $u \in X$

$$\begin{aligned}\Phi(u) &\geq \lambda_p \frac{1}{p}\|u\|^p \\ &\geq \lambda_p\left(\frac{1}{p}\|w\|^p + \langle \xi, u - w\rangle\right) \\ &= \frac{\lambda_p}{p}\|w\|^p + \langle \lambda_p\xi, u - w\rangle \\ &= \Phi(w) + \langle \lambda_p \xi, u - w\rangle.\end{aligned}$$

Thus $\lambda_p\xi \in \partial\Phi(w)$. Conversely, suppose that (2.1) holds and select $\xi \in \mathcal{J}_p(w), \zeta \in \partial\Phi(w)$ such that $0 = \zeta - \lambda_p\xi$. By (2.2) and (1.5)

$$\begin{aligned}p\Phi(w) &= \langle \zeta, w\rangle \\ &= \langle \lambda_p\xi, w\rangle \\ &= \lambda_p\langle \xi, w\rangle \\ &= \lambda_p\|w\|^p. \quad \square\end{aligned}$$

Remark 2.4. The first part of the proof above gives the slightly stronger implication: if $w \in X \setminus \{0\}$ satisfies (2.5), then $\lambda_p\mathcal{J}_p(w) \subset \partial\Phi(w)$. The second part of the proof can be used to establish the following inequality. If $u \in \mathrm{Dom}(\Phi) \setminus \{0\}$ satisfies $\partial\Phi(u) - \lambda\mathcal{J}_p(u) \ni 0$, then

$$\lambda = p\frac{\Phi(u)}{\|u\|^p} \geq \lambda_p.$$

So in this sense, λ_p is the “smallest eigenvalue” of $\partial\Phi$.

An important tool in our convergence proofs will be the projection onto the ray $\{\beta w \in X : \beta \geq 0\}$ for a given $w \in X \setminus \{0\}$. Let

$$P_w(u) := \{\alpha w \in X : \alpha \geq 0 \text{ and } \|u - \alpha w\| \leq \|u - \beta w\| \text{ for all } \beta \geq 0\}.$$

The following is a basic proposition.

Proposition 2.5. *Assume $w \in X \setminus \{0\}$ and $u \in X$.*
(i) $P_w(u)$ is non-empty, compact, and convex.
(ii) For each $\gamma \in \mathbb{R}$, $P_w(\gamma w) = \{\gamma^+ w\}$.
(iii) For $s > 0$, $P_w(su) = sP_w(u)$.

Proof. (i) As

$$[0, \infty) \ni \beta \mapsto \|u - \beta w\|$$

is continuous and tends to ∞ as $\beta \to \infty$, this function attains its minimum value. Thus, $P_w(u) \neq \emptyset$ for each $u \in X$. Now suppose $\alpha_k w \in P_w(u)$ for each $k \in \mathbb{N}$. Then

$$\|u - \alpha_k w\| \leq \|u\|, \quad k \in \mathbb{N}.$$

In particular, $\|\alpha_k w\| \leq \|u\| + \|\alpha_k w - u\| \leq 2\|u\|$ and so $0 \leq \alpha_k \leq 2\|u\|/\|w\|$. It follows that $(\alpha_k)_{k\in\mathbb{N}}$ has a convergent subsequence $(\alpha_{k_j})_{j\in\mathbb{N}}$ with limit $\alpha \geq 0$. Then

$$\|u - \alpha w\| = \lim_{j\to\infty} \|u - \alpha_{k_j} w\| \leq \|u - \beta w\|, \quad \beta \geq 0.$$

Consequently, $\alpha w \in P_w(u)$ which implies that $P_w(u)$ is compact.

Suppose $\alpha_1 w, \alpha_2 w \in P_w(u)$ and $c_1, c_2 \geq 0$ with $c_1 + c_2 = 1$. Then

$$\begin{aligned}
\|u - (c_1\alpha_1 w + c_2\alpha_2 w)\| &= \|c_1(u - \alpha_1 w) + c_2(u - \alpha_2 w)\| \\
&\leq c_1\|u - \alpha_1 w\| + c_2\|u - \alpha_2 w\| \\
&\leq c_1\|u - \beta w\| + c_2\|u - \beta w\| \\
&= \|u - \beta w\|
\end{aligned}$$

for each $\beta \geq 0$. As a result, $P_w(u)$ is convex.

(ii) Suppose $\gamma \leq 0$ and $\beta > 0$. Then

$$\|\gamma w - \beta\gamma\| = (\beta - \gamma)\|w\| > (-\gamma)\|w\| = \|\gamma w - 0w\|,$$

which implies $P_w(\gamma w) = \{0\}$. If $\gamma > 0$, then $\|\gamma w - \beta w\| = |\gamma - \beta|\|w\| = 0$ only if $\beta = \gamma$. Consequently, $P_w(\gamma w) = \{\gamma w\}$.

(*iii*) Assume $s > 0$ and observe $v \in P_w(su)$ if and only if

$$\|u - (v/s)\| \leq \|u - (\beta/s)w\|, \quad \beta \geq 0.$$

This inequality, in turn, holds if and only if $(v/s) \in P_w(u)$. □

By the above proposition, $P_w(u)$ is a compact line segment in $\{\beta w \in X : \beta \geq 0\}$. We define

$$\alpha_w(u) := \inf\{\alpha > 0 : \alpha w \in P_w(u)\},$$

which represents the distance between this line segment and the origin $0 \in X$. Below we state a few properties of α_w, the most important being that α_w is continuous at each $\gamma w \in X$.

Proposition 2.6. *Assume $w \in X \setminus \{0\}$.*
(i) For $u \in X$, $s > 0$, $\alpha_w(su) = s\alpha_w(u)$.
(ii) α_w is lower semicontinuous.
(iii) α_w is continuous at each $\gamma w \in X$, $\gamma \in \mathbb{R}$.

Proof. (*i*) follows from part (*iii*) of the previous proof. Indeed

$$s\alpha_w(u) = \inf\{s\alpha > 0 : \alpha w \in P_w(u)\} = \inf\{\beta > 0 : \beta w \in sP_w(u)\} = \alpha_w(su).$$

(*ii*) Assume $u_k \to u \in X$ and $\liminf_{k\to\infty} \alpha_w(u_k) = \lim_{j\to\infty} \alpha_w(u_{k_j}) =: \alpha_\infty$.

$$\|u - \alpha_\infty w\| = \lim_{j\to\infty} \|u_{k_j} - \alpha_w(u_{k_j})w\| \leq \lim_{j\to\infty} \|u_{k_j} - \beta w\| = \|u - \beta w\|$$

for each $\beta \geq 0$. Thus, $\alpha_\infty w \in P_w(u)$ and $\alpha_w(u) \leq \alpha_\infty$. Hence, α_w is lower semicontinuous.

(*iii*) Assume that $\gamma \in \mathbb{R}$ and $u_k \to \gamma w$. For each $k \in \mathbb{N}$, select $\alpha_k \geq 0$ such that $\alpha_k w \in P_w(u_k)$. Observe, $\alpha_k\|w\| \leq \|u_k\| + \|u_k - \alpha_k w\| \leq 2\|u_k\|$. Thus, $(\alpha_k)_{k\in\mathbb{N}}$ is bounded and thus has a convergent subsequence $(\alpha_{k_j})_{j\in\mathbb{N}}$ with limit α_∞. Note that for each $\beta \geq 0$

$$\begin{aligned}\|\gamma w - \alpha_\infty w\| &= \lim_{j\to\infty} \|u_{k_j} - \alpha_{k_j} w\| \\ &\leq \lim_{j\to\infty} \|u_{k_j} - \beta w\| \\ &= \|\gamma w - \beta w\|.\end{aligned}$$

As a result, $|\gamma - \alpha_\infty| = \min_{\beta\geq 0} |\gamma - \beta|$ and so $\alpha_\infty = \gamma^+$. As this limit is independent of the subsequence $(\alpha_{k_j})_{j\in\mathbb{N}}$, it must be that $\lim_{k\to\infty} \alpha_w(u_k) = \gamma^+ = \alpha_w(\gamma w)$. □

Let us now discuss some examples.

Example 2.7. Let $X = \mathbb{R}^n$ with the Euclidean norm $\|\cdot\|$. It is straightforward to verify that $\mathcal{J}_p(u) = \{\|u\|^{p-2}u\}$ for each $u \in \mathbb{R}^n$. In this case, (2.1) is

$$\partial\Phi(w) - \lambda_p\|w\|^{p-2}w \ni 0.$$

A typical Φ is

$$\Phi(u) = \frac{1}{p}N(u)^p \quad (u \in \mathbb{R}^n)$$

where N is a strictly convex norm on $\mathbb{R}^n$.

Example 2.8. Let $p = 2$ and X be a separable Hilbert space over $\mathbb{R}$ with inner product $(\cdot,\cdot)$. Here $\|u\|^2 = (u, u)$, and by the Riesz Representation Theorem, we may identify X with X^* and $\mathcal{J}_2$ with the identity mapping on X. Assume $(z_k)_{k\in\mathbb{N}} \subset X$ is an orthonormal basis for X and $(\sigma_k)_{k\in\mathbb{N}}$ is a nondecreasing sequence of nonnegative numbers that satisfy $\lim_{k\to\infty} \sigma_k = \infty$. For $u \in X$, let $u_k = (u, z_k)$. Define the subspace

$$Y := \left\{\sum_{k\in\mathbb{N}} u_k z_k \in X : \sum_{k\in\mathbb{N}} \sigma_k|u_k|^2 < \infty\right\}$$

and the operator

$$A : Y \mapsto X; \sum_{k\in\mathbb{N}} u_k z_k \mapsto \sum_{k\in\mathbb{N}} \sigma_k u_k z_k.$$

Note A is bijective if $\sigma_1 > 0$.

We set

$$\Phi(u) = \begin{cases} \frac{1}{2}(Au, u), & u \in Y, \\ +\infty, & \text{otherwise} \end{cases}$$

and also observe that Φ is strictly convex on Y if $\sigma_1 > 0$. Direct calculation gives

$$\partial\Phi(u) = \begin{cases} Au, & u \in Y \\ \emptyset, & \text{otherwise} \end{cases}.$$

Note in particular that

$$\lambda_2 = \inf\left\{\frac{(Au, u)}{\|u\|^2}, u \in Y \setminus \{0\}\right\} = \sigma_1,$$

and that (2.1) takes the form $Aw = \lambda_2 w$. It is also plain to see that λ_2 is simple if and only if

$$\sigma_1 < \sigma_2.$$

Example 2.9. Let $X = L^p(\Omega)$. In this case, $\mathcal{J}_p(u)$ is the singleton comprised of the function $|u|^{p-1}u \in L^q(\Omega)$ for each $u \in L^p(\Omega)$. A natural choice for Φ is

$$\Phi(u) = \begin{cases} \frac{1}{p}\int\limits_{\Omega} |Du|^p dx, & u \in W_0^{1,p}(\Omega) \\ +\infty, & \text{otherwise} \end{cases}. \tag{2.6}$$

Here $|Du| := \sqrt{\sum_{i=1}^n u_{x_i}^2}$ and $W_0^{1,p}(\Omega)$ is the closure of $C_c^\infty(\Omega)$ in the norm given by $\Phi(\cdot)^{1/p}$. Recall that Φ has compact sublevel sets by Rellich–Kondrachov compactness in Sobolev spaces.

For this example, it has been established that λ_p is simple [7,27,32]. The inequality (2.4)

$$\lambda_p \int\limits_{\Omega} |u|^p dx \le \int\limits_{\Omega} |Du|^p dx \quad \left(u \in W_0^{1,p}(\Omega)\right)$$

is known as Poincaré's inequality and ubiquitous in mathematical analysis. Moreover, (2.1) takes the form of the following PDE

$$\begin{cases} -\Delta_p w = \lambda_p |w|^{p-2} w & x \in \Omega \\ w = 0 & x \in \partial\Omega \end{cases}. \tag{2.7}$$

Here $\Delta_p \psi := \operatorname{div}(|D\psi|^{p-2} D\psi)$ is called the p-Laplacian. When $p = 2$, λ_2 is the first eigenvalue of the Dirichlet Laplacian.

Example 2.10. We may also extend the above example to fractional Sobolev spaces. Specifically, we can take

$$\Phi(u) = \begin{cases} \frac{1}{p}\iint\limits_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u(x) - u(y)|^p}{|x-y|^{n+ps}} dxdy, & u \in W_0^{s,p}(\Omega) \\ +\infty, & \text{otherwise} \end{cases} \tag{2.8}$$

for $u \in X$. Here $s \in (0,1)$ and $W_0^{s,p}(\Omega)$ is the closure of $C_c^\infty(\Omega)$ in the norm given by $\Phi(\cdot)^{1/p}$. See Theorem 7.1 of [17] for a proof that Φ has compact sublevel sets.

For this example, it has also been verified that λ_p is simple, see [11], [22] and [28]. In this case, (2.1) is

$$\begin{cases} (-\Delta_p)^s w = \lambda_p |w|^{p-2} w & x \in \Omega \\ w = 0 & x \in \mathbb{R}^n \setminus \Omega \end{cases}. \tag{2.9}$$

Here the operator $(-\Delta_p)^s$ is defined as the principal value

$$(-\Delta_p)^s \psi(x) := 2 \lim_{\epsilon \to 0^+} \int\limits_{\mathbb{R}^n \setminus B_\epsilon(x)} \frac{|\psi(x) - \psi(y)|^{p-2}(\psi(x) - \psi(y))}{|x - y|^{n+ps}} dy.$$

Also note when $p = 2$, $(-\Delta_p)^s$ is a power of the Dirichlet Laplacian.

Example 2.11. Let $X = L^p(\Omega)$ and suppose $\beta > 0$. For $u \in X$ set

$$\Phi(u) = \begin{cases} \frac{1}{p} \int\limits_{\Omega} |Du|^p dx + \beta \frac{1}{p} \int\limits_{\partial\Omega} |Tu|^p d\sigma, & u \in W^{1,p}(\Omega) \\ +\infty, & \text{otherwise} \end{cases}.$$

Here $W^{1,p}(\Omega)$ is the Sobolev space of $L^p(\Omega)$ functions whose weak first partial derivatives belong $L^p(\Omega)$. Recall $T : W^{1,p}(\Omega) \to L^p(\partial\Omega; \sigma)$ is the Sobolev trace operator and σ is $n - 1$ dimensional Hausdorff measure.

The least Rayleigh quotient λ_p is simple and minimizers of $\Phi(u)/\|u\|^p$ satisfy

$$\int\limits_{\Omega} |Dw|^{p-2} Dw \cdot D\phi dx + \beta \int\limits_{\partial\Omega} (|Tw|^{p-2} Tw) T\phi d\sigma = \lambda_p \int\limits_{\Omega} |w|^{p-2} w \phi dx \tag{2.10}$$

for each $\phi \in W^{1,p}(\Omega)$ (see [12] for a detailed discussion). Integrating by parts we find that (2.10) is a weak formulation of the Robin boundary value problem

$$\begin{cases} -\Delta_p w = \lambda_p |w|^{p-2} w & x \in \Omega \\ |Dw|^{p-2} Dw \cdot \nu + \beta |w|^{p-2} w = 0 & x \in \partial\Omega \end{cases}. \tag{2.11}$$

Here and in what follows ν is the outward unit normal vector field on $\partial\Omega$.

Example 2.12. Let $X = L^p(\Omega)/\mathcal{C}$, where $\mathcal{C} = \{v \in L^p(\Omega) : v(x) = \text{constant a.e. } x \in \Omega\}$. That is, for any two $u, v \in L^p(\Omega)$ with $u - v$ constant almost everywhere, the equivalence classes of u and v are the same X. We will take the liberty of identifying equivalence classes in X with a representative in $L^p(\Omega)$. Recall that X is equipped with the quotient norm

$$\|u\| := \inf_{v \in \mathcal{C}} \|u + v\|_{L^p(\Omega)} = \inf_{c \in \mathbb{R}} \left(\int\limits_{\Omega} |u + c|^p dx \right)^{1/p}$$

and

$$X^* = \left\{ \xi \in L^q(\Omega) : \int_\Omega \xi dx = 0 \right\}.$$

Note that for each $u \in X$,

$$\mathbb{R} \ni c \mapsto \int_\Omega |u + c|^p dx$$

is strictly convex and tends to $+\infty$ by sending c to either $+\infty$ or $-\infty$. Therefore, for each $u \in X$, there is a unique $c = c^u$ such that $\|u\| = \left(\int_\Omega |u + c^u|^p dx\right)^{1/p}$. Moreover,

$$\int_\Omega |u + c^u|^{p-2}(u + c^u) dx = 0.$$

It is also routine to verify that $\mathcal{J}_p(u) = \{|u + c^u|^{p-2}(u + c^u)\}$.

We define

$$\Phi(u) := \begin{cases} \dfrac{1}{p} \displaystyle\int_\Omega |Du|^p dx, & u \in W^{1,p}(\Omega) \\ +\infty, & \text{otherwise} \end{cases}$$

for $u \in X$. Note that Φ is strictly convex on its domain and $\Phi(u + c) = \Phi(u)$ for each constant c. In particular,

$$\begin{aligned}
\lambda_p &= \inf \left\{ \frac{\displaystyle\int_\Omega |Du|^p dx}{\displaystyle\inf_{c \in \mathbb{R}} \int_\Omega |u + c|^p dx} : u \in W^{1,p}(\Omega) \setminus \mathcal{C} \right\} \\
&= \inf \left\{ \frac{\displaystyle\int_\Omega |D(u + c^u)|^p dx}{\displaystyle\int_\Omega |u + c^u|^p dx} : u \in W^{1,p}(\Omega) \setminus \mathcal{C} \right\} \\
&= \inf \left\{ \frac{\displaystyle\int_\Omega |Dw|^p dx}{\displaystyle\int_\Omega |w|^p \, dx} : w \in W^{1,p}(\Omega) \setminus \{0\}, \quad \int_\Omega |w|^{p-2} w dx = 0 \right\}.
\end{aligned} \tag{2.12}$$

Minimizers in (2.12) satisfy the boundary value problem

$$\begin{cases} -\Delta_p w = \lambda_p |w|^{p-2} w & x \in \Omega \\ |Dw|^{p-2} Dw \cdot \nu = 0 & x \in \partial\Omega \end{cases} \tag{2.13}$$

in the weak sense, and of course, $\int_\Omega |w|^{p-2} w dx = 0$. Therefore, λ_p is the first nontrivial Neumann eigenvalue of the p-Laplacian. Moreover, λ_p is the optimal constant in another version of Poincaré's inequality

$$\lambda_p \inf_{c\in\mathbb{R}} \int_\Omega |u + c|^p dx \le \int_\Omega |Du|^p dx, \quad \left(u \in W^{1,p}(\Omega)\right).$$

It is not difficult to find domains Ω for which λ_p is not simple. However, there are conditions that can be assumed on Ω which result in a simple λ_p (see, for instance, Proposition 1.1 of [5]).

Example 2.13. Let $K \subset \mathbb{R}^n$ be compact and $X = C(K)$ be equipped with the norm $\|u\|_\infty := \sup\{|u(x)| : x \in K\}$. By a theorem of Riesz, X^* is the collection of signed Radon measures on K equipped with the total variation norm

$$\|\xi\|_{TV} := \sup \left\{ \int_K u(x) d\xi(x) : \|u\|_\infty \le 1 \right\}$$

(Corollary 7.18 in [21]). We leave it as an exercise to verify that $\xi \in X^*$ belongs to $\mathcal{J}_p(u)$ if and only if

$$\int_K h(x) d\xi(x) \le \max\{|u(x)|^{p-2} u(x) h(x) : x \in K \text{ such that } |u(x)| = \|u\|_\infty\}$$

for each $h \in X$.

Now assume $K = \overline{\Omega}$ and define Φ as in (2.6). Additionally suppose $p > n$ and recall that each function in $W^{1,p}_0(\Omega)$ has a continuous representative $C^{1-n/p}(\overline{\Omega})$; so without loss of generality we consider $W^{1,p}_0(\Omega) \subset C(\overline{\Omega})$. In particular, we note that this embedding is compact in X by the Arzelá–Ascoli theorem. Therefore, minimizers exist for the infimum

$$\lambda_p = \inf \left\{ \frac{\int_\Omega |Du|^p dx}{\|u\|_\infty^p} : u \in W^{1,p}_0(\Omega) \setminus \{0\} \right\}.$$

Minimizers of this Rayleigh quotient has been studied in [20] and in [26]. We now make a brief summary of what is known.

Computing the first variation of $\int_\Omega |Du|^p dx/\|u\|^p_\infty$ we find

$$\int\limits_\Omega |Du|^{p-2}Du\cdot D\phi dx = \lambda_p \max\{|u(x)|^{p-2}u(x)\phi(x) : x\in\overline{\Omega}, |u(x)| = \|u\|_\infty\} \tag{2.14}$$

for each $\phi \in W^{1,p}_0(\Omega)$. Notice that the left hand side of (2.14) is a linear functional of ϕ. As a result, for each pair $\phi_1, \phi_2 \in W^{1,p}_0(\Omega)$

$$\max_{|u|=\|u\|_\infty}\{|u|^{p-2}u(\phi_1+\phi_2)\} = \max_{|u|=\|u\|_\infty}\{|u|^{p-2}u\phi_1\} + \max_{|u|=\|u\|_\infty}\{|u|^{p-2}u\phi_2\}.$$

It follows that $|u|$ achieves its maximum value at a single $x_0 \in \Omega$ and so

$$\begin{cases} -\Delta_p u = \lambda_p |u(x_0)|^{p-2}u(x_0)\delta_{x_0} & x\in\Omega \\ \qquad u = 0 & x\in\partial\Omega \end{cases}. \tag{2.15}$$

Also observe that $|u(x_0)|^{p-2}u(x_0)\delta_{x_0} \in \mathcal{J}_p(u)$.

In [26], we proved that λ_p is simple when Ω is convex. In the case $\Omega = B_r(x_0)$, functions $u \in W^{1,p}_0(B_r(x_0))$ that minimize $\int_{B_r(x_0)} |Du|^p dx/\|u\|^p_\infty$ are necessarily of the form

$$u(x) = a\left(r^{\frac{p-n}{p-1}} - |x-x_0|^{\frac{p-n}{p-1}}\right), \quad x\in B_r(x_0)$$

for $a \in \mathbb{R}$, see [13,36].

Example 2.14. Recall that the trace operator $T : W^{1,p}(\Omega) \to L^p(\partial\Omega;\sigma)$ is a bounded linear mapping. We can use the methods in this paper to estimate the operator norm of T

$$\|T\| := \sup\left\{ \frac{\left(\int\limits_{\partial\Omega} |Tu|^p d\sigma\right)^{1/p}}{\left(\int\limits_\Omega (|Du|^p + |u|^p)dx\right)^{1/p}} : u \in W^{1,p}\setminus\{0\}\right\}. \tag{2.16}$$

In particular, we can use the fact that T is a compact mapping [8] and that $W^{1,p}(\Omega)$ is reflexive to show that there is at least one maximizer in (2.16). Finding such a maximizer is also known as the “Steklov problem” [4,16,19,35].

Maximizers in (2.16) satisfy

$$\int\limits_\Omega (|Dw|^{p-2}Dw\cdot D\phi + |w|^{p-2}w\phi)dx = \lambda_p \int\limits_{\partial\Omega} |Tw|^{p-2}Tw\cdot(T\phi)d\sigma$$

for $\phi \in W^{1,p}(\Omega)$. Here $\lambda_p := \|T\|^{-p}$, and observe that this is a weak formulation of the PDE

$$\begin{cases} -\Delta_p w + |w|^{p-2}w = 0 & x \in \Omega \\ \quad |Dw|^{p-2}Dw \cdot \nu = \lambda_p |w|^{p-2}w & x \in \partial\Omega \end{cases}. \tag{2.17}$$

Moreover, it has been established that the collection of maximizers is one dimensional [19], and so in this sense we say λ_p is simple.

3. Inverse iteration

In this section, we will study the convergence properties of solutions of the inverse iteration scheme (1.6)

$$\partial\Phi(u_k) - \mathcal{J}_p(u_{k-1}) \ni 0, \quad k \in \mathbb{N}.$$

Here $u_0 \in X$ is given. Once u_{k-1} is known, $u_k \in \mathrm{Dom}(\Phi)$ can be obtained by selecting $\xi_{k-1} \in \mathcal{J}_p(u_{k-1})$ and defining u_k as the unique minimizer of the functional

$$X \ni v \mapsto \Phi(v) - \langle \xi_{k-1}, v \rangle.$$

Indeed, $\Phi(v) - \langle \xi_{k-1}, v\rangle \geq \Phi(u_k) - \langle \xi_{k-1}, u_k\rangle$ for all $v \in X$ implies $\xi_{k-1} \in \partial\Phi(u_k)$.

We now proceed to derive various monotonicity and compactness properties of solutions that will be used in our proof of Theorem 1.2.

Lemma 3.1. *Assume $u_0 \in X$. Then*

$$\|u_k\| \leq \frac{1}{\mu_p}\|u_{k-1}\| \tag{3.1}$$

and

$$\Phi(u_k) \leq \frac{1}{\mu_p^p}\Phi(u_{k-1}) \tag{3.2}$$

for each $k \in \mathbb{N}$. In particular, $\left(\|\mu_p^k u_k\|\right)_{k\in\mathbb{N}}$ and $\left(\Phi(\mu_p^k u_k)\right)_{k\in\mathbb{N}}$ are nonincreasing sequences.

Proof. For each $k \in \mathbb{N}$, choose $\xi_{k-1} \in \mathcal{J}_p(u_{k-1}) \cap \partial\Phi(u_k)$. By (2.2) and (1.5),

$$\begin{aligned} p\Phi(u_k) &= \langle \xi_{k-1}, u_k \rangle \\ &\leq \|u_k\| \|\xi_{k-1}\|_* \\ &= \|u_k\| \|u_{k-1}\|^{p-1}. \end{aligned} \tag{3.3}$$

Combining this bound with inequality (2.4) gives

$$\lambda_p\|u_k\|^p \leq p\Phi(u_k) \leq \|u_k\|\|u_{k-1}\|^{p-1}.$$

Inequality (3.1) now follows.

We may assume $u_0 \in \mathrm{Dom}(\Phi)$; or else (3.2) clearly holds for $k = 1$. Moreover, if $\Phi(u_k) = 0$, (3.2) is immediate, so we assume otherwise. Continuing from (3.3) and again applying inequality (2.4) gives

$$\begin{aligned} p\Phi(u_k) &\leq \|u_k\|\|u_{k-1}\|^{p-1} \\ &\leq \left[\frac{1}{\lambda_p}p\Phi(u_k)\right]^{1/p}\left[\frac{1}{\lambda_p}p\Phi(u_{k-1})\right]^{1-1/p} \\ &= \frac{p}{\lambda_p}\left[\Phi(u_k)\right]^{1/p}\left[\Phi(u_{k-1})\right]^{1-1/p}. \end{aligned}$$

Therefore $\Phi(u_k)^{1-1/p} \leq \frac{1}{\lambda_p}\Phi(u_{k-1})^{1-1/p}$ and we have verified (3.2). □

As mentioned in the introduction, solution sequences of (1.6) admit the following fundamental monotonicity properties.

Lemma 3.2. *Assume $u_0 \in \overline{\mathrm{Dom}(\Phi)} \setminus \{0\}$. Then $u_k \in \mathrm{Dom}(\Phi) \setminus \{0\}$ for each $k \in \mathbb{N}$. Moreover,*

$$\frac{\Phi(u_k)}{\|u_k\|^p} \leq \frac{\Phi(u_{k-1})}{\|u_{k-1}\|^p} \tag{3.4}$$

and

$$\frac{\|u_k\|}{\|u_{k+1}\|} \leq \frac{\|u_{k-1}\|}{\|u_k\|} \tag{3.5}$$

for each $k \in \mathbb{N}$.

Proof. 1. Assume that $u_1 = 0$. By inequality (2.3), $\langle\xi, v\rangle = 0$ for any $\xi \in \partial\Phi(u_1)$ and $v \in \overline{\mathrm{Dom}(\Phi)}$. Selecting $\xi \in \partial\Phi(u_1) \cap \mathcal{J}_p(u_0)$ gives $0 = \langle\xi, u_0\rangle = \|u_0\|^p$ by (1.5). As a result, if $u_0 \in \overline{\mathrm{Dom}(\Phi)} \setminus \{0\}$, $u_1 \in \mathrm{Dom}(\Phi) \setminus \{0\}$. By induction on $k \in \mathbb{N}$, we conclude that $u_k \in \mathrm{Dom}(\Phi) \setminus \{0\}$.

2. We now proceed to verify (3.4). For $\xi_{k-1} \in \partial\Phi(u_k) \cap \mathcal{J}_p(u_{k-1})$,

$$\begin{aligned} \|u_{k-1}\|^p &= \langle\xi_{k-1}, u_{k-1}\rangle \\ &\leq [p\Phi(u_k)]^{1-1/p}[p\Phi(u_{k-1})]^{1/p} \end{aligned} \tag{3.6}$$

by (1.5) and inequality (2.3). Combining (3.3) and (3.6) gives

$$\begin{aligned}\frac{p\Phi(u_k)}{\|u_k\|^p} &\leq \frac{\|u_{k-1}\|^p}{\|u_k\|^{p-1}\|u_{k-1}\|}\\ &\leq \frac{[p\Phi(u_k)]^{1-1/p}[p\Phi(u_{k-1})]^{1/p}}{\|u_k\|^{p-1}\|u_{k-1}\|}\\ &= \left[\frac{p\Phi(u_k)}{\|u_k\|^p}\right]^{1-1/p}\left[\frac{p\Phi(u_{k-1})}{\|u_{k-1}\|^p}\right]^{1/p}.\end{aligned}$$

The inequality (3.4) is now immediate.

3. Employing (3.6), (3.4), and then (3.3) gives

$$\begin{aligned}\frac{\|u_k\|^p}{\|u_{k+1}\|^p} &\leq \frac{[p\Phi(u_{k+1})]^{1-1/p}[p\Phi(u_k)]^{1/p}}{\|u_{k+1}\|^p}\\ &\leq \left[p\Phi(u_k)\frac{\|u_{k+1}\|^p}{\|u_k\|^p}\right]^{1-1/p}\frac{[p\Phi(u_k)]^{1/p}}{\|u_{k+1}\|^p}\\ &= \frac{p\Phi(u_k)}{\|u_{k+1}\|\|u_k\|^{p-1}}\\ &\leq \frac{\|u_k\|\|u_{k-1}\|^{p-1}}{\|u_{k+1}\|\|u_k\|^{p-1}}\\ &= \frac{\|u_k\|}{\|u_{k+1}\|}\left[\frac{\|u_{k-1}\|}{\|u_k\|}\right]^{p-1}.\end{aligned}$$

We then conclude (3.5). □

Remark 3.3. Without the assumption $u_0 \in \overline{\text{Dom}(\Phi)} \setminus \{0\}$, the inequalities (3.4) and (3.5) are still valid provided their denominators are nonzero.

It is straightforward to check that if u_0 is a minimizer of $\Phi(u)/\|u\|^p$, then

$$u_k = \mu_p^{-k} u_0, \quad k \in \mathbb{N} \tag{3.7}$$

is a "separation of variables" solution of (1.6). We show in fact that this solution is unique. We will assume that λ_p is simple for the remainder of this section.

Corollary 3.4. *If $u_0 \neq 0$ is a minimizer of $\Phi(u)/\|u\|^p$, (3.7) is the unique solution sequence of (1.6).*

Proof. By induction, it suffices to verify this claim for $k = 1$. Recall $u_1 \neq 0$. By (3.4), u_1 is also a minimizer of $\Phi(u)/\|u\|^p$. For $\xi \in \partial\Phi(u_1) \cap \mathcal{J}_p(u_0)$

$$\lambda_p\|u_1\|^p = p\Phi(u_1) = \langle \xi, u_1 \rangle.$$

Therefore $\|\mu_p u_1\|^p = \langle \xi, \mu_p u_1 \rangle$, which implies $\xi \in \mathcal{J}_p(\mu_p u_1)$. It follows that $\|\mu_p u_1\| = \|u_0\|$, and as λ_p is simple, $u_1 = u_0/\mu_p$ or $u_1 = -u_0/\mu_p$.

Suppose $u_1 = -u_0/\mu_p$ and select $\eta \in \partial\Phi(-u_0/\mu_p) \cap \mathcal{J}_p(u_0)$. It follows that

$$\begin{aligned} \|u_0\|^p &= \langle \eta, u_0 \rangle \\ &= -\mu_p \left\langle \eta, -\frac{u_0}{\mu_p} \right\rangle \\ &= -\mu_p p \Phi\left(-\frac{u_0}{\mu_p}\right) \\ &= \frac{p}{\lambda_p}(-\Phi(-u_0)) \\ &< \frac{p}{\lambda_p}\Phi(u_0) \\ &= \|u_0\|^p \end{aligned}$$

since $u_0 \neq 0$ and Φ is strictly convex on its domain. Thus $u_1 = \mu_p^{-1}u_0$ and the claim is verified. □

Theorem 1.2 asserts that general solution sequences $(u_k)_{k\in\mathbb{N}}$ of (1.6) behave like (3.7) for large k. Two final ingredients in our proof will be a fundamental compactness assertion and a lemma involving the projection of solution sequences onto rays determined by minimizers of $\Phi(u)/\|u\|^p$. We now establish these two claims and then proceed directly to a proof of Theorem 1.2.

Lemma 3.5. *Assume $(g^j)_{j\in\mathbb{N}} \subset X$ converges to g and u^j is a solution of*

$$\partial\Phi(u^j) - \mathcal{J}_p(g^j) \ni 0 \tag{3.8}$$

for each $j \in \mathbb{N}$. Then there is a subsequence $(u^{j_\ell})_{\ell\in\mathbb{N}}$ that converges to a solution u of

$$\partial\Phi(u) - \mathcal{J}_p(g) \ni 0, \tag{3.9}$$

and $\Phi(u) = \lim_{\ell\to\infty}\Phi(u^{j_\ell})$.

Proof. As u^j is a solution of (3.8), we can mimic (3.3) and exploit (2.4) to derive

$$p\Phi(u^j) \leq \|g^j\|^{p-1}\|u^j\| \leq \|g^j\|^{p-1}\left(\frac{p\Phi(u^j)}{\lambda_p}\right)^{1/p}.$$

It follows that

$$p\Phi(u^j) \leq \frac{1}{\mu_p^p}\|g^j\|^p$$

for each $j \in \mathbb{N}$. Therefore, there is a subsequence $(u^{j_\ell})_{\ell\in\mathbb{N}}$ that converges to some $u \in \mathrm{Dom}(\Phi)$. Moreover, there is $\xi^\ell \in \partial\Phi(u^{j_\ell}) \cap \mathcal{J}_p(g^{j_\ell})$ with

$$\sup_{\ell\in\mathbb{N}}\|\xi^\ell\|_* = \sup_{\ell\in\mathbb{N}}\|g^{j_\ell}\|^{p-1} < \infty.$$

By Alaoglu's theorem, there is a subsequence $(\xi^{\ell_m})_{m\in\mathbb{N}}$ of $(\xi^{\ell})_{\ell\in\mathbb{N}}$ that converges weak-$*$ to some $\xi \in X^*$. By the convexity and lower semicontinuity of Φ and $\|\cdot\|^p/p$, it is routine to verify that $\xi \in \partial\Phi(u) \cap \mathcal{J}_p(g)$. That is, u satisfies (3.9). Moreover,

$$\lim_{m\to\infty} p\Phi(u^{j_{\ell_m}}) = \lim_{m\to\infty}\langle \xi^{\ell_m}, u^{j_{\ell_m}}\rangle = \langle \xi, u\rangle = p\Phi(u). \quad \square$$

Lemma 3.6. *Assume $w \in X \setminus \{0\}$ is a minimizer of $\Phi(z)/\|z\|^p$ and $C > 0$. There is $\delta = \delta(w, C) > 0$ with the following property. If $g \in X$ and $u \in Dom(\Phi)$ satisfy (3.9) and*

(i) $\Phi(g) \geq \Phi(w)$
(ii) $\|g\| \leq C$
(iii) $\alpha_w(g) \geq \frac{1}{2}$
(iv) $p\Phi(g)/\|g\|^p \leq \lambda_p + \delta$,

then

$$\alpha_w(\mu_p u) \geq \frac{1}{2}.$$

Proof. Assume the assertion is false. Then there is a $w_0 \in X \setminus \{0\}$ minimizing $\Phi(z)/\|z\|^p$ and $C_0 > 0$ such that: for each $j \in \mathbb{N}$ there are $g^j \in X$ and $u^j \in \mathrm{Dom}(\Phi)$ satisfying (3.8) and

(i) $\Phi(g^j) \geq \Phi(w_0)$
(ii) $\|g^j\| \leq C_0$
(iii) $\alpha_{w_0}(g^j) \geq \frac{1}{2}$
(iv) $p\Phi(g^j)/\|g^j\|^p \leq \lambda_p + \frac{1}{j}$,

while

$$\alpha_{w_0}(\mu_p u^j) < \frac{1}{2}. \tag{3.10}$$

As $(\Phi(g^j))_{j\in\mathbb{N}}$ is bounded, $(g^j)_{j\in\mathbb{N}}$ has a subsequence (that we will not relabel) that converges to $g \in X$. Moreover, it is straightforward to verify that $g \neq 0$, g minimizes $\Phi(z)/\|z\|^p$ and $\Phi(g) = \lim_{j\to\infty}\Phi(g^j)$. By Lemma 3.5, there is a subsequence $(u^{j_\ell})_{\ell\in\mathbb{N}}$ that converges to some u satisfying (3.9) and $\Phi(u) = \lim_{\ell\to\infty}\Phi(u^{j_\ell})$. As $g \neq 0$, $u \neq 0$ and by inequality (3.4),

$$\lambda_p \leq \frac{p\Phi(u)}{\|u\|^p} = \lim_{\ell\to\infty}\frac{p\Phi(u^{j_\ell})}{\|u^{j_\ell}\|^p} \leq \lim_{\ell\to\infty}\frac{p\Phi(g^{j_\ell})}{\|g^{j_\ell}\|^p} = \lambda_p.$$

Thus u minimizes $\Phi(z)/\|z\|^p$ and by Corollary 3.4, it must be that $u = g/\mu_p$.

Since λ_p is simple, g and w_0 are linearly dependent and $g = \gamma w_0$ for some $\gamma \in \mathbb{R}$. By part (ii) of Proposition 2.5, $\alpha_{w_0}(g) = \gamma^+$. By part (iii) of Proposition 2.6 and (ii) above

$$\gamma^+ = \alpha_{w_0}(g) = \lim_{j\to\infty} \alpha_{w_0}(g^j) \geq \frac{1}{2}.$$

Moreover,

$$\lambda_p \|g\|^p = \Phi(g) = \lim_{j\to\infty} \Phi(g^j) \geq \Phi(w_0) = \lambda_p \|w_0\|^p$$

and so $\|g\| \geq \|w_0\|$. This forces

$$\gamma \geq 1.$$

However, by (3.10) and part (iii) of Proposition 2.6

$$\gamma = \alpha_{w_0}(g) = \alpha_{w_0}(\mu_p u) = \lim_{j\to\infty} \alpha_{w_0}(\mu_p u^j) \leq \frac{1}{2}.$$

We are able to conclude by this contradiction. □

Proof of Theorem 1.2. Set $w_k := \mu_p^k u_k$ and observe that

$$0 \in \partial\Phi(w_k) - \lambda_p \mathcal{J}_p(w_{k-1}), \quad k \in \mathbb{N}.$$

Define $S = \lim_{k\to\infty} p\Phi(w_k)$ and $L = \lim_{k\to\infty} \|w_k\|$; these limits exist by Lemma 3.1. If $S = 0$, $\lim_{k\to\infty} w_k = 0$; so let us now assume $S > 0$.

As Φ has compact sublevel sets, $(w_k)_{k\in\mathbb{N}}$ has a convergent subsequence $\left(w_{k_j}\right)_{j\in\mathbb{N}}$ with limit w. We also have $\|w\| = L$, and by the lower semicontinuity of Φ, $p\Phi(w) \leq S$. Selecting $\xi_{k-1} \in \mathcal{J}_p(w_{k-1})$ and $\zeta_k \in \partial\Phi(w_k)$, such that $\zeta_k - \lambda_p \xi_{k-1} = 0$ for each $k \in \mathbb{N}$, and using (2.2) and (1.5), gives

$$\begin{aligned} p\Phi(w_{k_j}) &= \langle \zeta_{k_j}, w_{k_j} \rangle \\ &= \lambda_p \langle \xi_{k_j-1}, w_{k_j} \rangle \\ &\leq \lambda_p \|w_{k_j}\| \|\xi_{k_j-1}\|_* \\ &= \lambda_p \|w_{k_j}\| \|w_{k_j-1}\|^{p-1}. \end{aligned}$$

Thus,

$$S = \limsup_{j\to\infty} p\Phi(w_{k_j}) \leq \lambda_p \|w\|^p \leq p\Phi(w).$$

As a result, $S = p\Phi(w)$ and $\lambda_p \|w\|^p = p\Phi(w)$. Furthermore

$$\lim_{k\to\infty} \frac{p\Phi(u_k)}{\|u_k\|^p} = \lim_{k\to\infty} \frac{p\Phi(w_k)}{\|w_k\|^p} = \frac{p\Phi(w)}{\|w\|^p} = \lambda_p \tag{3.11}$$

and

$$\lim_{k\to\infty} \frac{\|u_{k-1}\|}{\|u_k\|} = \lim_{k\to\infty} \mu_p \frac{\|w_{k-1}\|}{\|w_k\|} = \mu_p.$$

We are only left to verify that the limit w is independent of the subsequence $(w_{k_j})_{j\in\mathbb{N}}$. To this end, we will employ Lemma 3.5 and Lemma 3.6. We first claim

$$w = \lim_{j\to\infty} w_{k_j+m} \tag{3.12}$$

for each $m \in \mathbb{N}$. Observe that

$$\partial\Phi(\mu_p^{-1} w_{k_j+1}) - \mathcal{J}_p(w_{k_j}) \ni 0, \quad j \in \mathbb{N}.$$

By Lemma 3.5, $(\mu_p^{-1} w_{k_j+1})_{j\in\mathbb{N}}$ has a subsequence that converges to $w_1 \in \mathrm{Dom}(\Phi)$ and $\partial\Phi(w_1) - \mathcal{J}_p(w) \ni 0$. It follows that $w_1 \neq 0$, and by inequality (3.4), $\lambda_p\|w_1\|^p = p\Phi(w_1)$. Corollary 3.4 then implies that $w_1 = \mu_p^{-1} w$ and so (3.12) holds for $m = 1$. The general assertion follows similarly by induction on $m \in \mathbb{N}$.

Next we claim that there is a $j_0 \in \mathbb{N}$ such that

$$\alpha_w(w_{k_j+m}) \geq \frac{1}{2} \tag{3.13}$$

whenever $j \geq j_0$, for each $m \in \mathbb{N}$. We will argue that any $j_0 \in \mathbb{N}$ chosen so large that

$$\frac{p\Phi(w_{k_j})}{\|w_{k_j}\|^p} \leq \lambda_p + \delta \quad \text{and} \quad \alpha_w(w_{k_j}) \geq \frac{1}{2}$$

for $j \geq j_0$ will suffice; here $\delta := \delta(\|u_0\|, w)$ is the positive number in the statement of Lemma 3.6. That such a j_0 exists follows from (3.11) and the continuity of α_w at w (part (*iii*) of Proposition 2.6).

By the monotonicity inequality (3.1), $\|w_{k_j}\| \leq \|u_0\| =: C$ for $j \in \mathbb{N}$ and by (3.4), $\Phi(w_{k_j}) \geq \Phi(w)$ for $j \in \mathbb{N}$. Therefore,

(*i*) $\Phi(w_{k_j}) \geq \Phi(w)$
(*ii*) $\|w_{k_j}\| \leq C$
(*iii*) $\alpha_w(w_{k_j}) \geq \frac{1}{2}$
(*iv*) $\dfrac{p\Phi(w_{k_j})}{\|w_{k_j}\|^p} \leq \delta + \lambda_p$

for $j \geq j_0$. By Lemma 3.6,

$$\alpha_w(w_{k_j+1}) \geq \frac{1}{2}$$

for $j \geq j_0$.

Now suppose (3.13) holds for some $m \in \mathbb{N}$ and

$$\frac{p\Phi(w_{k_j+m})}{\|w_{k_j+m}\|^p} \leq \lambda_p + \delta \quad (j \geq j_0)$$

hold for $m \in \mathbb{N}$. By the monotonicity inequalities (3.1) and (3.4),

(i) $\Phi(w_{k_j+m}) \geq \Phi(w)$
(ii) $\|w_{k_j+m}\| \leq C$
(iii) $\alpha_w(w_{k_j+m}) \geq \frac{1}{2}$
(iv) $\dfrac{p\Phi(w_{k_{j+m}})}{\|w_{k_{j+m}}\|^p} \leq \delta + \lambda_p$

for $j \geq j_0$. We appeal to Lemma 3.6 again to conclude

$$\alpha_w(w_{k_j+m+1}) \geq \frac{1}{2}$$

for $j \geq j_0$. In addition, (3.4) implies

$$\frac{p\Phi(w_{k_j+m+1})}{\|w_{k_j+m+1}\|^p} \leq \lambda_p + \delta \quad (j \geq j_0).$$

By induction, the claim (3.13) follows.

Now let $(w_{k_\ell})_{j\in\mathbb{N}}$ be another subsequence of $(w_k)_{k\in\mathbb{N}}$ that converges to some $w_1 \in \mathrm{Dom}(\Phi)$. The arguments above imply that $\|w_1\| = L$ and $p\Phi(w_1) = S$; in particular, $w_1 \neq 0$ and $\lambda_p\|w_1\|^p = p\Phi(w_1)$. Since λ_p is simple, $w_1 = w$ or $w_1 = -w$. Now suppose $w_1 = -w$ and choose a subsequence $(k_{\ell_j})_{j\in\mathbb{N}}$ of $(k_\ell)_{\ell\in\mathbb{N}}$ so that

$$k_{\ell_j} > k_j, \quad j \in \mathbb{N}.$$

For $m_j := k_{\ell_j} - k_j$, we have

$$\alpha_w(w_{k_{\ell_j}}) = \alpha_w(w_{k_j+m_j}) \geq \frac{1}{2}$$

by (3.13) for $j \geq j_0$. Passing to the limit and using the continuity of α_w at $-w$ (part (iii) of Proposition 2.6) gives

$$\alpha_w(-w) \geq \frac{1}{2}.$$

But this cannot be the case as $\alpha_w(-w) = 0$. As a result, every subsequence of $(w_k)_{k\in\mathbb{N}}$ has a further subsequence that converges to w. It follows that the sequence $(w_k)_{k\in\mathbb{N}}$ converges to w. □

Remark 3.7. Without assuming that λ_p is simple, our proof above verifies that if $S := \lim_k p\Phi(\mu_p^k u_k) > 0$ then

$$\lambda_p = \lim_{k\to\infty} \frac{p\Phi(u_k)}{\|u_k\|^p} \quad \text{and} \quad \mu_p = \lim_{k\to\infty} \frac{\|u_{k-1}\|}{\|u_k\|}.$$

Furthermore, we did not need to suppose that λ_p is simple in order to deduce the existence of a convergent subsequence $(\mu_p^{k_j} u_{k_j})_{j\in\mathbb{N}}$. Our argument also gives that if $S > 0$, then $\lim_j \mu_p^{k_j} u_{k_j}$ is a minimizer of $\Phi(u)/\|u\|^p$.

Remark 3.8. It may be that $\lim_{k\to\infty} \mu_p^k u_k = 0$. To see this, we recall example that we discussed in the introduction with $X = \mathbb{R}^n$ equipped with the Euclidean norm, $p = 2$ and $\Phi(u) = \frac{1}{2} Au \cdot u$. Here A is an $n \times n$, symmetric, positive definite matrix with eigenvalues

$$0 < \sigma_1 < \sigma_2 \le \cdots \le \sigma_n.$$

In this case, $\mu_2 = \lambda_2 = \sigma_1$. If u_0 is an eigenvector for A corresponding to σ_2, then $u_k = \sigma_2^{-k} u_0$. In particular, $\mu_2^k u_k = (\sigma_1/\sigma_2)^{-k} u_0 \to 0$ as $k \to \infty$.

Example 3.9. Let us continue our discussion of Example 2.8 and assume $0 < \sigma_1 < \sigma_2$ which ensures that λ_2 is simple. Observe that inverse iteration (1.6) takes the form

$$Au_k = u_{k-1}, \quad k \in \mathbb{N}.$$

If $u_0 = \sum_{j\in\mathbb{N}} a_j z_j \in X$. Then

$$u_k = \sum_{j\in\mathbb{N}} a_j \sigma_j^{-k} z_j.$$

Moreover,

$$\|\sigma_1^k u_k - a_1 z_1\|^2 = \sum_{j\ge 2} a_j^2 \left(\frac{\sigma_1}{\sigma_j}\right)^{2k} \to 0$$

and

$$\frac{1}{2}(A(\sigma_1^k u_k), \sigma_1^k u_k) = \frac{1}{2}\sigma_1 a_1^2 + \frac{1}{2}\sum_{j\ge 2} a_j^2 \left(\frac{\sigma_1}{\sigma_j}\right)^{2k} \sigma_j \to \frac{1}{2}\sigma_1 a_1^2 = \frac{1}{2}(A(a_1 z_1), a_1 z_1)$$

as $k \to \infty$; the interchanges of sum and limit follow routinely by dominated convergence.

Let us assume now that $a_1 = (u_0, z_1) \neq 0$. In this case,

$$\frac{(Au_k, u_k)}{\|u_k\|^2} = \frac{\sum_{j\in\mathbb{N}} a_j^2 \sigma_j^{-2k+1}}{\sum_{j\in\mathbb{N}} a_j^2 \sigma_j^{-2k}} = \frac{a_1^2\sigma_1 + \sum_{j\geq 2} a_j^2 \sigma_j \left(\frac{\sigma_1}{\sigma_j}\right)^{2k}}{a_1^2 + \sum_{j\geq 2} a_j^2 \left(\frac{\sigma_1}{\sigma_j}\right)^{2k}}.$$

Consequently,

$$\sigma_1 = \lim_{k\to\infty} \frac{(Au_k, u_k)}{\|u_k\|^2}.$$

Similarly, direct computation gives

$$\sigma_1 = \lim_{k\to\infty} \frac{\|u_{k-1}\|}{\|u_k\|}.$$

As $\sigma_1 = \lambda_2 = \mu_2$, these calculations offer an alternative proof of the conclusion of Theorem 1.2.

Example 3.10. We continue Example 2.9, where inverse iteration involves the study of the sequence of boundary value problems

$$\begin{cases} -\Delta_p u_k = |u_{k-1}|^{p-2}u_{k-1}, & x \in \Omega \\ \quad u_k = 0, & x \in \partial\Omega \end{cases}.$$

The function $u_0 \in L^p(\Omega)$ is given and $k \in \mathbb{N}$. By Theorem 1.2,

$$w = \lim_{k\to\infty} \mu_p^k u_k$$

exists in $W_0^{1,p}(\Omega)$; and if $w \neq 0$, then w satisfies (2.7),

$$\lambda_p = \lim_{k\to\infty} \frac{\int_\Omega |Du_k|^p dx}{\int_\Omega |u_k|^p dx}, \quad \text{and} \quad \mu_p = \lim_{k\to\infty} \frac{\left(\int_\Omega |u_{k-1}|^p dx\right)^{1/p}}{\left(\int_\Omega |u_k|^p dx\right)^{1/p}}.$$

This result was first verified in our previous work [23] and was motivated by the paper of R. Biezuner, G. Ercole, and E. Martins [9].

Example 3.11. Let us reconsider Example 2.10. We recall that for this example λ_p is simple. Moreover, inverse iteration involves the study of the sequence of PDE

$$\begin{cases} (-\Delta_p)^s u_k = |u_{k-1}|^{p-2}u_{k-1}, & x \in \Omega \\ u_k = 0, & x \in \mathbb{R}^n \setminus \Omega \end{cases}$$

for $u_0 \in L^p(\Omega)$ and $k \in \mathbb{N}$. Since λ_p is simple, Theorem 1.2 implies

$$w = \lim_{k\to\infty} \mu_p^k u_k$$

in $W_0^{s,p}(\Omega)$. And if $w \neq 0$, w satisfies (2.15),

$$\lambda_p = \lim_{k\to\infty} \frac{\iint\limits_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|u_k(x)-u_k(y)|^p}{|x-y|^{n+sp}} dxdy}{\int\limits_\Omega |u_k|^p dx} \quad \text{and} \quad \mu_p = \lim_{k\to\infty} \frac{\left(\int\limits_\Omega |u_{k-1}|^p dx\right)^{1/p}}{\left(\int\limits_\Omega |u_k|^p dx\right)^{1/p}}.$$

Example 3.12. Regarding Example 2.11, inverse iteration takes the form: $u_0 \in L^p(\Omega)$,

$$\begin{cases} -\Delta_p u_k = |u_{k-1}|^{p-2}u_{k-1} & x \in \Omega \\ |Du_k|^{p-2}Du_k \cdot \nu + \beta|u_k|^{p-2}u_k = 0 & x \in \partial\Omega \end{cases}$$

for $k \in \mathbb{N}$. By Theorem 1.2, the limit $w = \lim_{k\to\infty} \mu_p^k u_k$ exists in $W^{1,p}(\Omega)$. If $w \neq 0 \in L^p(\Omega)$, then w satisfies (2.11),

$$\lambda_p = \lim_{k\to\infty} \frac{\int\limits_\Omega |Du_k|^p dx + \beta \int\limits_{\partial\Omega} |Tu_k|^p d\sigma}{\int\limits_\Omega |u_k|^p dx}, \quad \text{and} \quad \mu_p = \lim_{k\to\infty} \frac{\left(\int\limits_\Omega |u_{k-1}|^p dx\right)^{1/p}}{\left(\int\limits_\Omega |u_k|^p dx\right)^{1/p}}.$$

Example 3.13. Let us revisit Example 2.12. In this case, the inverse iteration scheme starts with a given $u_0 \in L^p(\Omega)$ with $\int_\Omega |u_0|^{p-2}u_0 dx = 0$. Then we must solve

$$\begin{cases} -\Delta_p u_k = |u_{k-1}|^{p-2}u_{k-1} & x \in \Omega \\ |Du_k|^{p-2}Du_k \cdot \nu = 0 & x \in \partial\Omega \end{cases}$$

with $\int_\Omega |u_{k-1}|^{p-2}u_{k-1}dx = 0$ for $k \in \mathbb{N}$. If λ_p is simple, Theorem 1.2 implies that $w := \lim_{k\to\infty} \mu_p^k u_k$ exists in $W^{1,p}(\Omega)$ and satisfies (2.13) and $\int_\Omega |w|^{p-2}wdx = 0$. If additionally $w \neq 0 \in L^p(\Omega)$, then

$$\lambda_p = \lim_{k\to\infty} \frac{\int_\Omega |Du_k|^p dx}{\int_\Omega |u_k|^p dx} \quad \text{and} \quad \mu_p = \lim_{k\to\infty} \frac{\left(\int_\Omega |u_{k-1}|^p dx\right)^{1/p}}{\left(\int_\Omega |u_k|^p dx\right)^{1/p}}.$$

Example 3.14. Inverse iteration related to Example 2.13 is as follows: for $u_0 \in C(\overline{\Omega})$, solve

$$\begin{cases} -\Delta_p u_k = \xi_{k-1} & x \in \Omega \\ u_k = 0 & x \in \partial\Omega \end{cases}$$

for each $k \in \mathbb{N}$. Here $\xi_{k-1} \in \mathcal{J}_p(u_{k-1})$. In particular,

$$\int_\Omega |Du_k|^{p-2}Du_k \cdot D\phi dx = \int_\Omega \phi(x)d\xi_{k-1}(x)$$

for each $\phi \in W_0^{1,p}(\Omega)$ and

$$\int_{\overline{\Omega}} \phi(x)d\xi_{k-1}(x) \leq \max\{|u_{k-1}(x)|^{p-2}u_{k-1}(x)\phi(x) : x \in \overline{\Omega}, |u_{k-1}(x)| = \|u_{k-1}\|_\infty\}$$

for each $\phi \in C(\overline{\Omega})$.

As we previously explained, λ_p is simple when Ω is convex. In this case, $w(x) = \lim_{k\to\infty} \mu_p^k u_k(x)$ exists uniformly for $x \in \Omega$. If w does not vanish identically, w satisfies equation (2.15),

$$\lambda_p = \lim_{k\to\infty} \frac{\int_\Omega |Du_k|^p dx}{\|u_k\|_\infty^p}, \quad \text{and} \quad \mu_p = \lim_{k\to\infty} \frac{\|u_{k-1}\|_\infty}{\|u_k\|_\infty}.$$

Example 3.15. Let us recall Example 2.14, which involves the norm of the trace operator $T : W^{1,p}(\Omega) \to L^p(\partial\Omega; \sigma)$. We now describe an inverse iteration scheme for this example. For a given $u_0 \in W^{1,p}(\Omega)$ with $Tu_0 \neq 0 \in L^p(\partial\Omega; \sigma)$, find a sequence $(u_k)_{k\in\mathbb{N}}$ verifying

$$\int_\Omega (|Du_k|^{p-2}Du_k \cdot D\phi + |u_k|^{p-2}u_k\phi)dx = \int_{\partial\Omega} |Tu_{k-1}|^{p-2}Tu_{k-1} \cdot (T\phi)d\sigma$$

for $\phi \in W^{1,p}(\Omega)$. This version of inverse iteration is a weak formulation of the sequence of boundary value problems

$$\begin{cases} -\Delta_p u_k + |u_k|^{p-2}u_k = 0 & x \in \Omega \\ \quad |Du_k|^{p-2}Du_k \cdot \nu = |u_{k-1}|^{p-2}u_{k-1} & x \in \partial\Omega \end{cases}.$$

Such a solution sequence exists and u_k minimizes the functional

$$W^{1,p}(\Omega) \ni v \mapsto \frac{1}{p}\|v\|^p_{W^{1,p}(\Omega)} - \int\limits_{\partial\Omega} (|Tu_{k-1}|^{p-2}Tu_{k-1})(Tv)d\sigma$$

for each $k \in \mathbb{N}$.

It is possible to verify that $Tu_0 \neq 0$ implies $Tu_k \neq 0 \in L^p(\partial\Omega;\sigma)$ for all $k \in \mathbb{N}$. Moreover, we have the following monotonicity formulae

$$\|u_k\|_{W^{1,p}(\Omega)} \leq \frac{1}{\mu_p}\|u_{k-1}\|_{W^{1,p}(\Omega)}, \quad \|Tu_k\|_{L^p(\partial\Omega)} \leq \frac{1}{\mu_p}\|Tu_{k-1}\|_{L^p(\Omega)},$$

and

$$\frac{\|u_k\|^p_{W^{1,p}(\Omega)}}{\|Tu_k\|^p_{L^p(\partial\Omega)}} \leq \frac{\|u_{k-1}\|^p_{W^{1,p}(\Omega)}}{\|Tu_{k-1}\|^p_{L^p(\partial\Omega)}}.$$

Here $\mu_p := \lambda_p^{1/(p-1)} = \|T\|^{-q}$, and the arguments given in Lemma 3.1 and Lemma 3.2 are readily adapted in this setting.

Recall that λ_p is simple in the sense that any two functions for which equality holds in (2.16) are linearly dependent. The same line of reasoning given in Theorem 1.2 gives that $w := \lim_{k\to\infty} \mu_p^k u_k$ exists in $W^{1,p}(\Omega)$. If $Tw \neq 0 \in L^p(\partial\Omega;\sigma)$, then w satisfies the boundary value problem (2.17) and

$$\|T\| = \lambda_p^{-1/p} = \lim_{k\to\infty} \frac{\|Tu_k\|_{L^p(\partial\Omega)}}{\|u_k\|_{W^{1,p}(\Omega)}}.$$

Therefore, even though Example 2.14 did not exactly fit into our framework, the ideas that went into proving Theorem 1.2 yield an analogous result.

4. Curves of maximal slope

We will now pursue the large time behavior of solutions $v : [0,\infty) \to X$ of the doubly nonlinear evolution (1.7)

$$\mathcal{J}_p(\dot{v}(t)) + \partial\Phi(v(t)) \ni 0 \quad (a.e.\ t > 0).$$

As we discussed in the introduction, our plan is study the large time behavior of a more general class of paths called p-curves of maximal slope for Φ. These paths satisfy (1.7) when they are differentiable almost everywhere. This goal will require us to recall the concepts of absolute continuity in a Banach space, the metric derivative of an absolutely continuous path and the local slope of a convex functional. Our primary reference for this background material is the monograph by L. Ambrosio, N. Gigli, and G. Savaré [2], which gives a comprehensive account of curves of maximal slope in metric spaces. Other results for the large time behavior of doubly nonlinear flows can be found in [1,30,31,33].

For $r \in [1, \infty]$, a path $v : [0, T] \to X$ is *r-absolutely continuous* if there is $h \in L^r([0, T])$ such that

$$\|v(t) - v(s)\| \le \int_s^t h(\tau)d\tau \tag{4.1}$$

for each $s, t \in [0, T]$ with $s \le t$. In this case, we write $v \in AC^r([0, T]; X)$ and $v \in AC([0, T]; X)$ when $r = 1$. It turns out that

$$|\dot{v}|(t) := \lim_{\tau \to 0} \frac{\|v(t + \tau) - v(t)\|}{|\tau|}$$

exists for almost every $t > 0$ and $|\dot{v}| \le h$; furthermore, (4.1) holds with $|\dot{v}|$ replacing h (Theorem 1.1.2 of [2]). The function $|\dot{v}| \in L^r([0, T])$ is called the *metric derivative* of v. A path $v : [0, \infty) \to X$ is *locally r-absolutely continuous* if the restriction of v to $[0, T]$ is r-absolutely continuous for each $T > 0$. As above, we will write $v \in AC^r_{\text{loc}}([0, \infty); X)$ and $v \in AC_{\text{loc}}([0, \infty); X)$ when $r = 1$.

Suppose $\Psi : X \to (-\infty, \infty]$ is convex, proper and lower semicontinuous. The quantity

$$|\partial\Psi|(u) := \limsup_{z \to 0} \frac{(\Psi(u) - \Psi(u + z))^+}{\|z\|}$$

is the *local slope* of Ψ at u. The functional $u \mapsto |\partial\Psi|(u)$ is lower semicontinuous and is equal to the smallest norm that elements of $\partial\Psi(u)$ may assume

$$|\partial\Psi|(u) = \inf \{\|\xi\|_* : \xi \in \partial\Psi(u)\} \tag{4.2}$$

(Proposition 1.4.4 of [2]). If $v \in AC([0, T]; X)$ and $\Psi \circ v$ is absolutely continuous, then

$$\left|\frac{d}{dt}(\Psi \circ v)(t)\right| \le |\partial\Psi|(v(t))|\dot{v}|(t), \tag{4.3}$$

for almost every $t \in [0, T]$. A useful fact is that if the product $(|\partial\Psi| \circ v)|\dot{v}| \in L^1([0, T])$, then $\Psi \circ v$ is absolutely continuous (Remark 1.4.6 of [2]).

We now have all the necessary ingredients to define a curve of maximal slope.

Definition 4.1. A *p-curve of maximal slope for* Φ is a path $v \in AC_{\text{loc}}([0,\infty);X)$ that satisfies

$$\frac{d}{dt}(\Phi \circ v)(t) \leq -\frac{1}{p}|\dot{v}|^p(t) - \frac{1}{q}|\partial\Phi|^q(v(t)) \tag{4.4}$$

for almost every $t > 0$.

Observe that for any p-curve of maximal slope for Φ, $\Phi \circ v$ is nonincreasing. Therefore, $\Phi \circ v$ is differentiable except for at most countably many times $t > 0$, provided $v(0) \in \text{Dom}(\Phi)$. Combining (4.3) with (4.4) gives that

$$-|\partial\Phi|(v(t))|\dot{v}|(t) \leq \frac{d}{dt}(\Phi \circ v)(t) \leq -\frac{1}{p}|\dot{v}|^p(t) - \frac{1}{q}|\partial\Phi|^q(v(t))$$

for almost every $t > 0$. Consequently, equality holds in (4.4) and

$$\frac{d}{dt}\Phi(v(t)) = -|\dot{v}|^p(t) = -|\partial\Phi|^q(v(t)) \tag{4.5}$$

for almost every $t > 0$. In particular,

$$\int_s^t \left(\frac{1}{p}|\dot{v}|^p(\tau) + \frac{1}{q}|\partial\Phi|^q(v(\tau))\right) d\tau + \Phi(v(t)) = \Phi(v(s)), \quad 0 \leq s \leq t < \infty, \tag{4.6}$$

$v \in AC^p_{\text{loc}}([0,\infty);X)$ and $|\partial\Phi| \circ v \in L^q_{\text{loc}}[0,\infty)$.

Remark 4.2. An important point that will be used below is as follows. Suppose v is a p-curve of maximal slope. Since $|\partial\Phi| \circ v \in L^q_{\text{loc}}[0,\infty)$, $|\partial\Phi|(v(t))$ is finite for almost every $t \geq 0$. It must be that $\partial\Phi(v(t)) \neq \emptyset$ at any such time; for if $\partial\Phi(v(t)) = \emptyset$, then $|\partial\Phi|(v(t)) = \infty$ by (4.2).

Let us now argue that differentiable p-curves of maximal slope for Φ satisfy (1.7) and conversely; see also Proposition 1.4.1 of [2]. Along the way, we will use a routine fact that if $v \in AC_{\text{loc}}([0,\infty);X)$ is differentiable almost everywhere, then $\|\dot{v}(t)\| = |\dot{v}|(t)$ for almost every $t > 0$.

Proposition 4.3. *Suppose $v \in AC_{loc}([0,\infty);X)$ is differentiable almost everywhere and $v(0) \in Dom(\Phi)$. Then v is a p-curve of maximal slope for Φ if and only if v satisfies (1.7).*

Proof. Suppose v is an almost everywhere differentiable p-curve of maximal slope for Φ. As indicated in Remark 4.2, there exists $\xi(t) \in \partial\Phi(v(t))$ for almost every $t \geq 0$. By the chain rule and (4.5),

$$\frac{d}{dt}(\Phi \circ v)(t) = \langle \xi(t), \dot{v}(t) \rangle = -\|\dot{v}(t)\|^p$$

for almost every $t > 0$. This implies, $-\xi(t) \in \mathcal{J}_p(\dot{v}(t))$ and so v satisfies (1.7).

Conversely, suppose that v satisfies (1.7) and select $\xi(t) \in \partial\Phi(v(t)) \cap (-\mathcal{J}_p(\dot{v}(t)))$ for almost every $t \geq 0$. By the chain rule and (4.2),

$$\begin{aligned}\frac{d}{dt}(\Phi \circ v)(t) &= \langle \xi(t), \dot{v}(t) \rangle \\ &= -\frac{1}{p}\|\dot{v}(t)\|^p - \frac{1}{q}\|\xi(t)\|_*^q \\ &= -\frac{1}{p}|\dot{v}|^p(t) - \frac{1}{q}\|\xi(t)\|_*^q \\ &\leq -\frac{1}{p}|\dot{v}|^p(t) - \frac{1}{q}|\partial\Phi|^q(v(t))\end{aligned}$$

for almost every $t > 0$. □

We now resume our goal of proving Theorem 1.3, which characterizes the large time behavior of p-curves of maximal slope for Φ. We will not discuss the existence of such curves as this already has been established (see Chapters 1–3 in [2]) and because there has been a plethora of existence results for doubly nonlinear evolutions [3,6,14,15,29]. However, crucial to our proof of Theorem 1.3 is a compactness result (Lemma 4.9) which is inspired by previous existence results. We begin our study by deriving various estimates on p-curves of maximal slope for Φ. We assume for the remainder of this section that λ_p is simple.

Lemma 4.4. *Suppose that v is a p-curve of maximal slope for Φ with $v(0) \in Dom(\Phi)$. Then*

$$p\Phi(v(t)) \leq \frac{1}{\mu_p}|\dot{v}|^p(t) \tag{4.7}$$

for almost every $t \geq 0$.

Proof. Select $\xi(t) \in \partial\Phi(v(t))$ such that $\|\xi(t)\|_* = |\partial\Phi|(v(t))$ for almost every $t \geq 0$; such a $\xi(t)$ exists by Remark 4.2 and the fact that $\partial\Phi(v(t))$ is weak-$*$ closed (Proposition 1.4.4 in [2]). We have by (2.2) and (4.5)

$$\begin{aligned}p\Phi(v(t)) &= \langle \xi(t), v(t) \rangle \\ &\leq |\partial\Phi|(v(t))\|v(t)\| \\ &= |\dot{v}|^{p-1}(t)\|v(t)\| \\ &\leq |\dot{v}|^{p-1}(t)\left(\frac{p}{\lambda_p}\Phi(v(t))\right)^{1/p}.\end{aligned}$$

Consequently, (4.7) holds for almost every $t > 0$. □

Corollary 4.5. *Assume that v is a p-curve of maximal slope for Φ with $v(0) \in Dom(\Phi)$. Then*

$$\frac{d}{dt}\left[e^{p\mu_p t}\Phi(v(t))\right] \leq 0$$

for almost every $t \geq 0$. In particular,

$$\Phi(v(t)) \leq e^{-p\mu_p t}\Phi(v(0)) \tag{4.8}$$

for $t \geq 0$.

Proof. From the previous claim

$$\frac{d}{dt}\Phi(v(t)) = -|\dot{v}|^p(t) \leq -p\mu_p\Phi(v(t)),$$

and so $\frac{d}{dt}\left[e^{p\mu_p t}\Phi(v(t))\right] \leq 0$ for almost every $t \geq 0$. The inequality (4.8) is now immediate. □

It will also be important for us to estimate the derivative of $t \mapsto \|v(t)\|^p$, where v is a locally absolutely continuous path.

Lemma 4.6. *If $v \in AC_{loc}([0,\infty); X)$ and $p \geq 1$, then*

$$[0,\infty) \ni t \mapsto \frac{1}{p}\|v(t)\|^p$$

is locally absolutely continuous and

$$\left|\frac{d}{dt}\frac{1}{p}\|v(t)\|^p\right| \leq \|v(t)\|^{p-1}|\dot{v}|(t) \tag{4.9}$$

for almost every $t \geq 0$.

Proof. Suppose first that $p = 1$. Then by the triangle inequality

$$|\|v(t)\| - \|v(s)\|| \leq \|v(t) - v(s)\| \leq \int_s^t |\dot{v}|(\tau)d\tau,$$

for $0 \leq s \leq t < \infty$. Thus, $t \mapsto \|v(t)\|$ is locally absolutely continuous and

$$\left|\frac{d}{dt}\|v(t)\|\right| \leq |\dot{v}|(t)$$

for almost every $t \geq 0$.

Now assume $p > 1$. The argument just given above implies $t \mapsto \frac{1}{p}\|v(t)\|^p$ is locally absolutely continuous. Also recall that $\mathcal{J}_p$ is the subdifferential of the convex function $X \ni w \mapsto \frac{1}{p}\|w\|^p$ and that $\xi \in \mathcal{J}_p(w)$ if and only if $\langle \xi, w\rangle = \|\xi\|_*^q = \|w\|^p$. As a result, we can use (4.2) to compute the local slope

$$\left|\partial\left(\frac{1}{p}\|\cdot\|^p\right)\right|(w) = \|w\|^{p-1}, \quad w \in X.$$

Consequently, (4.9) follows from the chain rule bound (4.3) applied to $t \mapsto \frac{1}{p}\|v(t)\|^p$. □

In view of Proposition 4.3, it is straightforward to verify that if $g \in X \setminus \{0\}$ minimizes $\Phi(u)/\|u\|^p$, then

$$v(t) = e^{-\mu_p t} g, \quad t \geq 0 \tag{4.10}$$

is a p-curve of maximal slope for Φ. We will also prove that v given by (4.10) is the *unique* p-curve of maximal slope for Φ with initial condition $v(0) = g$. First, we will need to verify the following monotonicity property.

Proposition 4.7. *Assume that v is a p-curve of maximal slope for Φ and $v(t) \neq 0$ for each $t \geq 0$. Then*

$$\frac{d}{dt}\left\{\frac{p\Phi(v(t))}{\|v(t)\|^p}\right\} \leq 0 \tag{4.11}$$

for almost every $t > 0$.

Proof. Recall that both functions $t \mapsto \Phi(v(t))$ and $t \mapsto \|v(t)\|^p$ are locally absolutely continuous. Thus, the quotient rule gives

$$\frac{d}{dt}\left\{\frac{p\Phi(v(t))}{\|v(t)\|^p}\right\} = -p\frac{|\dot{v}|^p(t)}{\|v(t)\|^p} - \frac{p\Phi(v(t))}{(\|v(t)\|^p)^2}\frac{d}{dt}\|v(t)\|^p$$

at almost every $t > 0$. By Lemma 4.6,

$$\frac{d}{dt}\|v(t)\|^p \geq -p\|v(t)\|^{p-1}|\dot{v}|(t).$$

Consequently,

$$\begin{aligned}\frac{d}{dt}\left\{\frac{p\Phi(v(t))}{\|v(t)\|^p}\right\} &\leq -p\frac{|\dot{v}|^p(t)}{\|v(t)\|^p} + \frac{p\Phi(v(t))}{(\|v(t)\|^p)^2}p\|v(t)\|^{p-1}|\dot{v}|(t) \\ &= \frac{-p}{(\|v(t)\|^p)^2}\left\{|\dot{v}|^p(t)\|v(t)\|^p - p\Phi(v(t))\|v(t)\|^{p-1}|\dot{v}|(t)\right\}. \end{aligned} \tag{4.12}$$

As in the proof of Lemma 4.4

$$p\Phi(v(t)) \le |\partial\Phi|(v(t))\|v(t)\| = |\dot{v}|^{p-1}(t)\|v(t)\|$$

and so

$$p\Phi(v(t))\|v(t)\|^{p-1}|\dot{v}|(t) \le |\dot{v}|^{p}(t)\|v(t)\|^{p}.$$

Combining this inequality with (4.12) allows us to conclude this proof. □

Corollary 4.8. *Assume v is a p-curve of maximal slope for Φ with $v(0) \in X \setminus \{0\}$. If*

$$\lambda_p = \frac{p\Phi(v(0))}{\|v(0)\|^p},$$

then v is given by (4.10).

Proof. As $v(0) \neq 0$, $v(t) \neq 0$ for some interval of time; let $[0, T)$ be the largest such interval. For $t \in [0, T)$,

$$\lambda_p \le \frac{p\Phi(v(t))}{\|v(t)\|^p} \le \frac{p\Phi(v(0))}{\|v(0)\|^p} = \lambda_p$$

by (4.11). Thus $v(t)$ minimizes $\Phi(u)/\|u\|^p$, and since λ_p is simple, there is an absolutely continuous function $\alpha : [0, T) \to \mathbb{R}$ such that $v(t) = \alpha(t)v(0)$. Note $\alpha(0) = 1$ and $\alpha(t) > 0$ for $t \in (0, T)$. Also observe that since $t \mapsto \Phi(e^{\mu_p t}v(t)) = \left(e^{\mu_p t}\alpha(t)\right)^p \Phi(v(0))$ is nonincreasing, $t \mapsto e^{\mu_p t}\alpha(t)$ is nonincreasing. Thus,

$$\frac{d}{dt}e^{\mu_p t}\alpha(t) = e^{\mu_p t}(\dot{\alpha}(t) + \mu_p\alpha(t)) \le 0$$

and so $\dot{\alpha}(t) \le -\mu_p\alpha(t) < 0$ for almost every $t \in (0, T)$.

By (4.5), we have for almost every $t \in (0, T)$,

$$\begin{aligned}
0 &= \frac{d}{dt}\Phi(v(t)) + |\dot{v}|^p(t) \\
&= \frac{d}{dt}\Phi(\alpha(t)v(0)) + \|\dot{v}(t)\|^p \\
&= \Phi(v(0))\frac{d}{dt}[\alpha(t)^p] + |\dot{\alpha}(t)|^p\|v(0)\|^p \\
&= p\Phi(v(0))\alpha(t)^{p-1}\dot{\alpha}(t) + |\dot{\alpha}(t)|^p\|v(0)\|^p \\
&= \lambda_p\|v(0)\|^p\alpha(t)^{p-1}\dot{\alpha}(t) + |\dot{\alpha}(t)|^{p-2}\dot{\alpha}(t)\dot{\alpha}(t)\|v(0)\|^p \\
&= \|v(0)\|^p\dot{\alpha}(t)\left(|\mu_p\alpha(t)|^{p-2}\mu_p\alpha(t) + |\dot{\alpha}(t)|^{p-2}\dot{\alpha}(t)\right).
\end{aligned}$$

As a result, $\dot{\alpha}(t) = -\mu_p \alpha(t)$ and thus $\alpha(t) = e^{-\mu_p t}$. Consequently, $v(t) = e^{-\mu_p t} v(0)$ for $t \in [0, T)$. However, $v(T) = e^{-\mu_p T} v(0) \neq 0$ and therefore $T = +\infty$. We conclude that $v(t) = e^{-\mu_p t} v(0)$ for $t \in [0, \infty)$. □

Lemma 4.9. *Assume $(v^k)_{k\in\mathbb{N}}$ is a sequence of p-curves of maximal slope for Φ such that*

$$\sup_{k\in\mathbb{N}} \Phi(v^k(0)) < \infty. \tag{4.13}$$

Then there is a subsequence $(v^{k_j})_{j\in\mathbb{N}}$ and $v \in AC^p_{loc}([0,\infty); X)$ such that

$$\lim_{j\to\infty} \sup_{t\in[0,T]} \|v^{k_j}(t) - v(t)\| = 0 \quad \text{for each } T > 0, \tag{4.14}$$

$$\lim_{j\to\infty} |\dot{v}^{k_j}| = |\dot{v}| \quad \text{in } L^p_{loc}[0,\infty), \tag{4.15}$$

$$\lim_{j\to\infty} |\partial\Phi|(v^{k_j}) = |\partial\Phi|(v) \quad \text{in } L^q_{loc}[0,\infty), \tag{4.16}$$

and

$$\lim_{j\to\infty} \Phi(v^{k_j}(t)) = \Phi(v(t)), \quad \text{for each } t > 0. \tag{4.17}$$

Moreover, v is a p-curve of maximal slope for Φ with $v(0) \in Dom(\Phi)$.

Proof. By (4.6),

$$\int_0^t \left(\frac{1}{p} |\dot{v}^k|^p(\tau) + \frac{1}{q} |\partial\Phi|^q(v^k(\tau)) \right) d\tau + \Phi(v^k(t)) = \Phi(v^k(0))$$

for each $t \geq 0$ and $k \in \mathbb{N}$. Combining this identity with assumption (4.13) gives

$$\sup_{k\in\mathbb{N}} \left\{ \int_0^\infty |\dot{v}^k|^p(t)dt + \int_0^\infty |\partial\Phi|^q(v^k(t))dt + \sup_{t\in[0,\infty)} \Phi(v^k(t)) \right\} < \infty.$$

As a result, the sequence $(v^k)_{k\in\mathbb{N}}$ is equicontinuous and $(v^k(t))_{k\in\mathbb{N}}$ is precompact in X for each $t \geq 0$. It follows from a variant of the Arzelà–Ascoli Theorem (Lemma 1 of [34]) that there is a subsequence $(v^{k_j})_{j\in\mathbb{N}}$ converging to some $v : [0,\infty) \to X$ locally uniformly on $[0,\infty)$. That is, (4.14) holds; and since $\Phi(v(0)) \leq \liminf_{k\to\infty} \Phi(v^k(0))$ by lower semicontinuity, $v(0) \in \mathrm{Dom}(\Phi)$.

As $(|\dot{v}^k|)_{k\in\mathbb{N}}$ is bounded in $L^p[0,\infty)$, it has (up to a subsequence) a weak limit $h \in L^p[0,\infty)$. Note that for $0 \leq s \leq t < \infty$

$$\begin{aligned}\|v(t) - v(s)\| &= \lim_{j\to\infty} \|v^{k_j}(t) - v^{k_j}(s)\| \\ &\le \lim_{j\to\infty} \int_s^t |\dot{v}^{k_j}|(\tau)d\tau \\ &= \int_s^t h(\tau)d\tau.\end{aligned}$$

Thus, $|\dot{v}| \le h$ and $v \in AC^p_{\mathrm{loc}}([0,\infty), X)$. Moreover,

$$\int_E |\dot{v}|^p(\tau)d\tau \le \int_E (h(\tau))^p d\tau \le \liminf_{j\to\infty} \int_E |\dot{v}^{k_j}|^p(\tau)d\tau \tag{4.18}$$

for any Lebesgue measurable E. Similarly, as $w \mapsto |\partial\Phi|(w)$ is lower semicontinuous, Fatou's lemma gives

$$\liminf_{j\to\infty} \int_E |\partial\Phi|^q(v^{k_j}(\tau))d\tau \ge \int_E \liminf_{j\to\infty} |\partial\Phi|^q(v^{k_j}(\tau))d\tau \ge \int_E |\partial\Phi|^q(v(\tau))d\tau. \tag{4.19}$$

For $j \in \mathbb{N}$, select $\xi^j(t) \in \partial\Phi(v^{k_j}(t))$ such that $\|\xi^j(t)\|_* = |\partial\Phi|(v^{k_j}(t))$ for almost every $t \ge 0$. Note that

$$\begin{aligned}\Phi(v(t)) &\ge \Phi(v^{k_j}(t)) + \langle \xi^j(t), v(t) - v^{k_j}(t)\rangle \\ &\ge \Phi(v^{k_j}(t)) - |\partial\Phi|(v^{k_j}(t))\|v(t) - v^{k_j}(t)\|\end{aligned}$$

for almost every time $t > 0$. Since $(|\partial\Phi|(v^{k_j}))_{j\in\mathbb{N}}$ is bounded in $L^q_{\mathrm{loc}}[0,\infty)$ and v^{k_j} converges to v locally uniformly,

$$\int_E \Phi(v(t))dt \ge \limsup_{j\to\infty} \int_E \Phi(v^{k_j}(t))dt$$

for each bounded Lebesgue measurable $E \subset [0,\infty)$. By Fatou's lemma and the lower semicontinuity of Φ

$$\liminf_{j\to\infty} \int_E \Phi(v^{k_j}(t))dt \ge \int_E \liminf_{j\to\infty} \Phi(v^{k_j}(t))dt \ge \int_E \Phi(v(t))dt.$$

As a result, $\lim_{j\to\infty} \int_E \Phi(v^{k_j}(t))dt = \int_E \Phi(v(t))dt$. Since E was only assumed to be bounded and measurable,

$$\liminf_{j\to\infty} \Phi(v^{k_j}(t)) = \Phi(v(t)) \tag{4.20}$$

for almost every $t \geq 0$. As each function $t \mapsto \Phi(v^{k_j}(t))$ is nonincreasing and bounded, we may apply Helly's selection principle (Lemma 3.3.3 in [2]) to conclude the limit $f(t) := \lim_{j\to\infty} \Phi(v^{k_j}(t))$ exists for every $t \geq 0$ since it occurs for a subsequence. From (4.20) it follows that $f(t) = \Phi(v(t))$ for almost every $t \geq 0$ and by the lower semicontinuity of Φ, $f(t) \geq \Phi(v(t))$ for all $t \geq 0$.

For any given $t_0 > 0$, we may select a sequence of positive numbers $t_k \nearrow t_0$ such that $f(t_k) = \Phi(v(t_k))$ for each $k \in \mathbb{N}$. Indeed, the set of times t for which $f(t) = \Phi(v(t))$ for $t \in (t_0 - \delta, t_0)$ has full measure for each $\delta \in (0, t_0)$ and is thus nonempty. As f is nonincreasing, $f(t_0) \leq f(t_k)$. In addition (4.18) and (4.19) imply that $(|\partial\Psi| \circ v)|\dot{v}| \in L^1([0, T])$ so that $\Phi \circ v$ is absolutely continuous. Hence,

$$f(t_0) \leq \lim_{k\to\infty} f(t_k) = \lim_{k\to\infty} \Phi(v(t_k)) = \Phi(v(t_0)) \leq f(t_0).$$

Thus $f \equiv \Phi \circ v$ which implies $\lim_{j\to\infty} \Phi(v^{k_j}(t)) = \Phi(v(t))$ for all $t > 0$, as asserted in (4.17).

Let $t_0 < t_1$ and use (4.18) and (4.19) to send $j \to \infty$ in the equation

$$\int_{t_0}^{t_1} \left(\frac{1}{p}|\dot{v}^{k_j}|^p(\tau) + \frac{1}{q}|\partial\Phi|^q(v^{k_j}(\tau)) \right) d\tau + \Phi(v^{k_j}(t_1)) = \Phi(v^{k_j}(t_0))$$

to arrive at

$$\int_{t_0}^{t_1} \left(\frac{1}{p}|\dot{v}|^p(\tau) + \frac{1}{q}|\partial\Phi|^q(v(\tau)) \right) d\tau + \Phi(v(t_1)) \leq \Phi(v(t_0)). \tag{4.21}$$

Since $\Phi \circ v$ is nonincreasing and thus differentiable for almost every $t \geq 0$, this implies

$$\frac{d}{dt}\Phi(v(t)) \leq -\frac{1}{p}|\dot{v}|^p(t) - \frac{1}{q}|\partial\Phi|^q(v(t)), \quad \text{a.e. } t > 0.$$

Thus v is a curve of maximal slope for Φ. As noted above, this means that equality actually holds in (4.21) which implies (4.15) and (4.16). □

A final technical assertion is needed for our proof of Theorem 1.3. The claim below is a continuous time analog of the Lemma 3.6.

Lemma 4.10. *Assume $w \in X \setminus \{0\}$ is a minimizer of $\Phi(u)/\|u\|^p$ and $C > 0$. There is a $\delta = \delta(w, C) > 0$ with the following property. If v is curve of maximal slope for Φ and*

(i) $\Phi(v(0)) \geq \Phi(w)$,
(ii) $\|v(0)\| \leq C$,
(iii) $\alpha_w(v(0)) \geq \frac{1}{2}$,
(iv) $\frac{p\Phi(v(0))}{\|v(0)\|^p} \leq \lambda_p + \delta$,

then

$$\alpha_w\left(e^{\mu_p t}v(t)\right) \geq \frac{1}{2}, \quad t \in [0,1].$$

Proof. Assume the assertion does not hold. Then there is $w_0 \in X \setminus \{0\}$ that minimizes $\Phi(u)/\|u\|^p$ and constant C_0 for which there is a p-curve of maximal slope for Φ labeled v^j satisfying

(i) $\Phi(v^j(0)) \geq \Phi(w_0)$,
(ii) $\|v^j(0)\| \leq C_0$,
(iii) $\alpha_{w_0}(v^j(0)) \geq \frac{1}{2}$,
(iv) $\frac{p\Phi(v(0)^j)}{\|v^j(0)\|^p} \leq \lambda_p + \frac{1}{j}$,

while

$$\alpha_{w_0}\left(e^{\mu_p t_j}v^j(t_j)\right) < \frac{1}{2} \tag{4.22}$$

for some $t_j \in [0,1]$.

In view of the bounds (ii) and (iv), $\sup_{j\in\mathbb{N}} \Phi(v^j(0)) < \infty$. By Lemma 4.9, $(v^j)_{j\in\mathbb{N}}$ converges (up to a subsequence) uniformly in $t \in [0,1]$ to another p-curve of maximal slope v, with $v(0) \in \mathrm{Dom}(\Phi)$, and $\lim_{j\to\infty} \Phi(v^j(t)) = \Phi(v(t))$ for each $t \in [0,1]$. Sending $j \to \infty$ in (i) gives $\Phi(v(0)) \geq \Phi(w_0) > 0$, so $v(0) \neq 0$; and by sending $j \to \infty$ in (iv) gives that $v(0)$ minimizes $\Phi(u)/\|u\|^p$, so that $v(0) = \gamma w_0$ by simplicity. By Corollary 4.8, $v(t) = e^{-\mu_p t}v(0)$ for $t \in [0,1]$.

We also have

$$\lambda_p\|v(0)\|^p = p\Phi(v(0)) = \lim_{j\to\infty} p\Phi(v^j(0)) \geq \Phi(w_0) = \lambda_p\|w_0\|^p.$$

Thus, $\|v(0)\| \geq \|w_0\|$ or equivalently $|\gamma| \geq 1$. As $\gamma^+ = \alpha_{w_0}(v(0)) \geq \frac{1}{2} > 0$, it must actually be that $\gamma \geq 1$. However, we may use part (iii) of Proposition 2.6 to send $j \to \infty$ in (4.22) to get

$$\gamma = \alpha_{w_0}(v(0)) = \lim_{j\to\infty} \alpha_{w_0}\left(e^{\mu_p t_j}v^j(t_j)\right) \leq \frac{1}{2}.$$

As a result, the hypotheses of this lemma could not hold. Therefore, we have verified the claim. □

We are now finally in position to prove Theorem 1.3.

Proof of Theorem 1.3. We define $S := \lim_{t\to\infty} \Phi(e^{\mu_p t}v(t))$. Recall that this limit exists by Corollary 4.5. If $S = 0$, we conclude. So let us now assume $S > 0$, and let $(s_k)_{k\in\mathbb{N}}$ be a sequence of positive numbers that increase to $+\infty$. Set

$$w^k(t) := e^{\mu_p s_k}v(t + s_k), \quad t \geq 0.$$

By the homogeneity of (4.4), each w^k is p-curve of maximal slope for Φ with $w^k(0) = e^{\mu_p s_k} v(s_k)$. By (4.8), $\Phi(w^k(0)) \leq \Phi(v(0))$ for each $k \in \mathbb{N}$.

Lemma 4.9 implies there is a subsequence $(w^{k_j})_{j\in\mathbb{N}}$ converging locally uniformly to another p-curve of maximal slope for Φ labeled w with

$$\Phi(e^{\mu_p t} w(t)) = \lim_{j\to\infty} \Phi(e^{\mu_p t} w^{k_j}(t)) = \lim_{j\to\infty} \Phi\left(e^{\mu_p (s_{k_j}+t)} v(t + s_{k_j})\right) = S$$

for every $t \geq 0$. We compute

$$\begin{aligned} 0 &= \frac{d}{dt}\Phi(e^{\mu_p t} w(t)) \\ &= \frac{d}{dt} e^{p\mu_p t}\Phi(w(t)) \\ &= e^{p\mu_p t}\left(p\mu_p \Phi(w(t)) - |\dot{w}|^p(t)\right) \end{aligned}$$

for almost every $t \geq 0$.

In view of the proof of inequality (4.7), $w(t)$ must be a minimizer of $\Phi(u)/\|u\|^p$ for almost every $t \geq 0$. Furthermore, since $t \mapsto \|w(t)\|$ and $t \mapsto \Phi(w(t))$ are locally absolutely continuous, $w(t)$ is a minimizer of $\Phi(u)/\|u\|^p$ for every $t \geq 0$. By Corollary 4.8, $w(t) = e^{-\mu_p t} w(0)$. Moreover, by the monotonicity formula (4.11) and (4.17)

$$\lim_{t\to\infty} \frac{p\Phi(v(t))}{\|v(t)\|^p} = \lim_{j\to\infty} \frac{p\Phi(v(s_{k_j}))}{\|v(s_{k_j})\|^p} = \lim_{j\to\infty} \frac{p\Phi(w^{k_j}(0))}{\|w^{k_j}(0)\|^p} = \frac{p\Phi(w(0))}{\|w(0)\|^p} = \lambda_p.$$

In order to conclude that the limit $w(0)$ is independent of the sequence $(s_{k_j})_{j\in\mathbb{N}}$, we will make use of Lemma 4.10. Define

$$C := \left(\frac{p}{\lambda_p}\Phi(v(0))\right)^{1/p},$$

which is finite by hypothesis. Also recall that inequality (2.4) and Corollary 4.5 imply

$$\|e^{\mu_p t} v(t)\| \leq \left(\frac{p}{\lambda_p}\Phi(e^{\mu_p t} v(t))\right)^{1/p} \leq \left(\frac{p}{\lambda_p}\Phi(v(0))\right)^{1/p} = C$$

for $t \geq 0$. Now select $j_0 \in \mathbb{N}$ so large that

$$\frac{p\Phi(w^{k_j}(0))}{\|w^{k_j}(0)\|^p} \leq \lambda_p + \delta \quad \text{and} \quad \alpha_{w(0)}(w^{k_j}(0)) \geq \frac{1}{2}$$

for all $j \geq j_0$. Here $\delta = \delta(w(0), C) > 0$ is the number in the statement of Lemma 4.10.

We claim that in fact

$$\alpha_{w(0)}(e^{\mu_p t} w^{k_j}(t)) \geq \frac{1}{2}, \tag{4.23}$$

for all $t \geq 0$ and $j \geq j_0$. Observe that $\Phi(w^{k_j}(0)) \geq S = \Phi(w(0))$ and $\|w^{k_j}(0)\| \leq C$ for all $j \in \mathbb{N}$. So for any $j \geq j_0$, w^{k_j} satisfies the hypotheses Lemma 4.10 which implies that (4.23) holds for $t \in [0,1]$.

Now define $u(t) := e^{\mu_p} w^{k_j}(t+1)$. Note that u is a p-curve of maximal slope for Φ. Also notice

(*i*) $\Phi(u(0)) = \Phi(e^{\mu_p(1+s_{k_j})} v(s_{k_j}+1)) \geq S = \Phi(w(0))$,
(*ii*) $\|u(0)\| = \|e^{\mu_p} w^{k_j}(1)\| = \|e^{\mu_p(1+s_{k_j})} v(s_{k_j}+1)\| \leq C$,
(*iii*) $\alpha_{w(0)}(u(0)) = \alpha_{w(0)}(e^{\mu_p} w^{k_j}(1)) \geq \frac{1}{2}$,
(*iv*) and

$$\frac{p\Phi(u(0))}{\|u(0)\|^p} = \frac{p\Phi(w^{k_j}(1))}{\|w^{k_j}(1)\|^p} \leq \frac{p\Phi(w^{k_j}(0))}{\|w^{k_j}(0)\|^p} \leq \lambda_p + \delta.$$

By Lemma 4.10, $\alpha_{w(0)}(e^{\mu_p t} u(t)) = \alpha_{w(0)}(e^{\mu_p(t+1)} w^{k_j}(t+1)) \geq \frac{1}{2}$ for $t \in [0,1]$. As a result, (4.23) holds for $t \in [1,2]$ and thus for all $t \in [0,2]$ and for $j \geq j_0$. Continuing this argument we may establish (4.23) by induction on the intervals $[\ell, \ell+1]$ ($\ell \in \mathbb{N}$).

Now suppose there is another sequence of positive numbers $(t_\ell)_{\ell\in\mathbb{N}}$ increasing to infinity such that $(e^{\mu_p t_\ell} v(t_\ell))_{\ell\in\mathbb{N}}$ converges to a minimizer z of $\Phi(u)/\|u\|^p$. From our reasoning above, it must be that

$$\Phi(z) = \Phi(w(0)) = S. \tag{4.24}$$

As λ_p is simple, $z = \gamma w(0)$ for some $\gamma \in \mathbb{R} \setminus \{0\}$. Of course, if $\gamma > 0$ then $\gamma = 1$ by (4.24). Now suppose $\gamma < 0$ and select a subsequence $(t_{\ell_j})_{j\in\mathbb{N}}$ such that

$$t_{\ell_j} > s_{k_j} \quad j \in \mathbb{N}.$$

Next, substitute $t = t_{\ell_j} - s_{k_j} > 0$ in (4.23) to get

$$\alpha_{w(0)}\left(e^{\mu_p t_{\ell_j}} v(t_{\ell_j})\right) = \alpha_{w(0)}\left(e^{\mu_p(t_{\ell_j}-s_{k_j})} w^{k_j}(t_{\ell_j}-s_{k_j})\right) \geq \frac{1}{2}.$$

Letting $j \to \infty$ above gives

$$\alpha_{w(0)}(\gamma w(0)) \geq \frac{1}{2},$$

which cannot occur as $\alpha_{w(0)}(\gamma w(0)) = 0$ for $\gamma < 0$. Thus $z = w(0)$. Since the subsequential limit $w(0)$ is independent of the sequence $(s_{k_j})_{j\in\mathbb{N}}$, we conclude $\lim_{t\to\infty} e^{\mu_p t} v(t) = w(0)$. □

Remark 4.11. Without assuming that λ_p is simple, our argument above shows that if $S := \lim_{t\to\infty} \Phi(e^{\mu_p t}v(t)) > 0$ then

$$\lambda_p = \lim_{t\to\infty} \frac{p\Phi(v(t))}{\|v(t)\|^p}.$$

Moreover, we did not need to assume that λ_p is simple in order to conclude that there is a subsequence of positive numbers $(t_k)_{k\in\mathbb{N}}$ increasing to ∞ so that $(e^{\mu_p t_k}v(t_k))$ converges; and if $S > 0$, $\lim_k e^{\mu_p t_k}v(t_k)$ is a minimizer of $\Phi(u)/\|u\|^p$.

Example 4.12. We continue our discussion of Example 2.8 in the context of (4.4). Here X is a separable Hilbert space which is necessarily reflexive. In particular, all absolutely continuous paths with values X are differentiable almost everywhere, so all curves of maximal slope satisfy the doubly nonlinear evolution (1.7). In this setting, (1.7) takes the form

$$\dot{v}(t) + Av(t) = 0 \quad \text{(a.e. } t > 0\text{)}. \tag{4.25}$$

If $v(0) = \sum_{k\in\mathbb{N}} a_k z_k \in Y$, then

$$v(t) = \sum_{k\in\mathbb{N}} a_k e^{-\sigma_k t} z_k$$

is the corresponding solution of (4.25). Assuming $\sigma_1 < \sigma_2$, we have

$$e^{\sigma_1 t}v(t) = a_1 z_1 + \sum_{k\geq 2} a_k e^{-(\sigma_k-\sigma_1)t} z_k \to a_1 z_1$$

as $t \to \infty$. Moreover, if $a_1 = (v(0), z_1) \neq 0$

$$\frac{(Av(t), v(t))}{\|v(t)\|^2} = \frac{a_1^2\sigma_1 + \sum_{z\geq 2} \sigma_k a_k^2 e^{-2(\sigma_k-\sigma_1)t}}{a_1^2 + \sum_{z\geq 2} a_k^2 e^{-2(\sigma_k-\sigma_1)t}} \to \sigma_1$$

as $t \to \infty$. Recall that $\mu_2 = \lambda_2 = \sigma_1$ for this example, so these assertions are consistent with Theorem 1.3.

Example 4.13. Assume $X = L^p(\Omega)$ and Φ is given by (2.6). Recall that $L^p(\Omega)$ is reflexive so any p-curve of maximal slope for Φ satisfies the doubly nonlinear evolution (1.7). For this example, (1.7) is the PDE and boundary condition

$$\begin{cases} |v_t|^{p-2}v_t = \Delta_p v & \Omega \times (0, \infty) \\ v = 0 & \partial\Omega \times [0, \infty) \end{cases}.$$

Theorem 1.3 asserts that if $v(\cdot,0)\in W^{1,p}_0(\Omega)$,

$$w=\lim_{t\to\infty}e^{\mu_p t}v(\cdot,t)$$

exists in $W^{1,p}_0(\Omega)$. When $w\neq 0$, w satisfies (2.7) and

$$\lim_{t\to\infty}\frac{\displaystyle\int_\Omega |Dv(x,t)|^p dx}{\displaystyle\int_\Omega |v(x,t)|^p dx}=\lambda_p.$$

See our previous work [24] for a detailed discussion.

Example 4.14. Assume $X=L^p(\Omega)$ and now Φ is given by (2.8). As in the previous example, the evolution (1.7) corresponds to the PDE and boundary condition

$$\begin{cases}|v_t|^{p-2}v_t=-(-\Delta_p)^s v & \Omega\times(0,\infty)\\ v=0 & (\mathbb{R}^n\setminus\Omega)\times[0,\infty)\end{cases}.$$

Theorem 1.3 asserts that if $v(\cdot,0)\in W^{s,p}_0(\Omega)$,

$$w=\lim_{t\to\infty}e^{\mu_p t}v(\cdot,t) \tag{4.26}$$

exists in $W^{s,p}_0(\Omega)$. If $w\neq 0$, then w satisfies (2.9) and

$$\lim_{t\to\infty}\frac{\displaystyle\iint_{\mathbb{R}^n\times\mathbb{R}^n}\frac{|v(x,t)-v(y,t)|^p}{|x-y|^{n+ps}}dxdy}{\displaystyle\int_\Omega |v(x,t)|^p dx}=\lambda_p.$$

See our paper [25] for a recent account. In particular, we showed that the limit (4.26) also holds uniformly.

Example 4.15. Let us return to Example 2.11. Here the doubly nonlinear evolution is the PDE boundary value equation

$$\begin{cases}|v_t|^{p-2}v_t=\Delta_p v & \Omega\times(0,\infty)\\ |Dv|^{p-2}Dv\cdot\nu+\beta|v|^{p-2}v=0 & \partial\Omega\times[0,\infty)\end{cases}$$

and $v(\cdot,0)\in W^{1,p}(\Omega)$. By Theorem 1.2, the limit $w=\lim_{t\to\infty}e^{\mu_p t}v(\cdot,t)$ exists in $W^{1,p}(\Omega)$. If w doesn't vanish identically, then w satisfies (2.11) and

$$\lambda_p = \lim_{t\to\infty} \frac{\int\limits_\Omega |Dv(x,t)|^p dx + \beta \int\limits_{\partial\Omega} |Tv(x,t)|^p d\sigma(x)}{\int\limits_\Omega |v(x,t)|^p dx}.$$

Example 4.16. In Example 2.12, $L^p(\Omega)/\mathcal{C}$ is reflexive since $L^p(\Omega)$ is reflexive. The corresponding doubly nonlinear evolution is

$$\begin{cases} |v_t|^{p-2} v_t = \Delta_p v & \Omega \times (0,\infty) \\ |Dv|^{p-2} Dv \cdot \nu = 0 & \partial\Omega \times [0,\infty) \end{cases}$$

with $\int_\Omega |v(x,t)|^{p-2} v(x,t)dx = 0$ for $t \geq 0$. Here we also assume $v(\cdot,0) \in W^{1,p}(\Omega)$. Theorem 1.2 implies the limit $w = \lim_{t\to\infty} e^{\mu_p t} v(\cdot,t)$ exists in $W^{1,p}(\Omega)$. If this limit does not identically vanish, then

$$\lambda_p = \lim_{t\to\infty} \frac{\int\limits_\Omega |Dv(x,t)|^p dx}{\int\limits_\Omega |v(x,t)|^p dx}.$$

Example 4.17. Suppose that $p > n$, $X = C(\overline{\Omega})$ and Φ is again given by (2.6). It is known that in this case X does not satisfy the Radon–Nikodym property (Example 2.1.6 and Theorem 2.3.6 of [10]). Here p-curves of maximal slope satisfy (4.4)

$$\frac{d}{dt}(\Phi \circ v)(t) \leq -\frac{1}{p}|\dot{v}|^p(t) - \frac{1}{q}|\partial\Phi|^q(v(t))$$

for almost every $t > 0$ with

$$|\dot{v}|(t) := \lim_{h\to 0} \frac{\|v(\cdot,t+h) - v(\cdot,t)\|_\infty}{|h|}.$$

Now suppose that Ω is convex, so that the corresponding λ_p is simple. According to Theorem 1.3, if $v(\cdot,0) \in W_0^{1,p}(\Omega)$, then $w = \lim_{t\to\infty} e^{\mu_p t} v(\cdot,t)$ exists in $W_0^{1,p}(\Omega)$. Moreover, if $w \not\equiv 0$, then w satisfies the equation (2.15) and

$$\lambda_p = \lim_{t\to\infty} \frac{\int\limits_\Omega |Dv(x,t)|^p dx}{\|v(\cdot,t)\|_\infty^p}.$$

Example 4.18. The doubly nonlinear evolution associated with approximating the norm of the trace operator T, as presented in Example 2.14, is

$$\begin{cases} -\Delta_p v + |v|^{p-2}v = 0 & \Omega \times (0,\infty) \\ |Dv|^{p-2}Dv \cdot \nu + |v_t|^{p-2}v_t = 0 & \partial\Omega \times [0,\infty) \end{cases} \tag{4.27}$$

with $v(\cdot, 0) \in W^{1,p}(\Omega)$. The weak formulation for this equation is

$$\int_{\Omega} (|Dv(x,t)|^{p-2}Dv(x,t) \cdot D\phi(x) + |v(x,t)|^{p-2}v(x,t)\phi(x))dx$$
$$= \int_{\partial\Omega} |\partial_t(Tv)(\cdot,t)|^{p-2}\partial_t(Tv)(\cdot,t)(T\phi)d\sigma$$

for $\phi \in W^{1,p}(\Omega)$ and almost every time $t \in [0,\infty)$. Here a solution v satisfies

$$v \in L^\infty([0,\infty), W^{1,p}(\Omega)), \quad \text{and} \quad Tv \in AC^p_{\text{loc}}([0,\infty), L^p(\partial\Omega;\sigma)).$$

Recall that the corresponding optimal value $\lambda_p = \|T\|^{-p}$ is simple. By employing virtually the same arguments used to prove Theorem 1.3, we can show $w = \lim_{t\to\infty} e^{\mu_p} v(\cdot,t)$ exists in $W^{1,p}(\Omega)$. If $Tw \neq 0$, then w satisfies (2.17) and

$$\|T\| = \lambda_p^{-1/p} = \lim_{t\to\infty} \frac{\|Tv(t)\|_{L^p(\partial\Omega)}}{\|v(t)\|_{W^{1,p}(\Omega)}}.$$

Therefore, the doubly nonlinear evolution (4.27) can be used to approximate the operator norm of the Sobolev trace mapping.

We have proved the main results of this paper under the assumption that λ_p is simple. However, we suspect this is not necessary for the convergence we describe to occur. We conjecture that both Theorem 1.2 and Theorem 1.3 hold without this assumption.

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