Prescribing Initial Values for the Sticky Particle System (Survey)

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1 Introduction

The sticky particle system (SPS) is a system of partial differential equations which describes the motion of a collection of particles in $\mathbb{R}^d$ which move freely and interact only through perfectly inelastic collisions. Denoting ρ as the density of particles and v as an associated velocity field, the SPS consists of the *conservation of mass*

$$\partial_t \rho + \nabla \cdot (\rho v) = 0 \tag{1}$$

along with the *conservation of momentum*

$$\partial_t (\rho v) + \nabla \cdot (\rho v \otimes v) = 0. \tag{2}$$

These equations hold in $\mathbb{R}^d \times (0, \infty)$ and were first derived by the astronomer Yakov Zel'dovich in his work on the expansion of matter without pressure [17].

In this note, we will be concerned with determining whether or not solution pairs ρ and v exist for given initial conditions. In particular, we would like to prescribe an initial mass distribution ρ_0 and an initial velocity field v_0

$$\rho|_{t=0} = \rho_0 \quad \text{and} \quad v|_{t=0} = v_0 \tag{3}$$

and use the SPS to describe the evolution of the mass distribution ρ and associated velocity field v at later times. To this end, we will define a generalized solution and phrase our initial value problem using this notion.

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Since the total mass of any physical system we consider will be conserved, we will assume throughout that it is always equal to 1. As a result, it will be natural for us to employ the space $\mathcal{P}(\mathbb{R}^d)$ of Borel probability measures on $\mathbb{R}^d$. We recall that this space has a natural topology: $(\mu^k)_{k\in\mathbb{N}} \subset \mathcal{P}(\mathbb{R}^d)$ converges *narrowly* to $\mu \in \mathcal{P}(\mathbb{R}^d)$ provided

$$\lim_{k\to\infty}\int_{\mathbb{R}^d} g d\mu^k = \int_{\mathbb{R}^d} g d\mu$$

for each continuous and bounded $g : \mathbb{R}^d \to \mathbb{R}$. The notion of solution that we will use throughout this paper is as follows.

Definition 1 Suppose $\rho_0 \in \mathcal{P}(\mathbb{R}^d)$ and $v_0 : \mathbb{R}^d \to \mathbb{R}^d$ is continuous and bounded. A narrowly continuous $\rho : [0,\infty) \to \mathcal{P}(\mathbb{R}^d); t \mapsto \rho_t$ and Borel measurable $v : \mathbb{R}^d \times [0,\infty) \to \mathbb{R}^d$ is a *weak solution pair* of the SPS with initial conditions (3) provided the following hold.

- For each $T > 0$,

$$\int_0^T \int_{\mathbb{R}^d} |v|^2 d\rho_t dt < \infty.$$

- For each $\psi \in C_c^\infty(\mathbb{R}^d \times [0,\infty))$,

$$\int_0^\infty \int_{\mathbb{R}^d} (\partial_t \psi + \nabla\psi \cdot v) d\rho_t dt + \int_{\mathbb{R}^d} \psi(\cdot, 0) d\rho_0 = 0. \tag{4}$$

- For each $\varphi \in C_c^\infty(\mathbb{R}^d \times [0,\infty); \mathbb{R}^d)$,

$$\int_0^\infty \int_{\mathbb{R}^d} (\partial_t \varphi \cdot v + \nabla\varphi\, v \cdot v) d\rho_t dt + \int_{\mathbb{R}^d} \varphi(\cdot, 0) \cdot v_0 d\rho_0 = 0. \tag{5}$$

It is not hard to check that this definition extends the usual notion of a smooth solution pair. Indeed, if $\rho : \mathbb{R}^d \times [0,\infty) \to [0,\infty)$ and $v : \mathbb{R}^d \times [0,\infty) \to \mathbb{R}^d$ are continuously differentiable and satisfy the SPS, we can multiply (1) by $\psi \in C_c^\infty(\mathbb{R}^d \times [0,\infty))$ and (2) by $\varphi \in C_c^\infty(\mathbb{R}^d \times [0,\infty); \mathbb{R}^d)$ and integrate by parts in order to derive (4) and (5), respectively. It is also useful to have a more flexible notion of solution as classical solutions may not exist for a given pair of smooth initial conditions.

The problem that motivated this work is as follows.

Problem 1 Suppose $\rho_0 \in \mathcal{P}(\mathbb{R}^d)$ and $v_0 : \mathbb{R}^d \to \mathbb{R}^d$ is continuous and bounded. Determine whether or not there is a weak solution pair ρ and v of the SPS with these initial conditions, respectively.

This initial value problem was resolved in one spatial dimension ($d = 1$) in the pioneering works by E, Rykov and Sinai [8] and by Brenier and Grenier [3]. Much less progress has been made when $d > 1$. In this article, we will survey a few known results and explain some challenges with the SPS.

2 The Method of Characteristics

Suppose that $\rho_0 \in \mathcal{P}(\mathbb{R}^d)$ and $v_0 : \mathbb{R}^d \to \mathbb{R}^d$ are given. The simplest setting in which we can solve the SPS is when

$$\mathrm{id}_{\mathbb{R}^d} + tv_0 \text{ is invertible for each } t > 0.$$

This occurs, for example, if v_0 is continuously differentiable and monotone. Also note that this assumption implies that the rays $[0, \infty) \ni t \mapsto x + tv_0(x)$ and $[0, \infty) \ni t \mapsto y + tv_0(y)$ do not intersect for $x \neq y$. Let us show how this assumption leads to a solution pair.

For each $t \geq 0$, define $\rho_t \in \mathcal{P}(\mathbb{R}^d)$ via the formula

$$\int_{\mathbb{R}^d} g(y)d\rho_t(y) := \int_{\mathbb{R}^d} g(x + tv_0(x))d\rho_0(x) \tag{6}$$

for all continuous and bounded $g : \mathbb{R}^d \to \mathbb{R}$. The measure ρ_t is simply the mass distribution ρ_0 transported along the family of nonintersecting rays $[0, \infty) \ni t \mapsto x + tv_0(x)$ for $x \in \mathbb{R}^d$. Let us also specify $v : \mathbb{R}^d \times [0, \infty) \to \mathbb{R}^d$ implicitly by the equation

$$v(x + tv_0(x), t) = v_0(x).$$

This tells us that the velocity is constant along the straight line trajectories. It is straightforward to check that ρ and v is a weak solution pair with initial conditions ρ_0 and v_0.

Suppose in addition that ρ_0 has a smooth density, which we will identify with ρ_0 itself, and that v_0 is continuously differentiable. Using the change of variables theorem in (6), we find that the SPS admits the classical solution pair

$$\begin{cases} \rho(\cdot, t) := \left[\dfrac{\rho_0}{\det \nabla(\mathrm{id}_{\mathbb{R}^d} + tv_0)}\right] \circ (\mathrm{id}_{\mathbb{R}^d} + tv_0)^{-1} \\ \\ v(\cdot, t) := v_0 \circ (\mathrm{id}_{\mathbb{R}^d} + tv_0)^{-1}. \end{cases}$$

Unfortunately, once $\mathrm{id}_{\mathbb{R}^d} + tv_0$ fails to be injective, these formulae are no longer valid.

3 Sticky Particle Trajectories

Suppose there are N particles in $\mathbb{R}^d$ with masses $m_1, \ldots, m_N$ that sum to 1. In addition, suppose that these point masses move freely in space until they collide; when particles collide, they undergo perfectly inelastic collisions. For example, if the particles with masses $m_1, \ldots, m_k$ have respective velocities $v_1, \ldots, v_k \in \mathbb{R}^d$ just before they collide, they will join to form a single particle of mass $m_1 + \cdots + m_k$ which has velocity

$$\frac{m_1 v_1 + \cdots + m_k v_k}{m_1 + \cdots + m_k}$$

right after the collision.

We will denote *the sticky particle trajectories* $\gamma_1, \ldots, \gamma_N : [0, \infty) \to \mathbb{R}^d$ as the piecewise linear paths that track the position of the respective point masses discussed above. Specifically, $\gamma_i(t)$ is the location of the particle with mass m_i at time t. Note that this particle could be by itself or part of a larger mass if it collided with another particle before time t. It is not hard to show that these paths are well defined and satisfy the sticky particle condition

$$\gamma_i(s) = \gamma_j(s) \Longrightarrow \gamma_i(t) = \gamma_j(t) \tag{7}$$

for $t \geq s$ and $i, j = 1, \ldots, N$ (Proposition 2.1 and Corollary 2.4 of [10]).

One of the most important properties of sticky particle trajectories is as follows.

Proposition 1 (Proposition 2.5 of [10]) *Assume* $g : \mathbb{R}^d \to \mathbb{R}^d$. *Then*

$$\sum_{i=1}^{N} m_i g(\gamma_i(t)) \cdot \dot{\gamma}_i(t+) = \sum_{i=1}^{N} m_i g(\gamma_i(t)) \cdot \dot{\gamma}_i(s+)$$

for $0 \leq s \leq t$.

This averaging property embodies the conservation of momentum that particles experience in between and during collisions. To see this, we define

$$\rho_t := \sum_{i=1}^{N} m_i \delta_{\gamma_i(t)} \in \mathcal{P}(\mathbb{R}^d) \tag{8}$$

for each $t \geq 0$. Note that since $\gamma_1, \ldots, \gamma_N$ are continuous paths, $\rho : [0, \infty) \to \mathcal{P}(\mathbb{R}^d); t \mapsto \rho_t$ is narrowly continuous. Using (7), we may also set

$$v(x, t) = \begin{cases} \dot{\gamma}_i(t+), & x = \gamma_i(t) \\ 0, & \text{otherwise.} \end{cases} \tag{9}$$

It is a simple exercise to check that $v : \mathbb{R}^d \times [0, \infty) \to \mathbb{R}^d$ is Borel measurable. Moreover, the following assertion holds.

Proposition 2 (Corollary 2.6 and Section 2.3 of [10]) *Suppose*

$$\rho_0 := \sum_{i=1}^{N} m_i \delta_{\gamma_i(0)} \tag{10}$$

and $v_0 : \mathbb{R}^d \to \mathbb{R}^d$ *is continuous with*

$$v_0(\gamma_i(0)) = \dot{\gamma}_i(0+)$$

for $i = 1, \dots, N$. *Then* ρ *and* v *defined in* (8) *and* (9)*, respectively, is a weak solution pair of the SPS with initial conditions* (3)*. Moreover,*

$$\int_{\mathbb{R}^d} \frac{1}{2}|v(x,t)|^2 d\rho_t(x) \le \int_{\mathbb{R}^d} \frac{1}{2}|v(x,s)|^2 d\rho_s(x)$$

for each $0 \le s \le t$.

When $d = 1$, we have the additional property. We call it the *quantitative sticky particle property* as it quantifies (7).

Proposition 3 (Corollary 2.8 of [10]) *Suppose* $d = 1$. *For* $0 < s \le t < \infty$ *and* $i, j = 1, \dots, N$,

$$\frac{1}{t}|\gamma_i(t) - \gamma_j(t)| \le \frac{1}{s}|\gamma_i(s) - \gamma_j(s)|.$$

By this quantitative sticky particle property,

$$\begin{aligned} &\frac{d}{dt}\frac{1}{t^2}|\gamma_i(t) - \gamma_j(t)|^2 \\ &\quad = \frac{2}{t^2}\left[(v(\gamma_i(t),t) - v(\gamma_j(t),t))(\gamma_i(t) - \gamma_j(t)) - \frac{1}{t}|\gamma_i(t) - \gamma_j(t)|^2\right] \le 0 \end{aligned}$$

for almost every $t > 0$ and each $i, j = 1, \dots N$. As a result,

$$(v(x,t) - v(y,t))(x - y) \le \frac{1}{t}(x - y)^2 \tag{11}$$

for almost every $t > 0$ and each $x, y \in \text{supp}(\rho_t)$. We emphasize this entropy inequality only holds for $d = 1$.

4 Large Particle Limit

Suppose $\rho_0 \in \mathcal{P}(\mathbb{R}^d)$ is a given initial mass distribution. We may select a sequence $(\rho_0^k)_{k\in\mathbb{N}} \subset \mathcal{P}(\mathbb{R}^d)$ for which

$$\begin{cases} \text{each } \rho_0^k \text{ is of the form (10), and} \\ \rho_0^k \to \rho_0 \text{ narrowly as } k \to \infty \end{cases}$$

(Remark 5.1.2 in [1]). Let us additionally suppose $v_0 : \mathbb{R}^d \to \mathbb{R}^d$ is continuous and bounded. Using sticky particle trajectories, we can produce a weak solution pair ρ^k and v^k of the SPS with initial conditions ρ_0^k and v_0 for each $k \in \mathbb{N}$.

It is now natural to ask if there are subsequences $(\rho^k)_{k\in\mathbb{N}}$ and $(v^k)_{k\in\mathbb{N}}$ which converge in some sense to a weak solution pair ρ and v of the SPS with initial conditions ρ_0 and v_0. For this to work, we would need to send $k \to \infty$ along a subsequence in

$$\int_0^\infty \int_{\mathbb{R}^d} (\partial_t \psi + \nabla\psi \cdot v^k) d\rho_t^k dt + \int_{\mathbb{R}^d} \psi(\cdot, 0) d\rho_0^k = 0$$

for each $\psi \in C_c^\infty(\mathbb{R}^d \times [0,\infty))$ and in

$$\int_0^\infty \int_{\mathbb{R}^d} (\partial_t \varphi \cdot v^k + \nabla\varphi\, v^k \cdot v^k) d\rho_t^k dt + \int_{\mathbb{R}^d} \varphi(\cdot, 0) \cdot v_0 d\rho_0^k = 0$$

for each $\varphi \in C_c^\infty(\mathbb{R}^d \times [0,\infty); \mathbb{R}^d)$. The only estimate we have at our disposal is

$$\int_{\mathbb{R}^d} \frac{1}{2}|v^k(x,t)|^2 d\rho_t^k(x) \le \int_{\mathbb{R}^d} \frac{1}{2}|v_0(x)|^2 d\rho_0^k(x)$$

for $t \ge 0$ and $k \in \mathbb{N}$.

It turns out that this strategy only works in dimension 1, where we have the additional entropy estimate (11)

$$(v^k(x,t) - v^k(y,t))(x-y) \le \frac{1}{t}(x-y)^2$$

for each $x, y \in \text{supp}(\rho_t^k)$ and $t > 0$. We may interpret this estimate informally as the one sided derivative bound

$$\partial_x v^k(x,t) \le \frac{1}{t}$$

for ρ_t^k almost every $x \in \mathbb{R}$. This estimate is just enough to ensure that $(\rho^k)_{k\in\mathbb{N}}$ and $(v^k)_{k\in\mathbb{N}}$ have subsequences which converge in ways which allow us to conclude that their limits ρ and v indeed comprise a weak solution pair of the SPS with the desired initial data [10, 15].

The following existence theorem was first deduced by E, Rykov and Sinai [8] and by Brenier and Grenier [3]. We also note that there have been many other significant contributions to the initial value problem for the SPS in one spatial dimension including [2, 5, 6, 9–14].

Theorem 1 *Suppose $d = 1$, $\rho_0 \in \mathcal{P}(\mathbb{R})$ and $v_0 : \mathbb{R} \to \mathbb{R}$ continuous and bounded. There is a weak solution pair ρ and v of the SPS with initial conditions $\rho|_{t=0} = \rho_0$ and $v|_{t=0} = v_0$. Moreover, for almost every $t > 0$ and each $x, y \in \operatorname{supp}(\rho_t)$,*

$$(v(x,t) - v(y,t))(x - y) \le \frac{1}{t}(x - y)^2;$$

and for almost every $0 \le s \le t < \infty$,

$$\int_{\mathbb{R}} \frac{1}{2} v(x,t)^2 d\rho_t(x) \le \int_{\mathbb{R}} \frac{1}{2} v(x,s)^2 d\rho_s(x).$$

We remark that the uniqueness of a weak solution pair which satisfies the entropy inequality was first proved by Huang and Wang [9]. We also note that Nguyen and Tudorascu proved that there is a unique entropy solution pair provided the pth moment of ρ_0 is finite and $v_0 \in L^p(\rho_0)$ for $p \ge 2$ [15].

5 Instability

It seems the main issue with solving the initial value problem in several spatial dimensions is that the solutions to the SPS are unstable. For example, suppose the rays $[0, \infty) \ni t \mapsto x_1 + tv_1$ and $[0, \infty) \ni t \mapsto x_2 + tv_2$ intersect at time $s > 0$. If these rays initially describe the paths of two colliding particles with masses m_1 and m_2, respectively, the corresponding sticky particle trajectories are

$$\gamma_i(t) = \begin{cases} x_i + tv_i, & t \in [0, s] \\ z + (t - s)(m_1 v_1 + m_2 v_2), & t \in [s, \infty) \end{cases} \tag{12}$$

for $i = 1, 2$. Here $z = x_1 + sv_1 = x_2 + sv_2$ is the point where the particles collide.

When $d \ge 2$, we can replace x_2 with $\tilde{x}_2 \ne x_2$ and obtain two rays $[0, \infty) \ni t \mapsto x_1 + tv_1$ and $[0, \infty) \ni t \mapsto \tilde{x}_2 + tv_2$ which do not intersect. Furthermore, we can do so in a way that $\tilde{x}_2$ is as close to x_2 as desired. Therefore, a small change in initial conditions results in solutions which do not appear to be close to each other. This example also shows that the limit of a sequence of solution pairs of the SPS may not be a solution. Indeed if we send $\tilde{x}_2 \to x_2$, we obtain two intersecting rays $[0, \infty) \ni t \mapsto x_1 + tv_1$ and $[0, \infty) \ni t \mapsto x_2 + tv_2$ and not the sticky particle trajectories γ_1 and γ_2 in (12). We believe this simple observation is at the core of the what is preventing us from naively designing an approximation scheme as discussed in the previous section.

6 Lagrangian Variables

Not long after the initial value problem for the SPS was solved in one spatial dimension, Sever developed an interesting approach for the initial value problem in any spatial dimension [16] (see also [7]). This approach is based on the auxiliary initial value problem

$$\begin{cases} \dfrac{d}{dt}X(t) = \mathbb{E}_{\rho_0}[v_0|X(t)], & \text{a.e. } t \geq 0 \\ X_0 = \mathrm{id}_{\mathbb{R}^d}. \end{cases} \tag{13}$$

Here the unknown is an absolutely continuous path $X : [0, \infty) \to L^2(\rho_0; \mathbb{R}^d)$. We also recall that for each $t \geq 0$, the conditional expectation $\mathbb{E}_{\rho_0}[v_0|X(t)]$ is an $L^2(\rho_0; \mathbb{R}^d)$ map $g(X(t))$, where $g : \mathbb{R}^d \to \mathbb{R}^d$ is Borel and

$$\int_{\mathbb{R}^d} g(X(t)) \cdot h(X(t))d\rho_0 = \int_{\mathbb{R}^d} v_0 \cdot h(X(t))d\rho_0$$

for every bounded, continuous $h : \mathbb{R}^d \to \mathbb{R}$.

The key to linking this flow equation to the SPS is as follows. First define $\rho : [0, \infty) \to \mathcal{P}(\mathbb{R}^d); t \mapsto \rho_t$ via the formula

$$\int_{\mathbb{R}^d} hd\rho_t := \int_{\mathbb{R}^d} h \circ X(t)d\rho_0.$$

Next choose a Borel $v : \mathbb{R}^d \times [0, \infty) \to \mathbb{R}^d$ such that

$$v(X(t), t) = \mathbb{E}_{\rho_0}[v_0|X(t)] \quad \text{a.e. } t \geq 0.$$

Then ρ and v is a weak solution pair of the SPS with initial conditions (3).

Sever also argued that (13) has a solution which satisfies a natural sticky particle property (Theorem 4.2 of [16]). However, Bressan and Nguyen discovered that this result may fail to hold [4]. Specifically, they showed that there are initial conditions for which (13) does not have a solution as described by Sever's theorem. As a result, there is some controversy and much room for clarification with this method.

7 Kinetic Theory

We conclude this note by recalling an initial value problem in kinetic theory related to the SPS. The problem is to find $f : [0,\infty) \to \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d); t \mapsto f_t$ which satisfies

$$\begin{cases} \partial_t f + \xi \cdot \nabla_x f = 0, \text{ in } \mathbb{R}^d \times \mathbb{R}^d \times (0,\infty) \\ f|_{t=0} = (\mathrm{id}_{\mathbb{R}^d} \times v_0)_{\#}\rho_0 \end{cases}$$

in the weak sense. That is,

$$\int_0^\infty \int_{\mathbb{R}^d \times \mathbb{R}^d} (\partial_t \psi + \xi \cdot \nabla_x \psi)\, df_t(x,\xi) dt + \int_{\mathbb{R}^d} \psi(x, v_0(x), 0) d\rho_0(x) = 0$$

for each $\psi \in C_c^1(\mathbb{R}^d \times \mathbb{R}^d \times [0,\infty))$. The physical interpretation is that f_t is the distribution of particles in position and velocity space $\mathbb{R}^d \times \mathbb{R}^d$ at time $t \geq 0$.

It turns out that this initial value problem can easily be solved for any ρ_0 and v_0. If there is a solution f with

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} g(x,\xi) df_t(x,\xi) = \int_{\mathbb{R}^d} g(x, v(x,t)) d\rho_t(x)$$

for almost every $t \geq 0$, then ρ and v is a weak solution pair of the SPS which satisfies the initial conditions (3). If no such solution exists, we wonder if it is possible to select a solution f which can somehow be associated to the SPS in a most natural way.

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