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LIPSCHITZ REGULARITY FOR A HOMOGENEOUS DOUBLY NONLINEAR PDE*

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Abstract. We study the doubly nonlinear PDE $|\partial_t u|^{p-2}\,\partial_t u - \operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0$. This equation arises in the study of extremals of Poincaré inequalities in Sobolev spaces. We prove spatial Lipschitz continuity and Hölder continuity in time of order $(p-1)/p$ for viscosity solutions. As an application of our estimates, we obtain pointwise control of the large time behavior of solutions.

Key words. doubly nonlinear PDE, viscosity solutions, regularity

AMS subject classifications. 35K65, 35B45, 35D40

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1. Introduction.

We study the local regularity of viscosity solutions of the doubly nonlinear parabolic equation

$$|\partial_t u|^{p-2}\,\partial_t u - \Delta_p u = 0 \tag{1.1}$$

for $p \in [2, \infty)$. Here Δ_p is the p-Laplace operator

$$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u),$$

the first variation of the functional

$$u \mapsto \int_\Omega |\nabla u|^p dx.$$

The first occurrence of (1.1) that we have found is in a footnote in [KL96]. Our interest in (1.1) relies on the connection to the eigenvalue problem for the p-Laplacian. See our previous work [HL16a], [HL17] and also Theorem 3 in section 7. This eigenvalue problem amounts to studying extremals of the Rayleigh quotient

$$\lambda_p = \inf_{u \in W_0^{1,p}(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega |\nabla u|^p dx}{\displaystyle\int_\Omega |u|^p dx}. \tag{1.2}$$

Here $\Omega \subset \mathbb{R}^n$ is a bounded and open set. Extremals are often called *ground states*. This extremal problem is naturally equivalent to finding the optimal constant in the Poincaré inequality for $W_0^{1,p}(\Omega)$.

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1.1. Main results. The first of our results is spatial Lipschitz continuity and Hölder continuity in time of order $(p-1)/p$. These are proved using the Ishii–Lions method, introduced in [IL90]. To the best of our knowledge, this is the first pointwise regularity result for this equation. In order to state our first theorem we introduce the notation

$$Q_r(x_0,t_0) = B_r(x_0) \times (t_0 - r^{\frac{p}{p-1}}, t_0].$$

THEOREM 1. *Let $p \geq 2$, Ω be a bounded and open set in $\mathbb{R}^n$, and I a bounded interval. Suppose u is a viscosity solution of* (1.1) *in $\Omega \times I$. Then*

$$|u(x,t) - u(y,s)| \leq \frac{C(n,p)}{R} \|u\|_{L^\infty(Q_{2R}(x_0,t_0))} \left(|x-y| + |t-s|^{\frac{p-1}{p}}\right)$$

for any (x,t), $(y,s) \in Q_{R/2}(x_0,t_0)$ and for every R, x_0, and t_0 such that $Q_{2R}(x_0,t_0) \Subset \Omega \times I$.

Our second result concerns the large time behavior of solutions. This was investigated in our previous work [HL16a]. In particular, there exists a ground state w such that

$$e^{\lambda_p^{\frac{1}{p-1}} t} u(x,t) \to w(x)$$

in $W_0^{1,p}(\Omega)$, when u solves

$$\begin{cases} |\partial_t u|^{p-2}\,\partial_t u = \Delta_p u, & \Omega \times (0,\infty), \\ u = 0, & \partial\Omega \times [0,\infty), \\ u = g, & \Omega \times \{0\}. \end{cases} \tag{1.3}$$

See Theorem 3. As a consequence of this and Theorem 1, we obtain that this convergence is uniform.

THEOREM 2. *Let $p \geq 2$, Ω be a bounded and regular*[1] *domain, and assume that $g \in W_0^{1,p}(\Omega) \cap C(\overline{\Omega})$ satisfies $|g| \leq \phi$, where ϕ is a ground state. If u is a viscosity solution of* (1.3)*, then*

$$e^{\lambda_p^{\frac{1}{p-1}} t} u(x,t) \to w(x),$$

uniformly in $\overline{\Omega}$. Here w is either a ground state or identically zero.

We do not expect the estimates in Theorem 1 to be sharp. In our opinion, solutions are likely to be at least continuously differentiable in space, even though we are unable to verify this at the moment. Concerning time regularity, it may be a very delicate task to obtain any higher Hölder exponent. See the next section for a comparison with related equations.

1.2. Known results. Doubly nonlinear equations such as (1.1) have mostly been studied from a functional analytic point of view. See for instance [MRS13] and [Ste08]. However, the pointwise properties and, in particular, the regularity theory have not been developed. Needless to say, the nonlinearity in the time derivative presents a genuine challenge. A related result can be found in [HL16b], where Hölder estimates for some Hölder exponent are proved for the doubly nonlinear nonlocal equation

$$|\partial_t u|^{p-2}\,\partial_t u + (-\Delta_p)^s u = 0.$$

[1] Regular in the sense that any ground state is continuous up to the boundary. This is true for instance if $\partial\Omega$ is Lipschitz.

The large time behavior of solutions has a natural connection to the Poincaré inequality in the fractional Sobolev space $W^{s,p}$, the nonlocal counterpart of (1.2).

The related p-parabolic equation

$$\partial_t u - \Delta_p u = 0$$

has been given vast attention the past 30 years. In contrast to (1.1), this equation is not homogeneous. Due to the linearity in the time derivative, the notion of weak solutions turns out to be more useful than for (1.1). We refer to [DiB93] for an overview of the regularity theory. The best local regularity known is spatial $C^{1,\alpha}$-regularity for some $\alpha > 0$ and $C^{1/2}$-regularity in time. Neither of these exponents are known to be sharp. Due to the explicit solution

$$u = t \cdot n - \frac{p-1}{p}|x|^{\frac{p}{p-1}},$$

where n is the dimension, it is clear that solutions cannot be better than $C^{1,1/(p-1)}$ in space.

Recently, the Ishii–Lions method has been used for equations involving the p-Laplacian. In [IJS18], the authors used it to study the regularity of solutions of

$$\partial_t u = |\nabla u|^\kappa \Delta_p u, \quad \kappa \in (1-p, \infty).$$

In the recent papers [APR17] and [AP18] it was also used to analyze solutions of the equations

$$|\nabla u|^{2-p}\Delta_p u = f, \quad \partial_t u - |\nabla u|^{2-p}\Delta_p u = f.$$

1.3. The idea of the proof. For many elliptic or parabolic equations including (1.1), it is possible to prove a comparison principle. When working with viscosity solutions, this is usually accomplished by doubling the variables. This amounts to ruling out that

$$\sup_{x,y}\,(u(x) - v(y) - \phi(|x-y|)) > 0$$

when u is a subsolution, v is a supersolution, $u \le v$ on the boundary, and ϕ is appropriately chosen. For uniformly elliptic equations the choice $\phi(r) = r^2$ is suitable to prove a comparison principle.

It turns out that a similar approach can also give continuity estimates. This was first done in [IL90]. A spatial continuity estimate of order $\phi(r)$ for a solution u of (1.1) is saying that

$$\sup_{x,y,t}\,(u(x,t) - u(y,t) - \phi(|x-y|)) \le 0.$$

In order to prove this, we assume towards a contradiction that

$$\sup_{x,y,t}\,(u(x,t) - u(y,t) - \phi(|x-y|)) > 0.$$

In this paper, we work with the choices $\phi(r) \approx r|\ln r|$ and $\phi(r) \approx r$. This gives a log-Lipschitz and a Lipschitz estimate in the spatial variables. In contrast to the case $\phi(r) = r^2$, ϕ is here chosen so that it is strictly concave. It turns out that the spatial regularity can then be used to construct a suitable supersolution which yields the desired time regularity.

1.4. Plan of the paper. The plan of the paper is as follows. In section 2, we introduce some notation and the notion of viscosity solutions. This is followed by section 3, where we prove log-Lipschitz continuity in space. In section 4, we improve this to Lipschitz continuity. This result is then used in section 5, where we prove the corresponding Hölder regularity in time. We combine these results in section 6, where we prove our main regularity theorem. Finally, in section 7, we study the large time behavior.

2. Notation and prerequisites. Throughout the paper, we will use the notation

$$Q_r(x_0,t_0) = B_r(x_0) \times (t_0 - r^{\frac{p}{p-1}}, t_0]$$

and $Q_r = B_r(0) \times (-r^{\frac{p}{p-1}}, 0]$. These are cylinders reflecting the natural scaling of solutions to (1.1). We will also use the matrix norm

$$\|X\| = \sup_{|\xi|\leq 1} |X\xi|.$$

In addition, we will, for any subset of $Q \subset \mathbb{R}^{n+1}$, use the notation

$$\operatorname{osc}_Q u = \sup_Q u - \inf_Q u.$$

For completeness we include the definition of viscosity solutions.

Definition 1. *Let $\Omega \in \mathbb{R}^n$ be an open set and $I \in \mathbb{R}$ be a bounded interval. A function which is upper semicontinuous in $\Omega \times I$ is a* subsolution *of*

$$|\partial_t v|^{p-2}\partial_t v - \Delta_p v \leq 0 \text{ in } \Omega \times I$$

if the following holds: whenever $(x_0,t_0) \in \Omega \times I$ and $\phi \in C^{2,1}_{x,t}(B_r(x_0) \times (t_0 - r, t_0])$ for some $r > 0$ are such that

$$\phi(x_0,t_0) = v(x_0,t_0), \quad \phi(x,t) \geq v(x,t) \text{ for } (x,t) \in B_r(x_0) \times (t_0 - r, t_0],$$

then

$$|\partial_t \phi(x_0,t_0)|^{p-2}\partial_t \phi(x_0,t_0) - \Delta_p \phi\,(x_0,t_0) \leq 0.$$

A supersolution *is defined similarly and a* solution *is a function which is both a sub- and a super-solution.*

Remark 1. The notion of viscosity solutions may also be formulated in terms of so called jets: v is a viscosity subsolution in $\Omega \times I$ if $(\alpha, a, X) \in \overline{\mathcal{P}}^{2,+}_Q v\,(x_0,t_0)$ for $(x_0,t_0) \in Q$ for some cylinder $Q \subset \Omega \times I$ implies

$$|\alpha|^{p-2}\alpha - |a|^{p-2}\operatorname{tr}(L(a)X) \leq 0, \quad L(a) = I + (p-2)\frac{a \otimes a}{|a|^2}.$$

See [CIL92] and [DFO14] for further reading. Here and throughout the paper we will use the notation used in [DFO14].

In [HL16a], the following comparison principle is mentioned. The proof of this result is identical to, for instance, the proof of Theorem 4.7 of [JLM01].

Proposition 1. *Assume $v \in USC(\overline{\Omega} \times [0,T))$ and $w \in LSC(\overline{\Omega} \times [0,T))$. Suppose the inequality*

$$|\partial_t v|^{p-2}\partial_t v - \Delta_p v \leq 0 \leq |\partial_t w|^{p-2}\partial_t w - \Delta_p w, \quad \Omega \times (0,T),$$

holds in the sense of viscosity solutions and $v(x,t) \le w(x,t)$ *for* $(x,t) \in \partial\Omega \times [0,T)$ *and for* $(x,t) \in \Omega \times \{0\}$. *Then*

$$v \le w$$

in $\Omega \times (0,T)$.

3. Log-Lipschitz regularity. We start with a technical calculus result.

LEMMA 1. *Let*

$$\phi(r) = \begin{cases} -r \ln r, & 0 < r < e^{-1}, \\ e^{-1}, & r \ge e^{-1}. \end{cases}$$

Then $\phi(r) < 1/8$ *implies*

$$r < e^{-2}, \quad \phi(r) = -r\ln r, \quad \phi'(r) \ge 1, \quad |\phi''(r)| \le \phi'(r)/r.$$

Proof. First we note that ϕ is nondecreasing. Moreover,

$$\phi(e^{-2}) = 2e^{-2} > 1/8 > \phi(r).$$

Therefore, $r < e^{-2}$, which also implies that $\phi(r) = -r \ln r$. In addition,

$$\phi'(r) = -1 - \ln r$$

is a nonincreasing function and $\phi'(e^{-2}) = 1$. Therefore, $\phi'(r) \ge 1$.

Finally,

$$|\phi''(r)| = r^{-1} \le \phi'(r) r^{-1} = -r^{-1} - r^{-1} \ln r,$$

since $r < e^{-2}$. ∎

PROPOSITION 2. *Let* ϕ *be as in Lemma* 1. *Suppose* u *is a viscosity solution of* (1.1) *in* Q_2 *such that* $\operatorname{osc}_{Q_2} u \le 1$. *Then*

$$u(t,x) - u(t,y) \le A\phi(|x-y|) + \frac{B}{2}\left(|x|^2 + |y|^2 + t^2\right)$$

for $(x,t) \in Q_1$. *Here* $A = A(n,p)$ *and* B *are universal constants.*

Proof. Let

$$\Phi(x,y,t) = u(x,t) - u(y,t) - A\phi(|x-y|) - \frac{B}{2}\left(|x|^2 + |y|^2 + t^2\right).$$

In order to show the desired inequality, we assume towards a contradiction that Φ assumes a positive maximum at some $t \in [-1,0]$ and $x, y \in \overline{B}_1$. Since $\Phi(x,y,t) > 0$ we have

$$A\phi(|x-y|) + \frac{B}{2}\left(|x|^2 + |y|^2 + t^2\right) \le |u(t,x) - u(t,y)| \le 1. \tag{3.1}$$

Therefore, we may choose $B = 4$ ($B > 2$ is enough), so that $x, y \in B_1$ and $t \in (-1, 0]$. Let us introduce the notation

$$\hat{\delta} = \frac{x-y}{|x-y|}, \quad \delta = |x-y|.$$

By choosing $A > 8$ we see that (3.1) combined with Lemma 1 implies

$$\delta < e^{-2}, \quad \phi(\delta) = -\delta \ln\delta, \quad \phi'(\delta) \ge 1, \quad |\phi''(\delta)| \le \phi'(\delta)/\delta. \tag{3.2}$$

It also follows that

$$u(x,t) - u(y,t) > 0,$$

implying that $\delta \neq 0$.

Step 1: *Applying the theorem of sums.* From the parabolic theorem of sums (Theorem 8.3 in [CIL92] and Theorem 9 in [DFO14]), for any $\tau > 0$ there are $X, Y \in S(n)$, α_1, and α_2 such that[2]

$$(\alpha_1, A\phi'(\delta)\hat{\delta}, X) \in \overline{\mathcal{P}}^{2,+}_{Q_1}\left(u - \frac{B}{2}|\cdot|^2\right)(x,t),$$

$$(-\alpha_2, A\phi'(\delta)\hat{\delta}, Y) \in \overline{\mathcal{P}}^{2,-}_{Q_1}\left(u + \frac{B}{2}|\cdot|^2\right)(y,t),$$

$$\alpha_1 + \alpha_2 \geq Bt,$$

$$-\left[\tau + ||Z||\right]\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \leq \begin{bmatrix} X & 0 \\ 0 & -Y \end{bmatrix}, \tag{3.3}$$

and

$$\begin{bmatrix} X & 0 \\ 0 & -Y \end{bmatrix} \leq \begin{bmatrix} Z & -Z \\ -Z & Z \end{bmatrix} + \frac{1}{\tau}\begin{bmatrix} Z^2 & -Z^2 \\ -Z^2 & Z^2 \end{bmatrix}. \tag{3.4}$$

Here

$$Z = A\phi''(\delta)\hat{\delta}\otimes\hat{\delta} + A\frac{\phi'(\delta)}{\delta}\left(I - \hat{\delta}\otimes\hat{\delta}\right)$$

and thus

$$Z^2 = A^2\left(\phi''(\delta)\right)^2\hat{\delta}\otimes\hat{\delta} + A^2\left(\frac{\phi'(\delta)}{\delta}\right)^2\left(I - \hat{\delta}\otimes\hat{\delta}\right).$$

This implies in particular

$$(\alpha_1, a, X + BI) \in \overline{\mathcal{P}}^{2,+}_{Q_1} u\,(x,t), \quad (-\alpha_2, b, Y - BI) \in \overline{\mathcal{P}}^{2,-}_{Q_1} u\,(y,t), \tag{3.5}$$

where

$$a = A\phi'(\delta)\hat{\delta} + Bx, \quad b = A\phi'(\delta)\hat{\delta} - By.$$

We now choose

$$\tau = 4A\frac{\phi'(\delta)}{\delta}.$$

Step 2: *Basic estimates.* Since

$$Z = A\left(\phi''(\delta) - \frac{\phi'(\delta)}{\delta}\right)\hat{\delta}\otimes\hat{\delta} + A\frac{\phi'(\delta)}{\delta}I,$$

we see that

$$\|Z\| \leq A\frac{\phi'(\delta)}{\delta}, \quad \|Z^2\| \leq A^2\left(|\phi''(\delta)| + \frac{\phi'(\delta)}{\delta}\right)^2 \leq 4A^2\left(\frac{\phi'(\delta)}{\delta}\right)^2, \tag{3.6}$$

where we also used (3.2).

[2] $S(n)$ stands for symmetric $n \times n$ matrices.

It will be convenient to introduce the notation

$$q = A\phi'(\delta)\hat{\delta}.$$

Note that since $A > 8$ and $B = 4$

$$|a| \leq |q| + 4 = 2|q| + 4 - |q| \leq 2|q|,$$

where we used that by (3.2) we have

$$|q| = A|\phi'(\delta)| > 8.$$

Similarly,

$$|a| \geq |q| - 4 = |q|/2 - 4 + A|\phi'(\delta)|/2 \geq |q|/2.$$

The same arguments can be carried out for b. Hence,

$$|q|/2 \leq |a| \leq 2|q|, \quad |q|/2 \leq |b| \leq 2|q|. \tag{3.7}$$

By testing (3.3) and (3.4) with vectors of the form (ξ, ξ) and $(\xi, 0)$, where $\|\xi\| = 1$ we obtain that

$$\|X\|, \|Y\| \leq \max(\|Z\| + \tau, \|Z\| + \frac{1}{\tau}\|Z^2\|) \leq 5A\frac{\phi'(\delta)}{\delta}, \quad X - Y \leq 0, \tag{3.8}$$

where we used (3.6) and that $|\phi''(\delta)| \leq \phi'(\delta)/\delta$.

Step 3: *Using the equation.* From the equation together with (3.5) we obtain the two following inequalities:

$$\begin{aligned} |\alpha_1|^{p-2}\alpha_1 &\leq |a|^{p-2}\operatorname{tr}(L(a)(X + BI)), \\ -|\alpha_2|^{p-2}\alpha_2 &\geq |b|^{p-2}\operatorname{tr}(L(b)(Y - BI)), \end{aligned} \tag{3.9}$$

where

$$L(v) = I + (p-2)\frac{v \otimes v}{|v|^2}.$$

Subtracting these inequalities, we obtain

$$|\alpha_1|^{p-2}\alpha_1 + |\alpha_2|^{p-2}\alpha_2 \leq |a|^{p-2}\operatorname{tr}(L(a)(X + BI)) - |b|^{p-2}\operatorname{tr}(L(b)(Y - BI)). \tag{3.10}$$

The aim is now to estimate the left-hand side from below and the right-hand side from above, and obtain a contradiction when choosing A large enough. The idea is that there is at least one eigenvalue of $X - Y$ which is very negative when A is large enough. This will violate an inequality obtained from the equation.

Step 4: *Lower bound for the left-hand side.* First of all, by (3.7), (3.8), and (3.9)

$$|\alpha_i|^{p-2}\alpha_i \leq C|q|^{p-2}\left(A\frac{\phi'(\delta)}{\delta} + 1\right) \leq C|q|^{p-2}A\frac{\phi'(\delta)}{\delta}, \quad i = 1, 2, \quad C = C(n),$$

where we used that $|q| = A\phi'(\delta) \geq 8$ and $\phi'(\delta)/\delta \geq e^2$ by (3.2), so that the constant can be absorbed. From the above together with relation

$$\alpha_1 + \alpha_2 \geq Bt,$$

it follows that[3]

$$|\alpha_1|^{p-2}\alpha_1 \geq -C - C\left(\alpha_2^+\right)^{p-1} \geq -C|q|^{p-2}\left(A\frac{\phi'(\delta)}{\delta}+1\right) \geq -C|q|^{p-2}A\frac{\phi'(\delta)}{\delta},$$

where $C = C(p,n)$ and α_2^+ is the positive part of α_2. The same estimate holds also for α_2. Thus,

$$|\alpha_i|^{p-2} \leq C|q|^{\frac{(p-2)^2}{p-1}}A^{\frac{p-2}{p-1}}\left(\frac{\phi'(\delta)}{\delta}\right)^{\frac{p-2}{p-1}}, \quad i=1,2, \quad C = C(p,n).$$

This implies, via the inequality

$$\Big||\beta_1+\beta_2|^{p-2}(\beta_1+\beta_2) - |\beta_1|^{p-2}\beta_1\Big| \leq (p-1)|\beta_2|(|\beta_1|+|\beta_2|)^{p-2},$$

that

$$\begin{aligned} |\alpha_1|^{p-2}\alpha_1 + |\alpha_2|^{p-2}\alpha_2 &\geq |\alpha_1|^{p-2}\alpha_1 - |\alpha_1 - Bt|^{p-2}(\alpha_1 - Bt) \\ &\geq -C(|\alpha_1|+1)^{p-2} \\ &\geq -C_0|q|^{\frac{(p-2)^2}{p-1}}A^{\frac{p-2}{p-1}}\left(\frac{\phi'(\delta)}{\delta}\right)^{\frac{p-2}{p-1}}, \end{aligned} \tag{3.11}$$

where $C_0 = C_0(p,n)$ and where we again absorbed the constant due to the bounds from below on $|q|$ and $\phi'(\delta)/\delta$. From (3.10) and (3.11), we can thus conclude

$$|a|^{p-2}\operatorname{tr}(L(a)(X+BI)) - |b|^{p-2}\operatorname{tr}(L(b)(Y-BI)) \geq -C_0|q|^{\frac{(p-2)^2}{p-1}}A^{\frac{p-2}{p-1}}\left(\frac{\phi'(\delta)}{\delta}\right)^{\frac{p-2}{p-1}}, \tag{3.12}$$

where $C_0 = C_0(p,n)$.

Step 5: *Upper bound for the right-hand side.* We now turn our attention to the right-hand side. We split these terms into three parts:

$$\begin{aligned} &|a|^{p-2}\operatorname{tr}(L(a)(X+BI)) - |b|^{p-2}\operatorname{tr}(L(b)(Y-BI)) \\ &= |b|^{p-2}\operatorname{tr}(L(b)(X-Y)) + \operatorname{tr}((|a|^{p-2}L(a) - |b|^{p-2}L(b))X) \\ &\quad + \left(|a|^{p-2}\operatorname{tr}(L(a)BI) + |b|^{p-2}\operatorname{tr}(L(b)BI)\right) \\ &= T_1 + T_2 + T_3. \end{aligned}$$

Step 5(a): T_1. Testing inequality (3.4) with $(\hat{\delta}, -\hat{\delta})$ we see that by (3.2) and the choice of τ

$$\hat{\delta}\cdot(X-Y)\hat{\delta} \leq 4A\phi''(\delta) + \frac{4}{\tau}A^2\left(\phi''(\delta)\right)^2 \leq 2A\phi''(\delta),$$

so that at least one of the eigenvalues of $X - Y$ is smaller than $2A\phi''(\delta)$. From (3.8), we know that the rest are nonpositive. Hence,

$$T_1 \leq 2|b|^{p-2}A\phi''(\delta) \leq C_1A|q|^{p-2}\phi''(\delta) = -C_1A|q|^{p-2}\delta^{-1}, \quad C_1 = C_1(p), \tag{3.13}$$

where we used (3.7) and that the smallest eigenvalue of L is 1.

[3] Recall the inequality $(\beta_1+\beta_2)^{p-1} \leq 2^{p-2}(\beta_1^{p-1}+\beta_2^{p-1})$, $\quad \beta_1, \beta_2 \geq 0, \quad p \geq 2$.

Step 5(b): T_2*.* For T_2 we have

$$
\begin{aligned}
T_2 &\leq n\|X\|\||a|^{p-2}L(a) - |b|^{p-2}L(b)\| \leq C\|X\||q|^{p-3}|B(x+y)| \\
&\leq C_2 A\frac{\phi'(\delta)}{\delta}|q|^{p-3} \\
&= C_2|q|^{p-2}\delta^{-1},
\end{aligned}
\tag{3.14}
$$

where $C_2 = C_2(p,n)$, and where we used the mean value theorem (for the mapping $v \mapsto |v|^{p-2}L(v)$), the definition of q, (3.7), (3.8), that $x, y \in B_1$, and that $B = 4$. We also note that since

$$|a+s(b-a)| = |A\phi'(\delta)\hat{\delta} + Bx - sB(x+y)| \geq A\phi'(\delta) - B|x| - sB|x+y| \geq 8 - 3B \geq 2$$

for $s \in [0,1]$, the line between a and b does not pass through the origin.

Step 5(c): T_3*. For* T_3 *we have*

$$T_3 \leq CB(p-1)n|q|^{p-2} \leq C_3|q|^{p-2}, \quad C_3 = C_3(p,n), \tag{3.15}$$

where we have used (3.7)*.*

Step 6: *The contradiction.* Using (3.13)–(3.15) together with (3.12), we obtain

$$
\begin{aligned}
-C_0|q|^{\frac{(p-2)^2}{p-1}}A^{\frac{p-2}{p-1}}\left(\frac{\phi'(\delta)}{\delta}\right)^{\frac{p-2}{p-1}} &\leq -C_1A|q|^{p-2}\delta^{-1} + C_2|q|^{p-2}\delta^{-1} + C_3|q|^{p-2} \\
&= |q|^{p-2}\left(\frac{C_2 - C_1A}{\delta} + C_3\right)
\end{aligned}
$$

or, equivalently,

$$C_0|q|^{\frac{(p-2)^2}{p-1}}A^{\frac{p-2}{p-1}}\left(\frac{\phi'(\delta)}{\delta}\right)^{\frac{p-2}{p-1}} + |q|^{p-2}\left(\frac{C_2 - C_1A}{\delta} + C_3\right) \geq 0.$$

This will be a contradiction if A is chosen so that

$$|q|^{p-2}\left(\frac{C_2 - C_1A/2}{\delta} + C_3\right) < 0, \quad C_0|q|^{\frac{(p-2)^2}{p-1}}A^{\frac{p-2}{p-1}}\left(\frac{\phi'(\delta)}{\delta}\right)^{\frac{p-2}{p-1}} - \frac{C_1A|q|^{p-2}}{2\delta} < 0.$$

The first inequality is satisfied if we choose $A > 2(C_3 + C_2)/C_1$, which is a constant depending only on n and p. Using that $|q| = A|\phi'(\delta)|$, the second inequality can be simplified to

$$A > \frac{2C_0}{C_1}\delta^{\frac{1}{p-1}},$$

so that it is sufficient to choose $A > 2C_0/C_1$ which is a constant depending only on n and p. Hence, we arrive at a contradiction if

$$A > \max\left(2(C_3 + C_2)/C_1, 2C_0/C_1\right). \qquad \square$$

Corollary 1. *Suppose* u *is a viscosity solution of* (1.1) *in* Q_2 *such that* $\operatorname{osc}_{Q_2} u \leq 1$*. Then*

$$|u(x,t) - u(y,t)| \leq C|x-y||\ln|x-y||$$

for $t \in [-1, 0]$ *and* $x, y \in B_{\frac{1}{4}}$*. Here* $C = C(n,p)$*.*

Proof. First of all, by choosing $t = 0$ and $x = 0$ or $y = 0$ in Proposition 2, we obtain

$$|u(x,0) - u(0,0)| \leq C|x-y||\ln|x-y||, \quad x \in B_1, \quad C = C(n,p). \tag{3.16}$$

We now show how to obtain the desired regularity in the whole cylinder $B_{1/4} \times (-1,0)$. Let $(z_0, t_0) \in B_1 \times (-1,0)$ and define

$$v(x,t) := u\left(\frac{x}{2} + z_0, \frac{t}{2^{\frac{p}{p-1}}} + t_0\right).$$

Then v is a solution of (1.1) in Q_2. By construction, we also have

$$\operatorname{osc}_{Q_2} v \leq \operatorname{osc}_{Q_2} u \leq 1.$$

We may therefore apply (3.16) to v and obtain

$$\sup_{x \in B_r} |v(x,0) - v(0,0)| \leq C\, r|\ln r|, \quad 0 < r < 1.$$

In terms of u this implies

$$\sup_{x \in B_r(z_0)} |u(x,t_0) - u(z_0,t_0)| \leq Cr|\ln r|, \qquad 0 < r < \frac{1}{2}, \tag{3.17}$$

upon renaming the constant. We note that this holds for any $z_0 \in B_1, t_0 \in (-1,0)$. Now take any pair $x, y \in B_{1/4}$ and set $|x - y| = r$. We observe that $r < 1/2$ and we set $z = (x+y)/2$. Then we apply (3.17) with $z_0 = z$ and obtain

$$\begin{aligned} |u(x,t_0) - u(y,t_0)| &\leq |u(x,t_0) - u(z,t_0)| + |u(y,t_0) - u(z,t_0)| \\ &\leq 2 \sup_{w \in B_r(z)} |u(w,t_0) - u(z,t_0)| \\ &\leq 2\, C\, r \ln r = 2\, C|x-y||\ln|x-y||. \end{aligned}$$

which is the desired result. ∎

4. Lipschitz continuity. We first prove some properties of the function φ used in this section.

LEMMA 2. *Let*

$$\varphi(r) = \begin{cases} r - r^\gamma, & 0 < r \leq r_0 = \left(\frac{1}{\gamma}\right)^{\frac{1}{\gamma-1}}, \\ r_0 - r_0^\gamma, & \textit{otherwise,} \end{cases} \qquad \gamma \in \left(1, \min\left\{\frac{3}{2}, \frac{p}{p-1}\right\}\right).$$

Then

$$\varphi(r) < \left(\frac{1}{2\gamma}\right)^{\frac{1}{\gamma-1}} \left(1 - \frac{1}{2\gamma}\right)$$

implies

$$r < \left(\frac{1}{2\gamma}\right)^{\frac{1}{\gamma-1}} < 1, \quad \varphi(r) = r - r^\gamma, \quad \varphi'(r) \geq 1/2, \quad |\varphi''(r)| \leq \varphi'(r)/r.$$

Proof. First we note that φ is nondecreasing. Moreover,

$$\varphi\left(\left(\frac{1}{2\gamma}\right)^{\frac{1}{\gamma-1}}\right) = \left(\frac{1}{2\gamma}\right)^{\frac{1}{\gamma-1}}\left(1-\frac{1}{2\gamma}\right).$$

Therefore, $r < \left(\frac{1}{2\gamma}\right)^{\frac{1}{\gamma-1}}$ and by definition, $\varphi(r) = r - r^\gamma$. It is also straightforward to verify that

$$\varphi'(r) = 1 - \gamma r^{\gamma-1} \geq \frac{1}{2}$$

whenever $r \leq \left(\frac{1}{2\gamma}\right)^{\frac{1}{\gamma-1}}$. Finally,

$$|\varphi''(r)| = \gamma(\gamma-1)r^{\gamma-2} \leq \varphi'(r)r^{-1} = r^{-1} - \gamma r^{\gamma-2}$$

since $r^{\gamma-1} < 1/(2\gamma)$ together with $\gamma < 2$ implies $r^{\gamma-1} \leq \gamma^{-2}$. ∎

PROPOSITION 3. *Let φ be as in Lemma 2. Suppose u is a viscosity solution of* (1.1) *in Q_2 such that $\operatorname{osc}_{Q_2} u \leq 1$. Then*

$$u(x,t) - u(y,t) \leq A\varphi(|x-y|) + \frac{B}{2}\left(|x|^2 + |y|^2 + t^2\right)$$

for $(x,t) \in Q_1$. Here $A = A(n,p)$ and B are universal constants.

Proof. The proof is almost identical with the proof of Proposition 2. The main differences are the different modulus of continuity and that we use the log-Lipschitz regularity in our estimates. We spell out the details. Let

$$\Phi(x,y,t) = u(x,t) - u(y,t) - A\varphi(|x-y|) - \frac{B}{2}\left(|x|^2 + |y|^2 + t^2\right).$$

We will show that $\Phi(x,y,t) \leq 0$ for $t \in [-1,0]$ and $x,y \in B_1$. In order to do that we assume towards a contradiction that Φ has a positive maximum for $t \in [-1,0]$ and $x,y \in \overline{B}_1$ at (x,y,t). Since $\Phi(x,y,t) > 0$ we have

$$A\varphi(|x-y|) + \frac{B}{2}\left(|x|^2 + |y|^2 + t^2\right) \leq |u(x,t) - u(y,t)| \leq 1. \tag{4.1}$$

Therefore, by choosing $B = 33$ we can assure that $x,y \in B_{1/4}$ and $t \in (-1,0]$. Again, we let

$$\hat{\delta} = \frac{x-y}{|x-y|}, \quad \delta = |x-y|.$$

By choosing

$$A > \frac{1}{\left(\frac{1}{2\gamma}\right)^{\frac{1}{\gamma-1}}\left(1-\frac{1}{2\gamma}\right)},$$

estimate (4.1) and Lemma 2 imply

$$\delta < \left(\frac{1}{2\gamma}\right)^{\frac{1}{\gamma-1}} < 1, \quad \varphi(\delta) = \delta - \delta^\gamma, \quad \varphi'(\delta) \geq 1/2, \quad |\varphi''(\delta)| \leq \varphi'(\delta)/\delta. \tag{4.2}$$

By Corollary 1, u is log-Lipschitz in $B_{1/4} \times (-1,0)$. In particular, u is Hölder continuous with exponent $2\gamma - 2$. We may therefore use (4.1) to extract

$$|x|^2 + |y|^2 + t^2 \leq \frac{2}{B}|u(x,t) - u(y,t)| \leq \frac{2C}{B}|x-y|^{2\gamma-2}, \quad C = C(p,n),$$

or

$$|x|, |y|, |t| \leq \sqrt{\frac{2C}{B}} |x-y|^{\gamma-1}. \tag{4.3}$$

Step 1: *Theorem of sums.* From the parabolic theorem of sums (Theorem 8.3 in [CIL92] and Theorem 9 in [DFO14]) for any $\tau > 0$, there are $X, Y \in S(n)$, α_1, and α_2 such that

$$(\alpha_1, A\varphi'(\delta)\hat{\delta}, X) \in \overline{\mathcal{P}}^{2,+}_{Q_1} \left(u - \frac{B}{2}|\cdot|^2\right)(x,t),$$

$$(-\alpha_2, A\varphi'(\delta)\hat{\delta}, Y) \in \overline{\mathcal{P}}^{2,-}_{Q_1} \left(u + \frac{B}{2}|\cdot|^2\right)(y,t),$$

$$\alpha_1 + \alpha_2 \geq Bt,$$

$$-\left[\tau + ||Z||\right] \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \leq \begin{bmatrix} X & 0 \\ 0 & -Y \end{bmatrix} \tag{4.4}$$

and

$$\begin{bmatrix} X & 0 \\ 0 & -Y \end{bmatrix} \leq \begin{bmatrix} Z & -Z \\ -Z & Z \end{bmatrix} + \frac{1}{\tau} \begin{bmatrix} Z^2 & -Z^2 \\ -Z^2 & Z^2 \end{bmatrix}. \tag{4.5}$$

Here

$$Z = A\varphi''(\delta)\hat{\delta} \otimes \hat{\delta} + A\frac{\varphi'(\delta)}{\delta}\left(I - \hat{\delta} \otimes \hat{\delta}\right),$$

$$Z^2 = A^2 \left(\varphi''(\delta)\right)^2 \hat{\delta} \otimes \hat{\delta} + A^2 \left(\frac{\varphi'(\delta)}{\delta}\right)^2 \left(I - \hat{\delta} \otimes \hat{\delta}\right)$$

and we choose

$$\tau = 4A\frac{\varphi'(\delta)}{\delta}.$$

This implies in particular

$$(\alpha_1, a, X + BI) \in \overline{\mathcal{P}}^{2,+}_{Q_1} u\,(x,t), \quad (-\alpha_2, b, Y - BI) \in \overline{\mathcal{P}}^{2,-}_{Q_1} u\,(y,t), \tag{4.6}$$

where

$$a = A\varphi'(\delta)\hat{\delta} + Bx, \quad b = A\varphi'(\delta)\hat{\delta} - By.$$

Step 2: *Basic estimates.* Since

$$Z = A\left(\varphi''(\delta) - \frac{\varphi'(\delta)}{\delta}\right)\hat{\delta} \otimes \hat{\delta} + A\frac{\varphi'(\delta)}{\delta} I,$$

the last inequality in (4.2) implies

$$\|Z\| \leq A\frac{\varphi'(\delta)}{\delta}, \quad \|Z^2\| \leq 4A^2 \left(\frac{\varphi'(\delta)}{\delta}\right)^2. \tag{4.7}$$

We now introduce the notation

$$q = A\varphi'(\delta)\hat{\delta}.$$

By choosing $A \geq 200$ and using that $\varphi'(\delta) \geq 1/2$ (from (4.2)), we may as in the proof of Proposition 2, conclude

$$|q|/2 \leq a \leq 2|q|, \quad |q|/2 \leq b \leq 2|q|. \tag{4.8}$$

By testing (4.4) and (4.5) with vectors of the form (ξ, ξ) and $(\xi, 0)$, where $\|\xi\| = 1$, we obtain that

$$\|X\|, \|Y\| \leq \max\left(\|Z\| + \tau, \|Z\| + \frac{1}{\tau}\|Z^2\|\right) \leq 5A\frac{\varphi'(\delta)}{\delta}, \quad X - Y \leq 0, \tag{4.9}$$

where we used (4.7) and again that $|\varphi''(\delta)| \leq \varphi'(\delta)/\delta$.

Step 3: *Using the equation.* From the equation and (4.6) we obtain the two following inequalities:

$$|\alpha_1|^{p-2}\alpha_1 \leq |a|^{p-2}\operatorname{tr}(L(a)(X + BI)),$$
$$-|\alpha_2|^{p-2}\alpha_2 \geq |b|^{p-2}\operatorname{tr}(L(b)(Y - BI)),$$

where

$$L(v) = I + (p-2)\frac{v \otimes v}{|v|^2}.$$

Subtracting these inequalities, we obtain

$$|\alpha_1|^{p-2}\alpha_1 + |\alpha_2|^{p-2}\alpha_2 \leq |a|^{p-2}\operatorname{tr}(L(a)(X + BI)) - |b|^{p-2}\operatorname{tr}(L(b)(Y - BI)). \tag{4.10}$$

We will now estimate the left-hand side from below and the right-hand side from above, and obtain a contradiction by choosing A large enough.

Step 4: *Lower bound for the left-hand side.* The estimate of the left-hand side is identical to the estimate done in Step 4 in the proof of Proposition 2. This together with (4.10) yields

$$|a|^{p-2}\operatorname{tr}(L(a)(X + BI)) - |b|^{p-2}\operatorname{tr}(L(b)(Y - BI)) \geq -C_0|q|^{\frac{(p-2)^2}{p-1}}A^{\frac{p-2}{p-1}}\left(\frac{\varphi'(\delta)}{\delta}\right)^{\frac{p-2}{p-1}}, \tag{4.11}$$

where $C_0 = C_0(n,p)$.

Step 5: *Upper bound for the right-hand side.* We split these terms into three parts:

$$\begin{aligned}
&|a|^{p-2}\operatorname{tr}(L(a)(X + BI)) - |b|^{p-2}\operatorname{tr}(L(b)(Y - BI)) \\
&= |b|^{p-2}\operatorname{tr}(L(b)(X - Y)) \\
&\quad + \operatorname{tr}((|a|^{p-2}L(a) - |b|^{p-2}L(b))X) \\
&\quad + |a|^{p-2}\operatorname{tr}(L(a)BI) + |b|^{p-2}\operatorname{tr}(L(b)BI) \\
&= T_1 + T_2 + T_3.
\end{aligned}$$

Step 5(a): T_1. Testing inequality (4.5) with $(\hat{\delta}, -\hat{\delta})$, we see that by (4.2) and the choice of τ

$$\hat{\delta} \cdot (X - Y)\hat{\delta} \leq 4A\varphi''(\delta) + \frac{4}{\tau}A^2\left(\varphi''(\delta)\right)^2 \leq 2A\varphi''(\delta),$$

so that at least one of the eigenvalues of $X - Y$ is smaller than $2A\varphi''(\delta)$. From (4.9), we know that the rest are nonpositive. Hence,

$$T_1 \leq |b|^{p-2}A\varphi''(\delta) \leq CA|q|^{p-2}\varphi''(\delta) = -C_1A|q|^{p-2}\delta^{\gamma-2}, \quad C_1 = C_1(p), \tag{4.12}$$

where we used (4.8) and that the smallest eigenvalue of L is 1.

Step 5(b): T_2. For T_2 we have

$$\begin{aligned} T_2 \leq n\|X\|\||a|^{p-2}L(a) - |b|^{p-2}L(b)\| &\leq C\|X\||q|^{p-3}|B(x+y)| \\ &\leq C' A \frac{\varphi'(\delta)}{\delta}|q|^{p-3}\delta^{\gamma-1} \\ &= C_2|q|^{p-2}\delta^{\gamma-2}, \end{aligned} \tag{4.13}$$

$C_2 = C_2(p,n)$. Here we used the mean value theorem, the definition of q, (4.8), (4.9), and (4.3). We also note that since

$$|a+s(b-a)| = |A\varphi'(\delta)\hat{\delta}+Bx-sB(x+y)| \geq A\varphi'(\delta)-B|x|-sB|x+y| \geq 100-3B \geq 1$$

for $s \in [0,1]$, the line between a and b does not pass through the origin.

Step 5(c): T_3. For T_3 we have

$$T_3 \leq BC(p-1)n|q|^{p-2} \leq C_3|q|^{p-2}, \quad C_3 = C_3(p,n), \tag{4.14}$$

where we used (4.8).

Step 6: *The contradiction.* Using (4.12)–(4.14) together with (4.11), we obtain

$$-C_0|q|^{\frac{(p-2)^2}{p-1}}A^{\frac{p-2}{p-1}}\left(\frac{\varphi'(\delta)}{\delta}\right)^{\frac{p-2}{p-1}} \leq -C_1A|q|^{p-2}\delta^{\gamma-2} + C_2|q|^{p-2}\delta^{\gamma-2} + C_3|q|^{p-2}$$

or

$$0 \leq |q|^{p-2}\left((C_2-C_1A)\delta^{\gamma-2}+C_3\right) + C_0|q|^{\frac{(p-2)^2}{p-1}}A^{\frac{p-2}{p-1}}\left(\frac{\varphi'(\delta)}{\delta}\right)^{\frac{p-2}{p-1}}.$$

This is a contradiction if we choose A such that

$$0 > (C_2-C_1A/2)\delta^{\gamma-2}+C_3, \quad 0 > -C_1A/2|q|^{p-2}\delta^{\gamma-2}+C_0|q|^{\frac{(p-2)^2}{p-1}}A^{\frac{p-2}{p-1}}\left(\frac{\varphi'(\delta)}{\delta}\right)^{\frac{p-2}{p-1}}.$$

The first inequality holds if we choose $A > 2(C_3+C_2)/C_1$ and the second inequality is equivalent to

$$A > \frac{2C_0}{C_1}\delta^{\frac{p}{p-1}-\gamma},$$

once we recall $|q| = A\varphi'(\delta)$. Since $\delta < (1/(2\gamma))^{\frac{1}{\gamma-1}} < 1$ and $\gamma < p/(p-1)$, it is therefore sufficient to choose

$$A > \frac{2C_0}{C_1}$$

in order to have the second inequality. All in all, we arrive at a contradiction by choosing

$$A > \max\left(\frac{2C_0}{C_1}, 2(C_3+C_2)/C_1\right),$$

which is a constant depending only on n and p. ∎

That the result above implies the local Lipschitz regularity can be proved exactly as in Corollary 1.

COROLLARY 2. *Suppose u is a viscosity solution of* (1.1) *in Q_2 such that* $\operatorname{osc}_{Q_2} u \leq 1$. *Then*

$$|u(x,t)-u(y,t)| \leq C|x-y|$$

for $t \in [-1,0]$ and $x,y \in B_{\frac{1}{4}}$. Here $C = C(n,p)$.

Remark 2. By a simple covering argument we may also obtain an estimate

$$|u(x,t)-u(y,t)|\le C|x-y|, \quad C=C(n,p),$$

for $(x,t)\in Q_1$, for a solution u in Q_2 such that $\operatorname{osc}_{Q_2} u\le 1$.

Indeed, we can cover $\overline{Q_1}$ with finitely many cylinders of the form $B_{1/8}(x_i)\times(t_i-1/(2^{p/(p-1)}),t_i)$ where $x_i\in B_1$ and $t_i\in(-1,0)$. Corollary 2 applied to the functions

$$v_i(x,t)=u(x/2+x_i,1/(2^{p/(p-1)})t+t_i),$$

which are all solutions in Q_2, implies

$$|v_i(x,t)-v_i(y,t)|\le C|x-y|, \quad x,y\in B_{1/4}, t\in(-1,0).$$

Going back to u this implies

$$|u(x,t)-u(y,t)|\le C|x-y|, \quad x,y\in B_{1/8}(x_i), t\in(t_i-1/(2^{p/(p-1)}),t_i),$$

for any i, which is the desired estimate.

5. Hölder regularity in time. In this section we prove Hölder estimates in the t-variable. It amounts to constructing a suitable supersolution. See Lemma 3.1 in [IJS18] or Lemma 9.1 in [BBL02] for similar results.

PROPOSITION 4. *Suppose u is a viscosity solution of* (1.1) *in Q_2 such that $\|u\|_{L^\infty(Q_2)}\le 1$. Then*

$$|u(0,t)-u(0,s)|\le C|t-s|^{\frac{p-1}{p}}$$

for $t,s\in[-1,0]$.

Proof. Fix $t_0\in(-1,0)$. We claim that the following estimate holds:

$$u(x,t)-u(0,t_0)\le\phi(t,x):=\eta+A(t-t_0)+B|x|^{\frac{p}{p-1}} \tag{5.1}$$

for $t\in[t_0,0]$, $x\in B_1$, whenever A, B, and η satisfy

$$A=\left(\frac{p}{p-1}\right)n^{\frac{1}{p-1}}B, \quad B^{p-1}=\max\left(\frac{1}{p}\left(\frac{p-1}{p}\right)^{p-1}\frac{\|\nabla u\|^p_{L^\infty(Q_1)}}{\eta},2^{p-1}\right). \tag{5.2}$$

This is accomplished by making ϕ a supersolution and applying the comparison principle.

We first remark that for $x\in\partial B_1$, (5.1) reads

$$u(x,t)-u(0,t_0)\le\eta+A(t-t_0)+B,$$

which clearly holds if $B\ge 2$. In addition, when $t=t_0$ (5.1) reduces to

$$u(x,t_0)-u(0,t_0)\le\eta+B|x|^{\frac{p}{p-1}}. \tag{5.3}$$

By Corollary 2 and Remark 2, we know that u is Lipschitz in space in Q_1. Thus,

$$|u(x,t_0)-u(0,t_0)|\le\|\nabla u\|_{L^\infty(Q_1)}|x|, \quad \|\nabla u\|_{L^\infty(Q_1)}<C(n,p).$$

Hence, (5.3) is valid if

$$\|\nabla u\|_{L^\infty(Q_1)}|x|\le\eta+B|x|^{\frac{p}{p-1}},$$

which holds if

$$B^{p-1} \geq \frac{1}{p}\left(\frac{p-1}{p}\right)^{p-1} \frac{\|\nabla u\|^p_{L^\infty(Q_1)}}{\eta},$$

which holds due to (5.2). We have thus settled that (5.1) holds on the parabolic boundary of $B_1 \times [t_0, 0]$. We now see that

$$|\partial_t \phi|^{p-2}\partial_t \phi - \Delta_p \phi = A^{p-1} - \left(\frac{p}{p-1}\right)^{p-1} B^{p-1} n \geq 0$$

since $A = \left(\frac{p}{p-1}\right) n^{\frac{1}{p-1}} B$. Therefore, ϕ is a supersolution and (5.1) holds in $B_1 \times [t_0, 0]$ by the comparison principle (Proposition 1), given that (5.2) is satisfied.

In order to prove the assertion, we choose $\eta = \|\nabla u\|_{L^\infty(Q_1)} |t-t_0|^{\frac{p-1}{p}}$. We note that (5.2) implies with this choice of η that

$$B \leq \frac{1}{p^{\frac{1}{p-1}}} \frac{p-1}{p} \frac{\|\nabla u\|^{\frac{p}{p-1}}_{L^\infty(Q_1)}}{\eta^{\frac{1}{p-1}}} + 2^{p-1} \leq \frac{1}{p^{\frac{1}{p-1}}} \frac{p-1}{p} \frac{\|\nabla u\|_{L^\infty(Q_1)}}{|t-t_0|^{\frac{1}{p}}} + 2^{p-1}.$$

Then (5.1) and (5.2) imply

$$\begin{aligned} u(0,t) - u(0,t_0) &\leq \|\nabla u\|_{L^\infty(Q_1)} |t-t_0|^{\frac{p-1}{p}} + \left(\frac{p}{p-1}\right) n^{\frac{1}{p-1}} B(t-t_0) \\ &\leq C(1 + \|\nabla u\|_{L^\infty(Q_1)} + |t-t_0|^{\frac{1}{p}})|t-t_0|^{\frac{p-1}{p}} \\ &\leq C|t-t_0|^{\frac{p-1}{p}}, \quad C = C(n,p). \end{aligned}$$

Since this holds for u and $-u$, the reverse inequality also holds. ∎

Corollary 3. *Suppose u is a viscosity solution of* (1.1) *in Q_2 such that* $\operatorname{osc}_{Q_2} u \leq 1$. *Then*

$$|u(x,t) - u(x,s)| \leq C|t-s|^{\frac{p-1}{p}}$$

for $t, s \in [-1/2^{\frac{p}{p-1}}, 0]$ and $x, y \in B_1$. Here $C = C(n,p)$.

Proof. Let $z_0 \in B_1$ and define

$$v(x,t) := u\left(\frac{x}{2} + z_0, \frac{t}{2^{\frac{p}{p-1}}}\right).$$

Then v is a solution of (1.1) in Q_2. By construction, we also have

$$\operatorname{osc}_{Q_2} v \leq 1.$$

We may therefore apply Proposition 4 to v and obtain

$$|v(0,t) - v(0,s)| \leq C|s-t|^{\frac{p-1}{p}}, \quad s,t \in [-1,0].$$

In terms of u this implies

$$\left| u\left(z_0, \frac{t}{2^{\frac{p}{p-1}}}\right) - u\left(z_0, \frac{s}{2^{\frac{p}{p-1}}}\right) \right| \leq C|s-t|^{\frac{p-1}{p}}, \quad s,t \in [-1,0]$$

which is the desired result, upon renaming C. ∎

6. Proof of the regularity theorem. We have now everything needed for the proof of Theorem 1.

Proof of Theorem 1. Define

$$u_R(x,t) = \frac{u(Rx + x_0, R^{\frac{p}{p-1}}t + t_0)}{2\|u\|_{L^\infty(Q_{2R}(x_0,t_0))}}.$$

Then u_R solves (1.1) in Q_2 and $\operatorname{osc}_{Q_2} u \leq 1$. From Remark 2 and Corollary 3 we obtain

$$|u_R(x,t) - u_R(y,t)| \leq C|x-y|, \quad |u_R(x,t) - u_R(x,s)| \leq C|t-s|^{\frac{p-1}{p}}$$

for $x, y \in B_1, s, t \in (-1/2^{\frac{p}{p-1}}, 0)$. Coming back to u, this means

$$|u(x,t) - u(y,t)| \leq C\|u\|_{L^\infty(Q_{2R}(x_0,t_0))}\frac{|x-y|}{R}$$

and

$$|u(x,t) - u(x,s)| \leq C\|u\|_{L^\infty(Q_{2R}(x_0,t_0))}\frac{|t-s|^{\frac{p-1}{p}}}{R}$$

for all $x, y \in B_R(x_0)$ and $s, t \in (t_0 - (R/2)^{\frac{p}{p-1}}, t_0)$. The desired result now follows from the triangle inequality. □

7. The large time behavior. In [HL16a], we constructed the unique viscosity solution of (1.3). We also characterized the large time behavior of weak solutions (which thus also applies to viscosity solutions).

THEOREM 3. *Assume $g \in W_0^{1,p}(\Omega)$. Then for any weak solution u of* (1.3)*, the limit*

$$w := \lim_{t\to\infty} e^{\lambda_p^{\frac{1}{p-1}}t}u(\cdot,t)$$

exists in $W_0^{1,p}(\Omega)$ and is a p-ground state, provided $w \not\equiv 0$. In this case, $u(\cdot,t) \not\equiv 0$ for $t \geq 0$ and

$$\lambda_p = \lim_{t\to\infty}\frac{\displaystyle\int_\Omega |\nabla u(x,t)|^p dx}{\displaystyle\int_\Omega |u(x,t)|^p dx}.$$

We now have all the ingredients needed for the proof of Theorem 2.

Proof of Theorem 2. Let τ_k be an increasing sequence of positive numbers such that $\tau_k \to \infty$ as $k \to \infty$. Since any viscosity solution is also a weak solution, Theorem 3 establishes that

$$\lim_{k\to\infty} e^{\lambda_p^{\frac{1}{p-1}}\tau_k}u(\cdot,\tau_k) = w \tag{7.1}$$

in $W_0^{1,p}(\Omega)$, where w is either a ground state or identically zero. We also know that w does not depend on the sequence τ_k. It is therefore enough to prove that this sequence has a subsequence that convergences uniformly to w on $\overline{\Omega}$. Define

$$v^k(x,t) = e^{\lambda_p^{\frac{1}{p-1}}\tau_k} u(x, t+\tau_k).$$

We remark that $e^{-\lambda_p^{\frac{1}{p-1}}t}\phi$ is a solution of (1.1). By the comparison principle,

$$|v^k(x,t)| \le e^{\lambda_p^{\frac{1}{p-1}}}\phi(x) \tag{7.2}$$

for $(x,t) \in \Omega \times [-1,1]$ for all $k \in \mathbb{N}$ large enough. These bounds together with Theorem 1 give that v^k is uniformly bounded in $C^\alpha(B \times [0,1])$ for any ball $B \subset\subset \Omega$ for $\alpha \le (p-1)/p$. By a routine covering argument, v^k is then uniformly bounded in $C^\alpha(K \times [0,1])$ for any compact $K \subset\subset \Omega$. Since ϕ is continuous up to the boundary of Ω, (7.2) together with these local estimates implies that the sequence v^k is equicontinuous on $\overline{\Omega} \times [0,1]$ (see, for instance, the proof of Theorem 1.3 in [HL16b] for details). By the Arzelà–Ascoli theorem, we can extract a subsequence v^{k_j} such that

$$v^{k_j} \to e^{-\lambda_p^{\frac{1}{p-1}}t} w,$$

uniformly on $\overline{\Omega} \times [0,1]$. Letting $t = 0$, this establishes the desired existence of a uniformly convergent subsequence. ∎

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