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A TRAJECTORY MAP FOR THE PRESSURELESS EULER EQUATIONS

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ABSTRACT. We consider the dynamics of a collection of particles that interact pairwise and are restricted to move along the real line. Moreover, we focus on the situation in which particles undergo perfectly inelastic collisions when they collide. The equations of motion are a pair of partial differential equations for the particles' mass distribution and local velocity. We show that solutions of this system exist for given initial conditions by rephrasing these equations in Lagrangian coordinates and then by solving for the associated trajectory map.

1. Introduction

In this paper, we will study the dynamics of a collection of particles which interact pairwise and which moves along the real line. We will also suppose that when particles collide, they undergo perfectly inelastic collisions. The equations of motion for this type of physical system are the *pressureless Euler equations*

$$\begin{cases}\partial_t\rho + \partial_x(\rho v) = 0,\\ \partial_t(\rho v) + \partial_x(\rho v^2) = -\rho(W' * \rho),\end{cases} \tag{1.1}$$

which hold on $\mathbb{R}\times(0,\infty)$. The first equation expresses the conservation of mass, and the second expresses the conservation of momentum. Here ρ and v are the respective mass distribution and velocity field of particles and W is the interaction energy.

The central goal of this work is to describe how to find a pair ρ and v which solves (1.1) for given initial conditions

$$\rho|_{t=0} = \rho_0 \quad \text{and} \quad v|_{t=0} = v_0. \tag{1.2}$$

We typically will assume ρ_0 belongs to the space $\mathcal{P}(\mathbb{R}^2)$ of Borel probability measures on $\mathbb{R}$ and $v_0 : \mathbb{R}\to\mathbb{R}$ is continuous. To this end, we will first produce $X : [0,\infty) \to L^2(\rho_0)$ which satisfies the *pressureless Euler flow equation*

$$\dot{X}(t) = \mathbb{E}_{\rho_0}\left[v_0 - \int_0^t (W' * \rho_s)(X(s))ds \,\middle|\, X(t)\right] \quad \text{a.e. } t \geq 0 \tag{1.3}$$

and the initial condition

$$X(0) = \mathrm{id}_{\mathbb{R}} \tag{1.4}$$

ρ_0 almost everywhere. Here

$$\rho_t := X(t)_{\#}\rho_0, \quad t\geq 0, \tag{1.5}$$

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is the push forward of ρ_0 under $X(t)$, and $\mathbb{E}_{\rho_0}[g|X(t)]$ is the conditional expectation of a Borel $g:\mathbb{R}\to\mathbb{R}$ given $X(t)$.

To emphasize that $X(t)$ is a function on $\mathbb{R}$, we will sometimes write

$$X(t):\mathbb{R}\to\mathbb{R}; y\mapsto X(y,t).$$

The quantity $X(y,t)$ represents the time t position of a particle which was initially at position y. After showing a solution X exists, we will argue that there is a Borel function $v:\mathbb{R}\times[0,\infty)\to\mathbb{R}$ such that

$$\dot{X}(t)=v(X(t),t)\quad a.e.\ t\geq 0 \tag{1.6}$$

ρ_0 almost everywhere. In particular, we will see that

$$\rho:[0,\infty)\to\mathcal{P}(\mathbb{R}^2); t\mapsto\rho_t$$

and v together comprise an appropriately defined weak solution pair for the pressureless Euler system.

1.1. Main theorem.

Throughout this paper, we will assume the $\rho_0\in\mathcal{P}(\mathbb{R}^2)$ has finite second moment

$$\int_{\mathbb{R}}x^2d\rho_0(x)<\infty$$

and

$$v_0:\mathbb{R}\to\mathbb{R}\ \text{ is absolutely continuous.}$$

We will also suppose $W:\mathbb{R}\to\mathbb{R}$ is continuously differentiable, W is even

$$W(x)=W(-x),\quad x\in\mathbb{R},$$

and W' grows at most linearly

$$\sup_{x\in\mathbb{R}}\frac{|W'(x)|}{1+|x|}<\infty.$$

Moreover, we will suppose that W is semiconvex. That is,

$$W(x)+\frac{c}{2}x^2 \text{ is convex}$$

for some $c>0$. We recall that concave W corresponds to repulsive interaction between particles. Assuming that W is semiconvex forces $W''(x)\geq -c$ for Lebesgue almost every $x\in\mathbb{R}$, which in a sense limits repulsive interaction.

Theorem 1.1. *There is a locally Lipschitz continuous $X:[0,\infty)\to L^2(\rho_0)$ which satisfies the pressureless Euler flow equation* (1.3) *and the initial condition* (1.4). *Moreover, X has the following properties:*

(i) *For Lebesgue almost every $t,s\in[0,\infty)$ with $s\leq t$*

$$E(t)\leq E(s),$$

where

$$E(\tau):=\int_{\mathbb{R}}\frac{1}{2}\dot{X}(\tau)^2d\rho_0+\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{1}{2}W(X(y,\tau)-X(z,\tau))d\rho_0(y)d\rho_0(z).$$

(ii) *For $t\geq 0$ and $y,z\in\operatorname{supp}(\rho_0)$ with $y\geq z$,*

$$0\leq X(y,t)-X(z,t)\leq\cosh(\sqrt{c}t)(y-z)+\frac{1}{\sqrt{c}}\sinh(\sqrt{c}t)\int_z^y|v_0'(x)|dx.$$

(iii) *For* $0 < s \le t$ *and* $y, z \in \text{supp}(\rho_0)$,

$$\frac{|X(y,t) - X(z,t)|}{\sinh(\sqrt{c}t)} \le \frac{|X(y,s) - X(z,s)|}{\sinh(\sqrt{c}s)}.$$

A few remarks about the statement of this theorem are in order. Locally Lipschitz means that $X : [0,T] \to L^2(\rho_0)$ is Lipschitz continuous for each $T \ge 0$, and consequently,

$$\dot{X}(t) = \lim_{\tau \to 0} \frac{X(t+\tau) - X(t)}{\tau}$$

exists in $L^2(\rho_0)$ for almost every $t > 0$. The function E in condition (i) represents the total energy of the physical system being modeled by the pressureless Euler flow equation. Condition (ii) asserts that $X(t)$ is nondecreasing and absolutely continuous on the support of ρ_0,

$$\text{supp}(\rho_0) := \{y \in \mathbb{R} : \rho_0((y - \delta, y + \delta)) > 0 \text{ for all } \delta > 0\}.$$

Property (iii) asserts that X is quantitatively "sticky". That is, it quantifies the fact that if $X(y,s) = X(z,s)$, then $X(y,t) = X(z,t)$ for all $t \ge s$.

We will show that the existence of a weak solution of (1.1) for given initial conditions is a corollary of Theorem 1.1. In particular, we will verify that ρ defined in (1.5) and any Borel v which satisfies (1.6) is a weak solution of the pressureless Euler system whose energy

$$\int_{\mathbb{R}} \frac{1}{2} v(x,t)^2 d\rho_t(x) + \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{1}{2} W(x-y) d\rho_t(x) d\rho_t(y)$$

is essentially nonincreasing in t and which satisfies the one-sided Lipschitz condition

$$(v(x,t) - v(y,t))(x-y) \le \frac{\sqrt{c}}{\tanh(\sqrt{c}t)}(x-y)^2$$

for ρ_t almost every $x, y \in \mathbb{R}$.

1.2. **Prior work.** We have already established the existence of a weak solution pair to the pressureless Euler system with an even, continuously differentiable, semiconvex potential. In [15], we generated this solution via a Borel probability measure η on the space of continuous paths $\Gamma := C([0,\infty))$ endowed with the topology of local uniform convergence. Specifically, we constructed an η which satisfies: (i) for each bounded, continuous $h : \mathbb{R} \to \mathbb{R}$ and almost every $t \ge 0$

$$\int_\Gamma \dot{\gamma}(t) h(\gamma(t)) d\eta(\gamma) = \int_\Gamma \left[v_0(\gamma(0)) - \int_0^t (W' * \rho_s)(\gamma(s)) ds \right] h(\gamma(t)) d\eta(\gamma),$$

where $\rho_s \in \mathcal{P}(\mathbb{R}^2)$ is defined via

$$\int_{\mathbb{R}} h(x) d\rho_s(x) = \int_\Gamma h(\gamma(s)) d\eta(\gamma);$$

(ii) there is a Borel $v : \mathbb{R} \times [0,\infty) \to \mathbb{R}$, such that

$$\dot{\gamma}(t) = v(\gamma(t), t) \quad \text{a.e. } t > 0$$

for η almost every $\gamma \in \Gamma$. Then we checked that $\rho : [0,\infty) \to \mathcal{P}(\mathbb{R}^2); t \mapsto \rho_t$ and v is indeed a weak solution pair.

Along the way, we derived some specific information on η such as it is concentrated on absolutely continuous paths, it satisfies various energy estimates, and

$$\frac{|\gamma(t)-\xi(t)|}{\sinh(\sqrt{c}t)} \le \frac{|\gamma(s)-\xi(s)|}{\sinh(\sqrt{c}s)}$$

for $0 < s \le t$ and η almost every $\gamma, \xi \in \Gamma$. We consider Theorem 1.1 to be a refinement of the main result in [15] as it tells us that we can choose η as the push forward of ρ_0 under the map $\mathbb{R} \mapsto \Gamma; y \mapsto X(y,\cdot)$. Here $X(y,\cdot)$ is the path $t \mapsto X(y,t)$, which is continuous for ρ_0 almost every $y \in \mathbb{R}$. That is, η can be specified as

$$\int_\Gamma F(\gamma)d\eta(\gamma) = \int_{\mathbb{R}} F(X(y,\cdot))d\rho_0(y)$$

for each $F : \Gamma \to \mathbb{R}$ that is continuous and bounded.

There have been many other works on pressureless Euler type systems in one spatial dimension. Especially since they are special cases of the multidimensional systems of equations which arise in the study of galaxy formation [14,24]. One of the early mathematical works on this topic was by E, Rykov, and Sinai [9], where they studied the case $W(x) = |x|$ which corresponds to gravitational interaction between a collection of interacting particles constrained to move along the real line. We acknowledge that the existence of solutions for this particular case does not follow from Theorem 1.1 as $W(x) = |x|$ isn't continuously differentiable. Nevertheless, we will revisit this particular case below.

Another very influential study on this topic was done by Brenier, Gangbo, Savaré, and Westdickenberg [4]. In comparison to our work, they considered general interactions which could be attractive or repulsive. We also note that they recast the pressureless Euler equations in another coordinate system, and they were able to obtain precise information about solutions from the resulting differential inclusions. Other work with related approaches were done by Gangbo, Nguyen, and Tudorascu [12] and Nguyen and Tudorascu [22] on the Euler-Poisson system and by Brenier and Grenier [5], Natile and Savaré [21], and Cavalletti, Sedjro, and Westdickenberg [6] for the sticky particle system ($W \equiv 0$ in (1.1) or equation (1.7) below). We also recommend the additional references [2,7,13,17–20,23] for results on stationary solutions, local existence, uniqueness, and hydrodynamic limits related to pressureless Euler type systems.

The particular approach we take in this paper is motivated by the work of Dermoune [8] on the sticky particle system

$$\begin{cases} \partial_t\rho + \partial_x(\rho v) = 0, \\ \partial_t(\rho v) + \partial_x(\rho v^2) = 0. \end{cases} \tag{1.7}$$

In particular, Dermoune was the first to identify that

$$\dot{X}(t) = \mathbb{E}_{\rho_0}\left[v_0|X(t)\right] \quad \text{a.e. } t \ge 0 \tag{1.8}$$

is the natural equation for the sticky particle system in Lagrangian variables. We performed a thorough analysis of (1.8) in [16] and regard Theorem 1.1 as a significant generalization of the main results of [16].

1.3. **Euler-Poisson equations.** As mentioned above, we will also consider the *Euler-Poisson equations*

$$\begin{cases} \partial_t \rho + \partial_x(\rho v) = 0, \\ \partial_t(\rho v) + \partial_x(\rho v^2) = -\rho(\operatorname{sgn} * \rho). \end{cases} \tag{1.9}$$

Here

$$\operatorname{sgn}(x) := \begin{cases} 1, & x > 0, \\ 0, & x = 0, \\ -1, & x < 0, \end{cases}$$

and the associated interaction potential is $W(x) = |x|$. This system governs the dynamics of a collection of particles in which the force on each particle is proportional to the total mass to the right of the particle minus the total mass to the left of the particle; when particles collide, they undergo perfectly inelastic collisions [4, 9]. This is a simple model for gravitationally interacting particles which are constrained to move on the real line.

As we did for the pressureless Euler system, we will design a trajectory mapping X which satisfies the *Euler-Poisson flow equation*

$$\dot{X}(t) = \mathbb{E}_{\rho_0}\left[v_0 - \int_0^t (\operatorname{sgn} * \rho_s)(X(s))ds \middle| X(t) \right] \quad \text{a.e. } t \geq 0 \tag{1.10}$$

and the initial condition (1.4). However, since sgn is not continuous, we will have to argue a bit differently than we did to prove Theorem 1.1 in order to obtain the following theorem.

Theorem 1.2. *There is a Lipschitz continuous $X : [0, \infty) \to L^2(\rho_0)$ which satisfies the Euler-Poisson flow equation* (1.10) *and the initial condition* (1.4)*. Moreover, X has the following properties:*

(i) *For Lebesgue almost every $t, s \in [0, \infty)$ with $s \leq t$*

$$E(t) \leq E(s),$$

where

$$E(\tau) := \int_{\mathbb{R}} \frac{1}{2}\dot{X}(\tau)^2 d\rho_0 + \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{1}{2}|X(y,\tau) - X(z,\tau)| d\rho_0(y) d\rho_0(z).$$

(ii) *For $t \geq 0$ and $y, z \in \operatorname{supp}(\rho_0)$ with $y \geq z$,*

$$0 \leq X(y,t) - X(z,t) \leq y - z + t\int_z^y |v_0'(x)|dx.$$

(iii) *For $0 < s \leq t$ and $y, z \in \operatorname{supp}(\rho_0)$,*

$$\frac{1}{t}|X(y,t) - X(z,t)| \leq \frac{1}{s}|X(y,s) - X(z,s)|.$$

As with the pressureless Euler system, we will be able to generate a weak solution pair of the Euler-Poisson system from the solution X obtained in Theorem 1.2. Namely, ρ defined in (1.5) and any Borel v which satisfies (1.6) is a weak solution of the Euler-Poisson system whose total energy

$$\int_{\mathbb{R}} \frac{1}{2}v(x,t)^2 d\rho_t(x) + \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{1}{2}|x - y| d\rho_t(x) d\rho_t(y)$$

is nonincreasing in time and which fulfills the "entropy" inequality

$$(v(x,t) - v(y,t))(x-y) \le \frac{1}{t}(x-y)^2$$

for ρ_t almost every $x, y \in \mathbb{R}$.

The organization of this paper is as follows. First, we will review a few preliminaries needed for our study in section 2. Then we will show by a near explicit construction how to solve the pressureless Euler flow equation when the support of ρ_0 is finite in section 3. In section 4, we will analyze these special solutions and show they are compact in a certain sense. This compactness will allow us to solve the pressureless Euler flow equation for a general ρ_0 and consequently to solve the pressureless Euler equations for given initial conditions. Finally, in section 5, we will show how to alter the arguments we used for the pressureless Euler flow equation to solve the flow equation associated with the Euler-Poisson equations.

2. Preliminaries

In this section, we will briefly recall the facts we will need regarding the convergence of probability measures and conditional expectation.

2.1. **Convergence of probability measures.** As in the introduction, we denote $\mathcal{P}(\mathbb{R}^d)$ as the space of Borel probability measures on $\mathbb{R}^d$. We will also write $C_b(\mathbb{R}^d)$ for the space of bounded continuous functions on $\mathbb{R}^d$. We will say that a sequence $(\mu^k)_{k\in\mathbb{N}} \subset \mathcal{P}(\mathbb{R}^d)$ converges to μ in $\mathcal{P}(\mathbb{R}^d)$ *narrowly* provided

$$\lim_{k\to\infty} \int_{\mathbb{R}^d} g d\mu^k = \int_{\mathbb{R}^d} g d\mu \tag{2.1}$$

for every $g \in C_b(\mathbb{R}^d)$. It turns out that $(\mu^k)_{k\in\mathbb{N}}$ converges to μ narrowly if and only if $\lim_{k\to\infty} \mathscr{d}(\mu, \mu^k) = 0$, where $\mathscr{d}$ is a metric of the form

$$\mathscr{d} mu, \nu) := \sum_{j=1}^{\infty} \frac{1}{2^j} \left| \int_{\mathbb{R}^d} h_j d\mu - \int_{\mathbb{R}^d} h_j d\nu \right|, \quad \mu, \nu \in \mathcal{P}(\mathbb{R}^d). \tag{2.2}$$

Here each $h_j : \mathbb{R}^d \to \mathbb{R}$ satisfies

$$|h_j(x)| \le 1 \quad \text{and} \quad |h_j(x) - h_j(y)| \le |x-y|$$

for $x, y \in \mathbb{R}^d$ (Remark 5.1.1 of [1]). Furthermore, $(\mathcal{P}(\mathbb{R}^d), \mathscr{d})$ is a complete metric space.

We will need to be able to identify when a sequence of measures in $\mathcal{P}(\mathbb{R}^d)$ has a narrowly convergent subsequence. Fortunately, Prokhorov's theorem provides a necessary and sufficient condition; it asserts that $(\mu^k)_{k\in\mathbb{N}} \subset \mathcal{P}(\mathbb{R}^d)$ has a narrowly convergent subsequence if and only if there is $\varphi : \mathbb{R}^d \to [0, \infty]$ with compact sublevel sets such that

$$\sup_{k\in\mathbb{N}} \int_{\mathbb{R}^d} \varphi d\mu^k < \infty$$

(Theorem 5.1.3 of [1]). In addition, we will need to know when (2.1) holds for unbounded g. It turns out that if $g : \mathbb{R}^d \to \mathbb{R}$ is continuous and

$$\lim_{R\to\infty} \int_{|g|\ge R} |g| d\mu^k = 0$$

uniformly in $k \in \mathbb{N}$, then (2.1) holds (Lemma 5.1.7 of [1]). In this case, we say that $|g|$ is *uniformly integrable* with respect to the sequence $(\mu^k)_{k\in\mathbb{N}}$.

The following lemma will also prove to be useful.

Lemma 2.1 (Lemma 2.1 of [16]). *Suppose $(g^k)_{k\in\mathbb{N}}$ is a sequence of continuous functions on $\mathbb{R}^d$ which converges locally uniformly to g and $(\mu^k)_{k\in\mathbb{N}} \subset \mathcal{P}(\mathbb{R}^d)$ converges narrowly to μ. Further assume there is $h : \mathbb{R}^d \to [0,\infty)$ with compact sublevel sets, which is uniformly integrable with respect to $(\mu^k)_{k\in\mathbb{N}}$ and satisfies*

$$|g^k| \le h$$

for each $k \in \mathbb{N}$. Then

$$\lim_{k\to\infty} \int_{\mathbb{R}^d} g^k d\mu^k = \int_{\mathbb{R}^d} g d\mu.$$

2.2. **The push forward.** Suppose $f : \mathbb{R}^d \to \mathbb{R}^n$ is Borel measurable and $\mu \in \mathcal{P}(\mathbb{R}^d)$. We define the *push forward* of μ through f as the probability measure $f_{\#}\mu \in \mathcal{P}(\mathbb{R}^n)$ which satisfies

$$\int_{\mathbb{R}^n} g(y) d(f_{\#}\mu)(y) = \int_{\mathbb{R}^d} g(f(x)) d\mu(x)$$

for every $g \in C_b(\mathbb{R}^n)$. We note

$$f_{\#}\mu(A) = \mu(f^{-1}(A))$$

for Borel $A \subset \mathbb{R}^n$. Moreover, if f is continuous and $\mu^k \to \mu$ narrowly in $\mathcal{P}(\mathbb{R}^d)$, then

$$f_{\#}\mu^k \to f_{\#}\mu$$

in $\mathcal{P}(\mathbb{R}^n)$.

2.3. **Conditional expectation.** Suppose $\mu \in \mathcal{P}(\mathbb{R}^2)$, $g \in L^2(\mu)$, and $Y : \mathbb{R} \to \mathbb{R}$ is a Borel measurable function. A *conditional expectation* of g with respect to μ given Y is an $L^2(\mu)$ function $\mathbb{E}_\mu[g|Y]$ which satisfies two conditions: (i)

$$\int_{\mathbb{R}} \mathbb{E}_\mu[g|Y]\, h(Y) d\mu = \int_{\mathbb{R}} g\, h(Y) d\mu$$

for each Borel $h : \mathbb{R} \to \mathbb{R}$ with $h(Y) := h \circ Y \in L^2(\mu)$; and (ii)

$$\mathbb{E}_\mu[g|Y] = f(Y)$$

μ almost everywhere for a Borel $f : \mathbb{R} \to \mathbb{R}$ with $f(Y) \in L^2(\mu)$. The existence and μ almost everywhere uniqueness of a conditional expectation can be proved using the Radon-Nikodým theorem.

We emphasize that $X : [0,\infty) \to L^2(\rho_0)$ satisfies the pressureless Euler flow equation (1.3), provided the following two conditions hold for almost every $t \ge 0$: (i)

$$\int_{\mathbb{R}} g(X(t))\dot{X}(t) d\rho_0 = \int_{\mathbb{R}} g(X(t)) \left[v_0 - \int_0^t (W' * \rho_\tau)(X(\tau)) d\tau \right] d\rho_0$$

for each $g \in C_b(\mathbb{R})$; and (ii) there exists a Borel $u : \mathbb{R} \to \mathbb{R}$ for which

$$\dot{X}(t) = u(X(t))$$

ρ_0 almost everywhere.

3. Sticky particle trajectories

In this section, we will assume that ρ_0 is a convex combination of Dirac measures

$$\rho_0 := \sum_{i=1}^{N} m_i \delta_{x_i} \in \mathcal{P}(\mathbb{R}^2). \tag{3.1}$$

In particular, we suppose that $x_1, \dots, x_N \in \mathbb{R}$ are distinct and $m_1, \dots, m_N > 0$ with $\sum_{i=1}^N m_i = 1$. We also define

$$v_i := v_0(x_i)$$

for $i = 1, \dots, N$. It turns out that there is a natural ODE system related to the pressureless Euler flow equation, which is

$$\ddot{\gamma}_i(t) = -\sum_{j=1}^{N} m_j W'(\gamma_i(t) - \gamma_j(t)). \tag{3.2}$$

These are Newton's equations for N interacting particles with masses $m_1, \dots, m_N$; the positions of these particles are described by the trajectories $\gamma_1, \dots, \gamma_N$.

It turns out that a solution of the pressureless Euler flow equation can be built from these particle trajectories by first setting

$$X(x_i, t) = \gamma_i(t), \quad t \geq 0.$$

However, when trajectories intersect, we must modify the paths. Remarkably, the natural thing to do is to require that the corresponding particles undergo perfectly inelastic collisions when they collide. This amounts to requiring that the trajectories coincide and that their slopes average from the moment they intersect as in Figure 1. On any time interval when no collisions occur, the resulting trajectories will satisfy (3.2). We will call these paths *sticky particle trajectories* and we shall see that they are the building blocks for more general solutions.

The following proposition asserts that these trajectories exist and satisfy a few basic properties.

Proposition 3.1 (Proposition 2.1 of [15])**.** *There are continuous, piecewise C^2 paths*

$$\gamma_1, \dots, \gamma_N : [0, \infty) \to \mathbb{R}$$

with the following properties:

(i) *For $i = 1, \dots, N$ and all but finitely many $t \in (0, \infty)$, (3.2) holds.*

(ii) *For $i = 1, \dots, N$,*

$$\gamma_i(0) = x_i \quad \textit{and} \quad \dot{\gamma}_i(0+) = v_i.$$

(iii) *For $i, j = 1, \dots, N$, $0 \leq s \leq t$ and $\gamma_i(s) = \gamma_j(s)$ imply*

$$\gamma_i(t) = \gamma_j(t).$$

(iv) *If $t > 0$, $\{i_1, \dots, i_k\} \subset \{1, \dots, N\}$, and*

$$\gamma_{i_1}(t) = \cdots = \gamma_{i_k}(t) \neq \gamma_i(t)$$

for $i \notin \{i_1, \dots, i_k\}$, then

$$\dot{\gamma}_{i_j}(t+) = \frac{m_{i_1}\dot{\gamma}_{i_1}(t-) + \cdots + m_{i_k}\dot{\gamma}_{i_k}(t-)}{m_{i_1} + \cdots + m_{i_k}}$$

for $j = 1, \dots, k$.

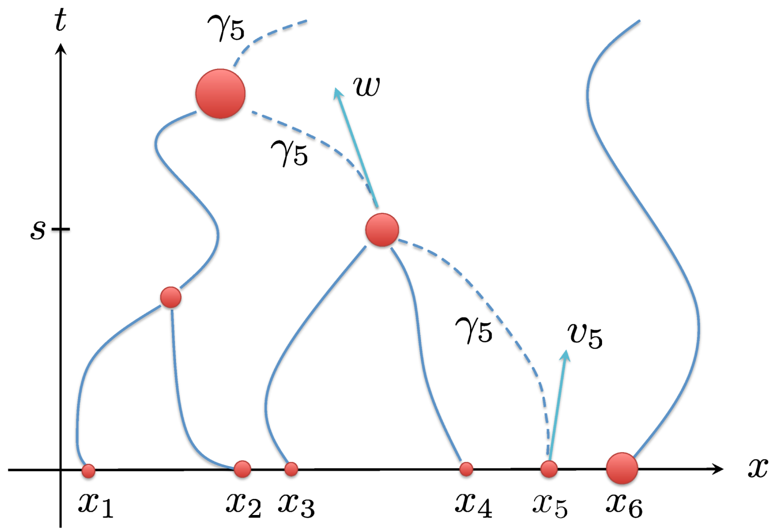

FIGURE 1. A schematic of sticky particle trajectories for $N = 6$. We have indicated the starting positions $x_1, \dots, x_6$ on the real line and we sketched the corresponding point masses larger than points to emphasize that their masses may be distinct. The path γ_5 that tracks point mass m_5 is shown in dashes along with its initial velocity v_5. Observe that there is a collision at time s between the point masses m_3, m_4, and m_5. As a result, the slope w satisfies $(m_3 + m_4 + m_5)w = m_3\dot\gamma_3(s-) + m_4\dot\gamma_4(s-) + m_5\dot\gamma_5(s-)$.

Remark 3.2. Using property (i), it is routine to check that $\dot\gamma_i(t\pm)$ both exist for each $t > 0$ and $i = 1, \dots, N$. Moreover,

$$\dot\gamma_i(t\pm) = \lim_{h\to 0^\pm} \frac{\gamma_i(t+h) - \gamma_i(t)}{h}.$$

A corollary of property (iv) above is what we call the *averaging property*. It is a general assertion about the conservation of momentum and is stated as follows.

Corollary 3.3 (Proposition 2.6 of [15]). *Suppose* $g : \mathbb{R} \to \mathbb{R}$ *and* $0 \le s < t$. *Then*

$$\sum_{i=1}^N m_i g(\gamma_i(t))\dot\gamma_i(t+)$$
$$= \sum_{i=1}^N m_i g(\gamma_i(t)) \left[\dot\gamma_i(s+) - \int_s^t \left(\sum_{j=1}^N m_j W'(\gamma_i(\tau) - \gamma_j(\tau))\right) d\tau\right].$$

3.1. **Quantitative stickiness.** Recall our standing assumption that there is a constant $c > 0$ chosen so that $W(x) + (c/2)x^2$ is convex. In terms of this constant, we can quantify (iii) in Proposition 3.1. Namely, we can estimate the distance $|\gamma_i(t) - \gamma_j(t)|$ in terms of the distance $|\gamma_i(s) - \gamma_j(s)|$ for $s \le t$. This is why we call the following assertion the *quantitative sticky particle property*.

Proposition 3.4 (Proposition 2.5 of [15])**.** *For each $i, j = 1, \dots, N$ and $0 < s \le t$*

$$\frac{|\gamma_i(t) - \gamma_j(t)|}{\sinh(\sqrt{c}t)} \le \frac{|\gamma_i(s) - \gamma_j(s)|}{\sinh(\sqrt{c}s)}.$$

An immediate corollary is as follows.

Proposition 3.5. *For each $0 < s \le t$, there is a function $f_{t,s} : \mathbb{R} \to \mathbb{R}$ for which*

$$\gamma_i(t) = f_{t,s}(\gamma_i(s))$$

for $i = 1, \dots, N$ and

$$|f_{t,s}(x) - f_{t,s}(y)| \le \frac{\sinh(\sqrt{c}t)}{\sinh(\sqrt{c}s)}|x - y| \tag{3.3}$$

for $x, y \in \mathbb{R}$.

Proof. By property (iii) of Proposition 3.1, the cardinality of the set

$$\{\gamma_1(t), \dots, \gamma_N(t)\}$$

is nonincreasing in t. It follows that there is a surjective function

$$g_{t,s} : \{\gamma_1(s), \dots, \gamma_N(s)\} \to \{\gamma_1(t), \dots, \gamma_N(t)\}; \gamma_i(s) \mapsto \gamma_i(t)$$

for $0 < s \le t$. By the quantitative sticky particle property, $g_{t,s}$ satisfies the Lipschitz condition (3.3). We can then extend $g_{t,s}$ to all of $\mathbb{R}$ in order to obtain the desired Lipschitz function $f_{t,s}$. □

3.2. **Energy estimates.** Sticky particle trajectories have nonincreasing energy. That is,

$$\begin{aligned}\frac{1}{2}\sum_{i=1}^N m_i\dot\gamma_i(t+)^2 + \frac{1}{2}\sum_{i,j=1}^N m_im_jW(\gamma_i(t) - \gamma_j(t)) \\ \le \frac{1}{2}\sum_{i=1}^N m_i\dot\gamma_i(s+)^2 + \frac{1}{2}\sum_{i,j=1}^N m_im_jW(\gamma_i(s) - \gamma_j(s))\end{aligned} \tag{3.4}$$

for $0 \le s < t$ (Proposition 2.8 of [15]). Using the semiconvexity of W, we can derive the subsequent kinetic energy estimates. We will express this result in terms of the increasing function

$$\vartheta(t) := e^{(c+1)t^2}\int_0^t e^{-(c+1)s^2}ds, \quad t \ge 0.$$

Lemma 3.6. *For each $t \ge 0$,*

$$\int_0^t \sum_{i=1}^N m_i\dot\gamma_i(s)^2 ds \le \left(\sum_{i=1}^N m_iv_i^2 + \frac{1}{2}\sum_{i,j=1}^N m_im_jW'(x_i - x_j)^2\right)\vartheta(t). \tag{3.5}$$

And for all but finitely many $t \ge 0$,

$$\sum_{i=1}^N m_i\dot\gamma_i(t)^2 \le \left(\sum_{i=1}^N m_iv_i^2 + \frac{1}{2}\sum_{i,j=1}^N m_im_jW'(x_i - x_j)^2\right)\vartheta'(t). \tag{3.6}$$

Proof. Due to the convexity of $x \mapsto W(x) + (c/2)x^2$,

$$\begin{aligned} W(\gamma_i(t) - \gamma_j(t)) &\geq W(x_i - x_j) + W'(x_i - x_j)(\gamma_i(t) - x_i - (\gamma_j(t) - x_j)) \\ &\qquad - \frac{c}{2}(\gamma_i(t) - x_i - (\gamma_j(t) - x_j))^2 \\ &\geq W(x_i - x_j) - \frac{1}{2}W'(x_i - x_j)^2 - \frac{c+1}{2}(\gamma_i(t) - x_i - (\gamma_j(t) - x_j))^2 \\ &\geq W(x_i - x_j) - \frac{1}{2}W'(x_i - x_j)^2 - (c+1)((\gamma_i(t) - x_i)^2 + (\gamma_j(t) - x_j)^2) \\ &\geq W(x_i - x_j) - \frac{1}{2}W'(x_i - x_j)^2 - (c+1)t\left(\int_0^t \dot{\gamma}_i(s)^2 ds + \int_0^t \dot{\gamma}_j(s)^2 ds\right). \end{aligned}$$

Combining these lower bounds with (3.4) at $s = 0$ gives

$$\begin{aligned} \sum_{i=1}^N m_i \dot{\gamma}_i(t)^2 \leq \sum_{i=1}^N m_i v_i^2 + \frac{1}{2}\sum_{i,j=1}^N m_i m_j W'(x_i - x_j)^2 \\ + 2(c+1)t\int_0^t \sum_{i=1}^N m_i \dot{\gamma}_i(s)^2 ds \end{aligned} \tag{3.7}$$

for all but finitely many $t \geq 0$. As a result,

$$\begin{aligned} &\frac{d}{dt} e^{-(c+1)t^2} \int_0^t \sum_{i=1}^N m_i \dot{\gamma}_i(s)^2 ds \\ &\quad = e^{-(c+1)t^2}\left(\sum_{i=1}^N m_i \dot{\gamma}_i(t)^2 - 2(c+1)t\int_0^t \sum_{i=1}^N m_i \dot{\gamma}_i(s)^2 ds\right) \\ &\quad \leq e^{-(c+1)t^2}\left(\sum_{i=1}^N m_i v_i^2 + \frac{1}{2}\sum_{i,j=1}^N m_i m_j W'(x_i - x_j)^2\right) \end{aligned}$$

for all but finitely many $t \geq 0$. We can then integrate from 0 to t to derive (3.5). Inequality (3.6) follows from (3.5) and (3.7). □

3.3. **Stability estimate.** We need one more estimate that depends on the following elementary lemma.

Lemma 3.7. *Suppose $T > 0$ and $y : [0, T) \to \mathbb{R}$ is continuous and piecewise C^2. Further assume*

$$\dot{y}(t+) \leq \dot{y}(t-) \tag{3.8}$$

for each $t \in (0, T)$ and that there is $c > 0$ for which

$$\ddot{y}(t) \leq cy(t)$$

for all but finitely many $t \in (0, T)$. Then

$$y(t) \leq \cosh(\sqrt{c}t)y(0) + \frac{1}{\sqrt{c}}\sinh(\sqrt{c}t)\dot{y}(0+)$$

for $t \in [0, T)$.

Proof. By a routine scaling argument, it suffices to verify this assertion for $c = 1$. To this end, we suppose in addition that there are times $0 < t_1 < \cdots < t_n$ for which y is C^2 on the intervals $(0, t_1), \ldots, (t_n, T)$.

Define

$$u(t) := \frac{y(t)}{\cosh(t)}, \quad t \in (0,T),$$

so that $y(t) = u(t)\cosh(t)$. Observe

$$\dot{u}(t) = \frac{\dot{y}(t)}{\cosh(t)} - \frac{y(t)}{\cosh(t)^2}\sinh(t)$$

and

$$\begin{aligned}\ddot{y}(t) &= \ddot{u}(t)\cosh(t) + 2\dot{u}(t)\sinh(t) + u(t)\cosh(t)\\ &= \ddot{u}(t)\cosh(t) + 2\dot{u}(t)\sinh(t) + y(t)\\ &\le y(t)\end{aligned}$$

for $t \in (0,T)\setminus\{t_1,\dots,t_n\}$. Consequently,

$$\frac{d}{dt}\left(\dot{u}(t)\cosh(t)^2\right) = \cosh(t)\left(\ddot{u}(t)\cosh(t) + 2\dot{u}(t)\sinh(t)\right) \le 0 \tag{3.9}$$

for $t \in (0,T)\setminus\{t_1,\dots,t_n\}$.

In view of (3.9),

$$\dot{u}(t)\cosh(t)^2 \le \dot{u}(0+) = \dot{y}(0+)$$

for $t \in (0,t_1)$. Multiplying through by $\operatorname{sech}(t)^2$ and integrating from 0 to t gives

$$u(t) \le y(0) + \dot{y}(0+)\tanh(t), \quad t \in [0,t_1].$$

That is,

$$y(t) = \cosh(t)u(t) \le \cosh(t)y(0) + \sinh(t)\dot{y}(0+) \tag{3.10}$$

for $t \in [0,t_1]$.

By (3.8) and (3.9), we likewise have

$$\dot{u}(t)\cosh(t)^2 \le \dot{u}(t_1+)\cosh(t_1)^2 \le \dot{u}(t_1-)\cosh(t_1)^2 \le u(0+) = \dot{y}(0+)$$

for $t \in (t_1,t_2)$. Again we multiply through by $\operatorname{sech}(t)^2$ and integrate from t_1 to $t \in (t_1,t_2)$ to get

$$\begin{aligned}u(t) &\le u(t_1) + \dot{y}(0+)(\tanh(t) - \tanh(t_1))\\ &\le y(0) + \dot{y}(0+)\tanh(t_1) + \dot{y}(0+)(\tanh(t) - \tanh(t_1))\\ &\le y(0) + \dot{y}(0+)\tanh(t).\end{aligned}$$

In particular, (3.10) holds for $t \in [t_1,t_2]$. We can argue similarly to show that (3.10) also holds on the intervals $[t_2,t_3],\dots,[t_n,T)$. □

This leads to a stability estimate.

Proposition 3.8. *Suppose $i,j \in \{1,\dots,N\}$, $x_i \ge x_j$, and $t \ge 0$. Then*

$$\gamma_i(t) - \gamma_j(t) \le \cosh(\sqrt{c}t)(x_i - x_j) + \frac{1}{\sqrt{c}}\sinh(\sqrt{c}t)\int_{x_j}^{x_i}|v_0'(x)|dx.$$

Proof. Without loss of generality, we may assume $x_1 \le \dots \le x_N$ so that the sticky particle trajectories are ordered $\gamma_1 \le \dots \le \gamma_N$. Under this assumption, it suffices to verify

$$\gamma_{i+1}(t) - \gamma_i(t) \le \cosh(\sqrt{c}t)(x_{i+1} - x_i) + \frac{1}{\sqrt{c}}\sinh(\sqrt{c}t)|v_{i+1} - v_i| \tag{3.11}$$

for $t \geq 0$. For if $j, k \in \{1, \dots, N\}$ with $k > j$,

$$\begin{aligned}\gamma_k(t) - \gamma_j(t) &= \sum_{i=j}^{k-1}(\gamma_{i+1}(t) - \gamma_i(t)) \\ &\leq \sum_{i=j}^{k-1}\left(\cosh(\sqrt{c}t)(x_{i+1} - x_i) + \frac{1}{\sqrt{c}}\sinh(\sqrt{c}t)|v_{i+1} - v_i|\right) \\ &= \cosh(\sqrt{c}t)(x_k - x_j) + \frac{1}{\sqrt{c}}\sinh(\sqrt{c}t)\sum_{i=j}^{k-1}|v_{i+1} - v_i| \\ &\leq \cosh(\sqrt{c}t)(x_k - x_j) + \frac{1}{\sqrt{c}}\sinh(\sqrt{c}t)\sum_{i=j}^{k-1}\int_{x_i}^{x_{i+1}}|v_0'(x)|dx \\ &= \cosh(\sqrt{c}t)(x_k - x_j) + \frac{1}{\sqrt{c}}\sinh(\sqrt{c}t)\int_{x_j}^{x_k}|v_0'(x)|dx.\end{aligned}$$

To this end, we fix $i \in \{1, \dots, N\}$ and set

$$T := \inf\{t \geq 0 : \gamma_{i+1}(t) - \gamma_i(t) = 0\}.$$

In order to verify (3.11), it is enough to show

$$\begin{aligned}\gamma_{i+1}(t) - \gamma_i(t) &\leq \cosh(\sqrt{c}t)(x_{i+1} - x_i) \\ &\quad + \frac{1}{\sqrt{c}}\sinh(\sqrt{c}t)|v_{i+1} - v_i|, \quad t \in [0, T].\end{aligned} \tag{3.12}$$

We will do so by applying the previous lemma to the restriction of the function

$$y(t) := \gamma_{i+1}(t) - \gamma_i(t), \quad t \geq 0,$$

to $[0, T)$. In particular, we note that $y(t) = 0$ for $t \geq T$ whenever T is finite.

We first claim

$$\dot{\gamma}_{i+1}(s+) \leq \dot{\gamma}_{i+1}(s-). \tag{3.13}$$

Note that if γ_{i+1} does not have a first intersection time at $s \in (0, T)$, then γ_{i+1} is C^1 near s and so

$$\dot{\gamma}_{i+1}(s) = \dot{\gamma}_{i+1}(s+) = \dot{\gamma}_{i+1}(s-).$$

Alternatively let us suppose γ_{i+1} has a first intersection time at s. As a result, there are trajectories $\gamma_{i+2}, \dots, \gamma_{i+r}$ $(r \geq 2)$ such that

$$\gamma_{i+1}(s) = \gamma_{i+2}(s) = \cdots = \gamma_{i+r}(s)$$

and

$$\dot{\gamma}_{i+j}(s+) = \frac{m_{i+1}\dot{\gamma}_{i+1}(s-) + \cdots + m_{i+r}\dot{\gamma}_{i+r}(s-)}{m_{i+1} + \cdots + m_{i+r}}, \tag{3.14}$$

$j = 1, \dots, r$.

Also note that as $\gamma_{i+1} \leq \gamma_{i+j}$ for $j = 2, \dots, r$,

$$\frac{\gamma_{i+1}(s+h) - \gamma_{i+1}(s)}{h} \geq \frac{\gamma_{i+j}(s+h) - \gamma_{i+j}(s)}{h}$$

for all $h < 0$ small. By Remark 3.2, we can send $h \to 0^-$ and conclude

$$\dot{\gamma}_{i+1}(s-) \geq \dot{\gamma}_{i+j}(s-).$$

It then follows from (3.14) (with $j = 1$) that

$$\dot\gamma_{i+1}(s+) \le \frac{m_{i+1}\dot\gamma_{i+1}(s-) + \cdots + m_{i+r}\dot\gamma_{i+1}(s-)}{m_{i+1} + \cdots + m_{i+r}} = \dot\gamma_{i+1}(s-),$$

which is (3.13). A similar argument gives

$$\dot\gamma_i(s+) \ge \dot\gamma_i(s-) \tag{3.15}$$

for each $s \in (0, T)$. Combining (3.13) and (3.15),

$$\dot y(s+) = \dot\gamma_{i+1}(s+) - \dot\gamma_i(s+) \le \dot\gamma_{i+1}(s-) - \dot\gamma_i(s-) = \dot y(s-)$$

for all $s \in (0, T)$.

As $x \mapsto W'(x) + cx$ is nondecreasing,

$$\begin{aligned}\ddot y(t) &= \ddot\gamma_i(t) - \ddot\gamma_j(t)\\ &= -\sum_{k=1}^N m_k\left(W'(\gamma_i(t) - \gamma_k(t)) - W'(\gamma_j(t) - \gamma_k(t))\right)\\ &\le \sum_{k=1}^N m_k c\left(\gamma_i(t) - \gamma_j(t)\right)\\ &= c(\gamma_i(t) - \gamma_j(t))\\ &= y(t)\end{aligned}$$

for all but finitely many $t > 0$. Therefore, (3.12) follows from Lemma 3.7. $\square$

3.4. **Associated trajectory map.** We are now ready to show how to design a solution of (1.3) with ρ_0 given by (3.1). For $t \ge 0$, we define

$$X(t) : \{x_1, \dots, x_N\} \to \mathbb{R}; x_i \mapsto \gamma_i(t).$$

We will also write

$$X(x_i, t) = \gamma_i(t)$$

for $i = 1, \dots, N$ and $t \ge 0$. The following proposition details all the important features of X.

Proposition 3.9. *The mapping X has the following properties:*

(i) $X(0) = \mathrm{id}_{\mathbb{R}}$ *and*

$$\dot X(t) = \mathbb{E}_{\rho_0}\left[v_0 - \int_0^t (W' * \rho_s)(X(s))ds \middle| X(t)\right]$$

for all but finitely many $t \ge 0$. Both equalities hold on the support of ρ_0.

(ii) $E(t) \le E(s)$ *for $s \le t$. Here*

$$E(\tau) := \int_{\mathbb{R}} \frac{1}{2}\dot X(\tau+)^2 d\rho_0 + \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{1}{2} W(X(y,\tau) - X(z,\tau))d\rho_0(y)d\rho_0(z).$$

(iii) $X : [0,\infty) \to L^2(\rho_0); t \mapsto X(t)$ *is locally Lipschitz continuous.*

(iv) *For $t \ge 0$ and $y, z \in \mathrm{supp}(\rho_0)$ with $y \le z$,*

$$0 \le X(z,t) - X(y,t) \le \cosh(\sqrt{c}t)(z - y) + \frac{1}{\sqrt{c}}\sinh(\sqrt{c}t)\int_y^z |v_0'(x)|dx.$$

(v) *For each $0 < s \le t$ and $y, z \in \mathrm{supp}(\rho_0)$,*

$$\frac{|X(y,t) - X(z,t)|}{\sinh(\sqrt{c}t)} \le \frac{|X(y,s) - X(z,s)|}{\sinh(\sqrt{c}s)}.$$

(vi) *For each* $0 < s \le t$, *there is a function* $f_{t,s} : \mathbb{R} \to \mathbb{R}$ *which satisfies the Lipschitz condition* (3.3) *and*

$$X(y,t) = f_{t,s}(X(y,s))$$

for $y \in \operatorname{supp}(\rho_0)$.

Proof. Part (i): As $X(x_i, 0) = x_i$,

$$X(0) = \mathrm{id}_{\mathbb{R}}$$

on $\operatorname{supp}(\rho_0)$. Also note that if $g : \mathbb{R} \to \mathbb{R}$ and $t \ge 0$, then Corollary 3.3 gives

$$\begin{aligned}
\int_{\mathbb{R}} g(X(t))\dot{X}(t+)d\rho_0 &= \sum_{i=1}^{N} m_i g(\gamma_i(t))\dot{\gamma}_i(t+) \\
&= \sum_{i=1}^{N} m_i g(\gamma_i(t)) \left[\dot{\gamma}_i(0+) - \int_0^t \left(\sum_{j=1}^{N} m_j W'(\gamma_i(\tau) - \gamma_j(\tau)) \right) d\tau \right] \\
&= \int_{\mathbb{R}} g(X(t)) \left[v_0 - \int_0^t (W' * \rho_\tau)(X(\tau)) d\tau \right] d\rho_0.
\end{aligned}$$

In particular,

$$\int_{\mathbb{R}} g(X(t))\dot{X}(t)d\rho_0 = \int_{\mathbb{R}} g(X(t)) \left[v_0 - \int_0^t (W' * \rho_\tau)(X(\tau)) d\tau \right] d\rho_0$$

for all but finitely many $t \ge 0$.

Define

$$\begin{cases} v(x,t) = \dot{\gamma}_i(t+), & x = \gamma_i(t), \\ \quad\quad\;\; = 0, & \text{otherwise}. \end{cases}$$

By parts (iii) and (iv) of Proposition 3.1, v is well defined. Moreover, it is routine to check that $v : \mathbb{R} \times [0, \infty) \to \mathbb{R}$ is Borel measurable. Furthermore,

$$\dot{X}(t) = v(X(t), t)$$

on the support of ρ_0 for all but finitely many $t \ge 0$. It follows that X satisfies the pressureless Euler flow equation (1.3) for all but finitely many $t \ge 0$.

Part (ii): In view of (3.4),

$$\begin{aligned}
E(t) &= \int_{\mathbb{R}} \frac{1}{2}\dot{X}(t+)^2 d\rho_0 + \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{1}{2} W(X(y,t) - X(z,t)) d\rho_0(y) d\rho_0(z) \\
&= \frac{1}{2} \sum_{i=1}^{N} m_i \dot{\gamma}_i(t+)^2 + \frac{1}{2} \sum_{i,j=1}^{N} m_i m_j W(\gamma_i(t) - \gamma_j(t)) \\
&\le \frac{1}{2} \sum_{i=1}^{N} m_i \dot{\gamma}_i(s+)^2 + \frac{1}{2} \sum_{i,j=1}^{N} m_i m_j W(\gamma_i(s) - \gamma_j(s)) \\
&= \int_{\mathbb{R}} \frac{1}{2}\dot{X}(t+)^2 d\rho_0 + \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{1}{2} W(X(y,t) - X(z,t)) d\rho_0(y) d\rho_0(z) \\
&= E(s).
\end{aligned}$$

Part (iii): By the energy estimate (3.6),

(3.16)

$$\begin{aligned}\int_{\mathbb{R}}(X(t)-X(s))^2 d\rho_0 &\le (t-s)\int_s^t\int_{\mathbb{R}}\dot X(\tau)^2 d\rho_0 d\tau\\ &= (t-s)\int_s^t\sum_{i=1}^N m_i\dot\gamma_i(\tau)^2 d\tau\\ &\le (t-s)(\vartheta(t)-\vartheta(s))\left(\sum_{i=1}^N m_i v_i^2+\frac{1}{2}\sum_{i,j=1}^N m_i m_j W'(x_i-x_j)^2\right)\\ &= (t-s)(\vartheta(t)-\vartheta(s))\left(\int_{\mathbb{R}} v_0^2 d\rho_0+\frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} W'(x-y)^2 d\rho_0(x)d\rho_0(y)\right).\end{aligned}$$

Since ϑ is smooth, $X:[0,\infty)\to L^2(\rho_0)$ is locally Lipschitz continuous.

Parts (iv), (v), and (vi): Part (iv) follows from Proposition 3.8, part (v) is a corollary of Proposition 3.4, and part (vi) is due to Corollary 3.5. □

Observe that as $v_0:\mathbb{R}\to\mathbb{R}$ is absolutely continuous,

$$\omega(r) := \sup\left\{\int_a^b |v_0'(x)|dx \;:\; 0\le b-a\le r\right\} \tag{3.17}$$

tends to 0 as $r\to 0^+$ and is sublinear. By part (iv) of the above proposition,

$$|X(y,t)-X(z,t)|\le \cosh(\sqrt{c}t)|y-z|+\frac{1}{\sqrt{c}}\sinh(\sqrt{c}t)\omega(|y-z|) \tag{3.18}$$

for y,z belonging to the support of ρ_0. As a result, $X(t)$ is uniformly continuous on the support of ρ_0. In particular, we may extend $X(t)$ to a uniformly continuous function on $\mathbb{R}$ which satisfies (3.18) and agrees with $X(t)$ on the support of ρ_0. Without any loss of generality, we will identify $X(t)$ with this extension and assume $X(t)$ is uniformly continuous on $\mathbb{R}$.

4. Existence of solutions

Now let $\rho_0\in\mathcal{P}(\mathbb{R}^2)$ with

$$\int_{\mathbb{R}} x^2 d\rho_0(x)<\infty.$$

We can select a sequence $(\rho_0^k)_{k\in\mathbb{N}}\subset\mathcal{P}(\mathbb{R}^2)$ such that each ρ_0^k is a convex combination of Dirac measures, $\rho_0^k\to\rho_0$ narrowly, and

$$\lim_{k\to\infty}\int_{\mathbb{R}} x^2 d\rho_0^k(x)=\int_{\mathbb{R}} x^2 d\rho_0(x). \tag{4.1}$$

We recommend, for instance, the reference [3] for a discussion on how to design such a sequence.

In view of Proposition 3.9, there is a locally Lipschitz continuous mapping $X^k : [0,\infty) \to L^2(\rho_0^k)$ which satisfies

$$\begin{cases} \dot{X}^k(t) = \mathbb{E}_{\rho_0^k}\left[v_0 - \int_0^t (W' * \rho_s^k)(X^k(s))ds \middle| X^k(t)\right] & \text{a.e. } t \geq 0, \\ X^k(0) = \mathrm{id}_{\mathbb{R}} \end{cases}$$

for each $k \in \mathbb{N}$. Here

$$\rho_t^k := X^k(t)_{\#}\rho_0^k, \quad t \geq 0.$$

In this section, we will show that there is a subsequence of $(X^k)_{k\in\mathbb{N}}$ which converges in an appropriate sense to a solution of the pressureless Euler flow equation (1.3) and satisfies (1.4). Then we will show how to use this solution to generate a corresponding weak solution of the pressureless Euler equations.

4.1. **Compactness.**

We will prove that $(X^k)_{k\in\mathbb{N}}$ has a subsequence which converges in a strong sense. The limit mapping will be our candidate for a solution of the pressureless Euler flow equation.

Proposition 4.1. *There is a subsequence $(X^{k_j})_{j\in\mathbb{N}}$ and a locally Lipschitz $X : [0,\infty) \to L^2(\rho_0)$ such that*

$$\lim_{j\to\infty} \int_{\mathbb{R}} h(\mathrm{id}_{\mathbb{R}}, X^{k_j}(t))d\rho_0^{k_j} = \int_{\mathbb{R}} h(\mathrm{id}_{\mathbb{R}}, X(t))d\rho_0 \tag{4.2}$$

for each $t \geq 0$ and continuous $h : \mathbb{R}^2 \to \mathbb{R}$ with

$$\sup_{(x,y)\in\mathbb{R}^2} \frac{|h(x,y)|}{1+x^2+y^2} < \infty.$$

Furthermore, X has the following properties:

(i) *For $t \geq 0$ and $y, z \in \mathrm{supp}(\rho_0)$ with $y \leq z$,*

$$0 \leq X(z,t) - X(y,t) \leq \cosh(\sqrt{c}t)(z-y) + \frac{1}{\sqrt{c}}\sinh(\sqrt{c}t)\int_y^z |v_0'(x)|dx.$$

(ii) *For each $0 < s \leq t$ and $y, z \in \mathrm{supp}(\rho_0)$,*

$$\frac{|X(y,t) - X(z,t)|}{\sinh(\sqrt{c}t)} \leq \frac{|X(y,s) - X(z,s)|}{\sinh(\sqrt{c}s)}.$$

(iii) *For each $0 < s \leq t$, there is a function $f_{t,s} : \mathbb{R} \to \mathbb{R}$ which satisfies the Lipschitz condition (3.3) and*

$$X(y,t) = f_{t,s}(X(y,s))$$

for $y \in \mathrm{supp}(\rho_0)$.

Proof. 1. Weak convergence: Define

$$\sigma_t^k := (\mathrm{id}_{\mathbb{R}}, X^k(t))_{\#}\rho_0^k$$

for $t \geq 0$ and $k \in \mathbb{N}$. By (3.16),

$$\left(\int_{\mathbb{R}} (X^k(t))^2 d\rho_0^k\right)^{1/2} \leq \left(\int_{\mathbb{R}} (X^k(t) - X^k(0))^2 d\rho_0^k\right)^{1/2} + \left(\int_{\mathbb{R}} (X^k(0))^2 d\rho_0^k\right)^{1/2}$$
$$\leq \sqrt{t\vartheta(t)} \left(\int_{\mathbb{R}} v_0^2 d\rho_0^k + \frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} W'(x-y)^2 d\rho_0^k(x) d\rho_0^k(y)\right)^{1/2}$$
$$+ \left(\int_{\mathbb{R}} x^2 d\rho_0^k(x)\right)^{1/2}.$$

And as v_0 and W' grow at most linearly, there are constants $A, B \geq 0$

$$\left(\int_{\mathbb{R}} (X^k(t))^2 d\rho_0^k\right)^{1/2} \leq A\sqrt{t\vartheta(t)} + B$$

for each $t \geq 0$ and $k \in \mathbb{N}$.

It follows that

$$\sup_{k\in\mathbb{N}} \iint_{\mathbb{R}^2} (x^2 + y^2) d\sigma_t^k(x,y) < \infty \tag{4.3}$$

for each $t \geq 0$. As a result, $(\sigma_t^k)_{k\in\mathbb{N}}$ is narrowly precompact. Moreover, for Lipschitz continuous $h : \mathbb{R}^2 \to \mathbb{R}$,

$$\iint_{\mathbb{R}^2} h(x,y) d\sigma_t^k(x,y) - \iint_{\mathbb{R}^2} h(x,y) d\sigma_s^k(x,y)$$
$$= \int_{\mathbb{R}} h(\mathrm{id}_{\mathbb{R}}, X^k(t)) d\rho_0^k - \int_{\mathbb{R}} h(\mathrm{id}_{\mathbb{R}}, X^k(s)) d\rho_0^k$$
$$\leq \mathrm{Lip}(h) \int_{\mathbb{R}} |X^k(t) - X^k(s)| d\rho_0^k$$
$$\leq \mathrm{Lip}(h) \int_s^t \left(\int_{\mathbb{R}} |\dot{X}^k(\tau)| d\rho_0^k\right) d\tau$$
$$\leq \mathrm{Lip}(h) \int_s^t (A\sqrt{\tau\vartheta(\tau)} + B) d\tau.$$

In terms of the metric (2.2), we then have

$$d(\sigma_t^k, \sigma_s^k) \leq \int_s^t (A\sqrt{\tau\vartheta(\tau)} + B) d\tau$$

for $k \in \mathbb{N}$ and $0 \leq s \leq t$.

By the Arzelà-Ascoli theorem, there is a subsequence $(\sigma^{k_j})_{j\in\mathbb{N}}$ and a narrowly continuous path $\sigma : [0,\infty) \to \mathcal{P}(\mathbb{R}^2); t \mapsto \sigma_t$ such that $\sigma_t^{k_j} \to \sigma_t$ uniformly for t belonging to compact subsets of $[0,\infty)$. We can also use this narrow convergence and (4.3) to show

$$\lim_{j\to\infty} \iint_{\mathbb{R}^2} |y| d\sigma_t^{k_j}(x,y) = \iint_{\mathbb{R}^2} |y| d\sigma_t(x,y)$$

for each $t \geq 0$. Further,

$$\int_{\mathbb{R}} \phi d\rho_0 = \lim_{j\to\infty} \int_{\mathbb{R}} \phi d\rho_0^{k_j} = \lim_{j\to\infty} \iint_{\mathbb{R}^2} \phi(x) d\sigma_t^{k_j}(x,y) = \iint_{\mathbb{R}^2} \phi(x) d\sigma_t(x,y)$$

for $\phi \in C_b(\mathbb{R})$.

By the disintegration theorem (Theorem 5.3.1 of [1]), there is a family of probability measures $(\zeta_t^x)_{x\in\mathbb{R}}$ for which

$$\iint_{\mathbb{R}^2} h(x,y)d\sigma_t(x,y) = \int_{\mathbb{R}}\left(\int_{\mathbb{R}} h(x,y)d\zeta_t^x(y)\right)d\rho_0(x).$$

Set

$$X(x,t) := \int_{\mathbb{R}} y d\zeta_t^x(y), \quad (x,t)\in\mathbb{R}\times[0,\infty),$$

and as before we will write $X(t):\mathbb{R}\to\mathbb{R}; x\mapsto X(x,t)$. Observe

$$\begin{aligned}
\lim_{j\to\infty}\int_{\mathbb{R}}\phi X^{k_j}(t)d\rho_0^{k_j} &= \lim_{j\to\infty}\int_{\mathbb{R}^2}\phi(x)yd\sigma_t^{k_j}(x,y)\\
&= \int_{\mathbb{R}^2}\phi(x)yd\sigma_t(x,y)\\
&= \int_{\mathbb{R}}\left(\int_{\mathbb{R}}\phi(x)yd\zeta_t^x(y)\right)d\rho_0(x)\\
&= \int_{\mathbb{R}}\phi(x)\left(\int_{\mathbb{R}}yd\zeta_t^x(y)\right)d\rho_0(x)\\
&= \int_{\mathbb{R}}\phi X(t)d\rho_0
\end{aligned} \tag{4.4}$$

for each $t\geq 0$.

2. Strong convergence: By (3.18),

$$|X^k(y,t)-X^k(z,t)| \leq \cosh(\sqrt{c}t)|y-z| + \frac{1}{\sqrt{c}}\sinh(\sqrt{c}t)\omega(|y-z|)$$

for $y,z\in\mathbb{R}$. As $\lim_{r\to 0^+}\omega(r)=0$, $(X^k(t))_{k\in\mathbb{N}}$ is uniformly equicontinuous. Moreover,

$$|X^k(y,t)| \leq |X^k(y,t)-X^k(z,t)| + |X^k(z,t)|$$

and integrating over $z\in\mathbb{R}$ gives

$$\begin{aligned}
|X^k(y,t)| &\leq \int_{\mathbb{R}}|X^k(y,t)-X^k(z,t)|d\rho_0^k(z) + \int_{\mathbb{R}}|X^k(z,t)|d\rho_0^k(z)\\
&\leq \cosh(\sqrt{c}t)\int_{\mathbb{R}}|y-z|d\rho_0^k(z) + \frac{1}{\sqrt{c}}\sinh(\sqrt{c}t)\int_{\mathbb{R}}\omega(|y-z|)d\rho_0^k(z)\\
&\quad + \int_{\mathbb{R}}|X^k(z,t)|d\rho_0^k(z)\\
&\leq \cosh(\sqrt{c}t)\int_{\mathbb{R}}|y-z|d\rho_0^k(z) + \frac{1}{\sqrt{c}}\sinh(\sqrt{c}t)\int_{\mathbb{R}}\omega(|y-z|)d\rho_0^k(z)\\
&\quad + A\sqrt{t\vartheta(t)} + B.
\end{aligned}$$

By (4.1) and the at most sublinear growth of ω, there are constants $a,b\geq 1$ such that

$$|X^k(y,t)| \leq ae^{bt^2}(|y|+1) \tag{4.5}$$

for $k\in\mathbb{N}$, $y\in\mathbb{R}$, and $t\geq 0$.

It follows that a subsequence of $(X^{k_j}(t))_{j\in\mathbb{N}}$ (which we will not relabel) converges locally uniformly on $\mathbb{R}$ to a continuous function $Y:\mathbb{R}\to\mathbb{R}$. It is easy to check

$$\lim_{j\to\infty}\int_{\mathbb{R}}\phi X^{k_j}(t)d\rho_0^{k_j} = \int_{\mathbb{R}}\phi Y d\rho_0$$

for each $\phi \in C_b(\mathbb{R})$. So if $(X^{k_j}(t))_{j\in\mathbb{N}}$ has another subsequence which converges locally uniformly on $\mathbb{R}$ to Z, then

$$\int_{\mathbb{R}} \phi Y d\rho_0 = \int_{\mathbb{R}} \phi Z d\rho_0.$$

In particular, $Y = Z$ ρ_0 almost everywhere. Since Y and Z are continuous, it is routine to check that $Y = Z$ on the entire support of ρ_0.

By (4.4), we also have that $X(t) = Y$ ρ_0 almost everywhere. Without loss of generality, we can redefine $X(t) = Y$ to ensure that $X^{k_j}(t) \to X(t)$ locally uniformly on $\mathbb{R}$. By estimate (4.5) and Lemma 2.1, we also have

$$\lim_{j\to\infty} \int_{\mathbb{R}} X^{k_j}(t)^2 d\rho_0^{k_j} = \int_{\mathbb{R}} X(t)^2 d\rho_0.$$

In particular,

$$\lim_{j\to\infty} \iint_{\mathbb{R}^2} (x^2+y^2) d\sigma_t^{k_j}(x,y) = \iint_{\mathbb{R}^2} (x^2+y^2) d\sigma_t(x,y),$$

which when combined the narrow converge $\sigma_t^{k_j} \to \sigma_t$ in $\mathcal{P}(\mathbb{R}^2)$ gives (4.2).

In view of (3.16), we also have

$$\begin{aligned}\int_{\mathbb{R}} (X(t)-X(s))^2 d\rho_0 &= \lim_{j\to\infty} \int_{\mathbb{R}} (X^{k_j}(t)-X^{k_j}(s))^2 d\rho_0^{k_j} \\ &\le (t-s)(\vartheta(t)-\vartheta(s)) \lim_{j\to\infty} \left(\int_{\mathbb{R}} v_0^2 d\rho_0^{k_j} + \frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} W'(x-y)^2 d\rho_0^{k_j}(x) d\rho_0^{k_j}(y) \right) \\ &\le (t-s)(\vartheta(t)-\vartheta(s)) \left(\int_{\mathbb{R}} v_0^2 d\rho_0 + \frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} W'(x-y)^2 d\rho_0(x) d\rho_0(y) \right).\end{aligned}$$

Here we used that v_0 and W' are continuous and grow at most linearly. Therefore, the mapping $X : [0,\infty) \to L^2(\rho_0)$ is locally Lipschitz continuous.

3. Properties of the limit: Suppose $y, z \in \text{supp}(\rho_0)$ with $y < z$. As $\rho^{k_j} \to \rho_0$ narrowly in $\mathcal{P}(\mathbb{R}^2)$, there are sequences $y^j, z^j \in \text{supp}(\rho_0^{k_j})$ such that $y^j \to y$ and $z^j \to z$ (Proposition 5.1.8 in [1]). Without any loss of generality, we may suppose that $y^j < z^j$ for all $j \in \mathbb{N}$ as this occurs for all j large enough. By part (iv) of Proposition 3.9,

$$0 \le X^{k_j}(z^j,t) - X^{k_j}(y^j,t) \le \cosh(\sqrt{c}t)(z^j-y^j) + \frac{1}{\sqrt{c}}\sinh(\sqrt{c}t)\int_{y^j}^{z^j} |v_0'(x)| dx.$$

Since $X^{k^j}(t) \to X(t)$ locally uniformly, we can send $j \to \infty$ in order to conclude property (i) of this theorem. Property (ii) can be proved similarly.

Now let $0 < s \le t$ and $y \in \text{supp}(\rho_0)$. As above, we may select $y^j \in \text{supp}(\rho_0^{k_j})$ such that $y^j \to y$. Appealing to part (vi) of Proposition 3.9, there is a sequence of functions $(f_{t,s}^{k_j})_{j\in\mathbb{N}}$ in which each element satisfies the Lipschitz condition (3.3) and

$$X^{k_j}(y^j,t) = f_{t,s}^{k_j}(X^{k_j}(y^j,s)) \tag{4.6}$$

for each $j \in \mathbb{N}$. In particular,

$$\begin{aligned}|f_{t,s}^{k_j}(x)| &\le |f_{t,s}^{k_j}(x) - f_{t,s}^{k_j}(X^{k_j}(y^j,s))| + |f_{t,s}^{k_j}(X^{k_j}(y^j,s))| \\ &\le \frac{\sinh(\sqrt{c}t)}{\sinh(\sqrt{c}s)}|x - X^{k_j}(y^j,s)| + |X^{k_j}(y^j,t)|.\end{aligned}$$

As $X^{k^j}(t) \to X(t)$ and $X^{k^j}(s) \to X(s)$ locally uniformly, $|f^{k_j}_{t,s}(x)|$ is bounded above for x belonging to compact subsets of $\mathbb{R}$ independently of $j \in \mathbb{N}$. As a result, $(f^{k_j}_{t,s})_{j\in\mathbb{N}}$ has a locally uniformly convergent subsequence. Therefore, we can send $j \to \infty$ along an appropriate subsequence in (4.6) to obtain part (iii). □

For the remainder of this section, let X denote the mapping obtained in Proposition 4.1.

Corollary 4.2. *For Lebesgue almost every $t > 0$, there is a Borel function $u : \mathbb{R} \to \mathbb{R}$ for which*

$$\dot{X}(t) = u(X(t))$$

ρ_0 almost everywhere.

Proof. Let t be a time such that

$$\dot{X}(t) = \lim_{n\to\infty} n\left(X(t + 1/n) - X(t)\right) \tag{4.7}$$

in $L^2(\rho_0)$. We recall that since $X : [0,\infty) \to L^2(\rho_0)$ is locally Lipschitz, the set of all such t has full measure in $[0,\infty)$. Moreover, we may suppose that the limit (4.7) holds ρ_0 almost everywhere in $\mathbb{R}$ as it does for a subsequence. By Theorem 1.19 in [11], we may further assume that the limit (4.7) holds everywhere on some Borel $S \subset \mathbb{R}$ with $\rho_0(S) = 1$.

In view of part (iii) of Proposition 4.1,

$$\dot{X}(t) = \lim_{n\to\infty} u_n\left(X(t)\right)$$

on S. Here

$$u_n := n(f_{t+1/n,t} - \mathrm{id}_{\mathbb{R}})$$

is a Borel measurable function for each $n \in \mathbb{N}$. Consequently, $u_n\left(X(t)\right)|_S$ is measurable with respect to the Borel sigma-sub algebra generated by $X(t)|_S$,

$$\mathcal{F} := \{\{y \in S : X(y,t) \in B\} : B \subset \mathbb{R} \text{ Borel}\}.$$

As $\dot{X}(t)|_S$ is a pointwise limit of $\mathcal{F}$ measurable functions, it is $\mathcal{F}$ measurable itself (Proposition 2.7 in [11]). It follows that there is a Borel function $u : \mathbb{R} \to \mathbb{R}$ such that

$$\dot{X}(t)|_S = u(X(t))|_S.$$

That is, $\dot{X}(t) = u(X(t))$ ρ_0 almost everywhere. □

Proof of Theorem 1.1. 1. Initial condition: The limit (4.2) taken when $t = 0$ implies $X(0) = \mathrm{id}_{\mathbb{R}}$ ρ_0 almost everywhere.

2. Flow equation: We next claim

$$\int_s^t \int_{\mathbb{R}} \dot{X}(\tau)h(X(\tau))d\rho_0 d\tau = \int_s^t \int_{\mathbb{R}} \left[v_0 - \int_0^\tau (W' * \rho_\xi)(X(\xi))d\xi\right] h(X(t))d\rho_0 d\tau \tag{4.8}$$

for each continuous $h : \mathbb{R} \to \mathbb{R}$ which satisfies

$$\sup_{x\in\mathbb{R}} \frac{|h(x)|}{1 + |x|} < \infty$$

and each $0 \le s \le t$. In view of Corollary 4.2, this would imply that X is a solution of the pressureless Euler flow equation. To this end, we set $F(y) := \int_0^y h(x)dx$ for $y \in \mathbb{R}$ and note that F grows at most quadratically. By Proposition 4.1,

$$\begin{aligned}
\lim_{j\to\infty}\int_s^t\int_{\mathbb{R}}\dot{X}^{k_j}(\tau)h(X^{k_j}(\tau))d\rho_0^{k_j}d\tau &= \lim_{j\to\infty}\int_{\mathbb{R}}\left(\int_s^t \dot{X}^{k_j}(\tau)h(X^{k_j}(\tau))d\tau\right)d\rho_0^{k_j}\\
&= \lim_{j\to\infty}\int_{\mathbb{R}}\left(\int_s^t \frac{d}{d\tau}F(X^{k_j}(\tau))d\tau\right)d\rho_0^{k_j}\\
&= \lim_{j\to\infty}\int_{\mathbb{R}}\left(F(X^{k_j}(t)) - F(X^{k_j}(s))\right)d\rho_0^{k_j}\\
&= \int_{\mathbb{R}}\left(F(X(t)) - F(X(s))\right)d\rho_0.
\end{aligned}$$

Consequently,

$$\lim_{j\to\infty}\int_s^t\int_{\mathbb{R}}\dot{X}^{k_j}(\tau)h(X^{k_j}(\tau))d\rho_0^{k_j}d\tau = \int_s^t\int_{\mathbb{R}}\dot{X}(\tau)h(X(\tau))d\rho_0 d\tau. \tag{4.9}$$

Let us fix $\tau \ge 0$ for the moment and consider the integral

$$\begin{aligned}
&\int_{\mathbb{R}}\left[v_0 - \int_0^\tau (W' * \rho_\xi^{k_j})(X^{k_j}(\xi))d\xi\right]h(X^{k_j}(\tau))d\rho_0^{k_j}\\
&\quad= \int_{\mathbb{R}} v_0 h(X^{k_j}(\tau))d\rho_0^{k_j}\\
&\qquad - \int_0^\tau\left[\int_{\mathbb{R}}\int_{\mathbb{R}} W'(X^{k_j}(y,\xi) - X^{k_j}(z,\xi))h(X^{k_j}(y,\tau))d\rho_0^{k_j}(z)d\rho_0^{k_j}(y)\right]d\xi.
\end{aligned}$$

In view of Proposition 4.1 and the at most linear growth of v_0,

$$\lim_{j\to\infty}\int_{\mathbb{R}} v_0 h(X^{k_j}(\tau))d\rho_0^{k_j} = \int_{\mathbb{R}} v_0 h(X(\tau))d\rho_0$$

as $j \to \infty$. Also observe

$$\lim_{j\to\infty} W'(X^{k_j}(y,\xi) - X^{k_j}(z,\xi))h(X^{k_j}(y,t)) = W'(X(y,\xi) - X(z,\xi))h(X(y,\tau))$$

locally uniformly for $(y,z) \in \mathbb{R}^2$ and each fixed $\xi \in [0,\tau]$.

Choosing C so large that

$$|h(x)| + |W'(x)| \le C(1+|x|) \quad (x \in \mathbb{R})$$

gives

$$\begin{aligned}
&|W'(X^{k_j}(y,\xi) - X^{k_j}(z,\xi))h(X^{k_j}(y,\tau))|\\
&\quad\le C(1+|X^{k_j}(y,\xi)| + |X^{k_j}(z,\xi)|)\cdot C(1+|X^{k_j}(y,\tau)|)\\
&\quad\le C(1+ae^{b\xi^2}(|y|+1) + ae^{b\xi^2}(|z|+1))\cdot C(1+ae^{b\tau^2}(|y|+1))\\
&\quad\le C^2(1+ae^{b\tau^2}(|y|+1) + ae^{b\tau^2}(|z|+1))^2\\
&\quad\le (Cae^{b\tau^2})^2(3+|y|+|z|)^2\\
&\quad\le (Cae^{b\tau^2})^2 2(9+y^2+z^2)\\
&\quad\le 18(Cae^{b\tau^2})^2(1+y^2+z^2)
\end{aligned} \tag{4.10}$$

for $j \in \mathbb{N}$ and $\xi \in [0,\tau]$. Here a, b are the constants from inequality (4.5).

We can then appeal to Lemma 2.1 to conclude

$$\lim_{j\to\infty}\int_{\mathbb{R}}\int_{\mathbb{R}} W'(X^{k_j}(y,\xi)-X^{k_j}(z,\xi))h(X^{k_j}(y,\tau))d\rho_0^{k_j}(z)d\rho_0^{k_j}(y) = \int_{\mathbb{R}}\int_{\mathbb{R}} W'(X(y,\xi)-X(z,\xi))h(X(y,\tau))d\rho_0(z)d\rho_0(y) \tag{4.11}$$

for each $\xi\in[0,\tau]$. What's more, (4.10) implies

$$\left|\int_{\mathbb{R}}\int_{\mathbb{R}} W'(X^{k_j}(y,\xi)-X^{k_j}(z,\xi))h(X^{k_j}(y,\tau))d\rho_0^{k_j}(z)d\rho_0^{k_j}(y)\right| \le 18(Cae^{b\tau^2})^2\left(1+2\int_{\mathbb{R}} y^2 d\rho_0^{k_j}(y)\right)$$

for $\xi\in[0,\tau]$. The limit (4.1) and a standard variant of dominated convergence (Theorem 1.20 in [10]) together give

$$\lim_{j\to\infty}\int_0^\tau\int_{\mathbb{R}}\int_{\mathbb{R}} W'(X^{k_j}(y,\xi)-X^{k_j}(z,\xi))h(X^{k_j}(y,\tau))d\rho_0^{k_j}(z)d\rho_0^{k_j}(y)d\xi = \int_0^\tau\int_{\mathbb{R}}\int_{\mathbb{R}} W'(X(y,\xi)-X(z,\xi))h(X(y,\tau))d\rho_0(z)d\rho_0(y)d\xi.$$

As a result,

$$\lim_{j\to\infty}\int_{\mathbb{R}}\left[v_0-\int_0^\tau (W'*\rho_\xi^{k_j})(X^{k_j}(\xi))d\xi\right]h(X^{k_j}(\tau))d\rho_0^{k_j} = \int_{\mathbb{R}}\left[v_0-\int_0^\tau (W'*\rho_\xi)(X(\xi))d\xi\right]h(X(\tau))d\rho_0. \tag{4.12}$$

Since

$$\left|\int_{\mathbb{R}}\left[v_0-\int_0^\tau (W'*\rho_\xi^{k_j})(X^{k_j}(\xi))d\xi\right]h(X^{k_j}(\tau))d\rho_0^{k_j}\right| \le \int_{\mathbb{R}}|h(X^{k_j}(\tau))||v_0|d\rho_0^{k_j}+\tau\cdot 18(Cae^{b\tau^2})^2\left(1+2\int_{\mathbb{R}} y^2 d\rho_0^{k_j}(y)\right),$$

and the integrals $\int_{\mathbb{R}}|h(X^{k_j}(\tau))||v_0|d\rho_0^{k_j}$, $\int_{\mathbb{R}} y^2 d\rho_0^{k_j}(y)$ are bounded and converge for each $\tau\in[s,t]$ as $j\to\infty$, we likewise have

$$\lim_{j\to\infty}\int_s^t\int_{\mathbb{R}}\left[v_0-\int_0^\tau (W'*\rho_\xi^{k_j})(X^{k_j}(\xi))d\xi\right]h(X^{k_j}(\tau))d\rho_0^{k_j}d\tau = \int_s^t\int_{\mathbb{R}}\left[v_0-\int_0^\tau (W'*\rho_\xi)(X(\tau))d\xi\right]h(X(\tau))d\rho_0 d\tau.$$

The claim (4.8) then follows by sending $j\to\infty$ in

$$\int_s^t\int_{\mathbb{R}}\dot{X}^{k_j}(\tau)h(X^{k_j}(\tau))d\rho_0^{k_j}d\tau = \int_s^t\int_{\mathbb{R}}\left[v_0-\int_0^\tau (W'*\rho_\xi^{k_j})(X^{k_j}(\xi))d\xi\right]h(X^{k_j}(\tau))d\rho_0^{k_j}d\tau.$$

3. Properties of the solution: Assertions (ii) and (iii) of this claim follow from parts (i) and (ii) of Proposition 4.1, respectively. So we will now focus on verifying

assertion (i) which states that

$$E(\tau) := \int_{\mathbb{R}} \frac{1}{2}\dot{X}(\tau)^2 d\rho_0 + \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{1}{2} W(X(y,\tau) - X(z,\tau))d\rho_0(y)d\rho_0(z)$$

is essentially nonincreasing on $[0,\infty)$. Recall that

$$E^j(\tau) := \int_{\mathbb{R}} \frac{1}{2}\dot{X}^{k_j}(\tau+)^2 d\rho_0^{k_j} + \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{1}{2} W(X^{k_j}(y,\tau) - X^{k_j}(z,\tau))d\rho_0^{k_j}(y)d\rho_0^{k_j}(z)$$

is nonincreasing by part (iii) of Proposition 4.1.

As X^{k_j} solves the pressureless Euler flow equation, we can integrate by parts to find

$$\begin{aligned}\int_s^t \int_{\mathbb{R}} \dot{X}^{k_j}(\tau)^2 d\rho_0^{k_j} d\tau &= \int_s^t \int_{\mathbb{R}} \dot{X}^{k_j}(\tau)\left[v_0 - \int_0^\tau (W' * \rho_s^{k_j})(X^{k_j}(s))ds\right] d\rho_0^{k_j} d\tau \\ &= \int_{\mathbb{R}} \left(\int_s^t \dot{X}^{k_j}(\tau)\left[v_0 - \int_0^\tau (W' * \rho_s^{k_j})(X^{k_j}(s))ds\right] d\tau\right) d\rho_0^{k_j} \\ &= \int_{\mathbb{R}} X^{k_j}(\tau)\left[v_0 - \int_0^\tau (W' * \rho_s^{k_j})(X^{k_j}(s))ds\right] d\rho_0^{k_j}\Big|_{\tau=s}^{\tau=t} \\ &\quad + \int_s^t \int_{\mathbb{R}} X^{k_j}(\tau)(W' * \rho_\tau^{k_j})(X^{k_j}(\tau))d\rho_0^{k_j} d\tau.\end{aligned}$$

The limit (4.11) with $\tau = t, s$ and $h = \mathrm{id}_{\mathbb{R}}$ and the limit (4.12) with $h = \mathrm{id}_{\mathbb{R}}$ give

(4.13)
$$\begin{aligned}\lim_{j\to\infty} \int_s^t \int_{\mathbb{R}} \dot{X}^{k_j}(\tau)^2 d\rho_0^{k_j} d\tau &= \int_{\mathbb{R}} X(\tau)\left[v_0 - \int_0^\tau (W' * \rho_s)(X(s))ds\right] d\rho_0\Big|_{\tau=s}^{\tau=t} \\ &\quad + \int_s^t \int_{\mathbb{R}} X(\tau)(W' * \rho_\tau)(X(\tau))d\rho_0 d\tau \\ &= \int_s^t \int_{\mathbb{R}} \dot{X}(\tau)^2 d\rho_0 d\tau\end{aligned}$$

for each $0 \le s \le t$.

Next, we claim that

(4.14)
$$\begin{aligned}\lim_{j\to\infty} \int_{\mathbb{R}}\int_{\mathbb{R}} W(X^{k_j}(y,\tau) - X^{k_j}(z,\tau))d\rho_0^{k_j}(y)d\rho_0^{k_j}(z) \\ = \int_{\mathbb{R}}\int_{\mathbb{R}} W(X(y,\tau) - X(z,\tau))d\rho_0(y)d\rho_0(z)\end{aligned}$$

for each $\tau \ge 0$. Note that W grows at most quadratically. Indeed,

$$W(x) \ge W(0) + W'(0)x - \frac{c}{2}x^2$$

for $x \in \mathbb{R}$ as W is semiconvex. Likewise

$$W(0) \ge W(x) + W'(x)(0-x) - c\frac{(0-x)^2}{2},$$

and so

$$W(x) \leq W(0) + xW'(x) + \frac{c}{2}x^2.$$

Since $W'(x)$ grows at most linearly, it must be that

$$|W(x)| \leq D(1 + x^2) \tag{4.15}$$

for some constant $D > 0$.

Combining (4.15) and (4.5), it is routine to check

$$|W(X^{k_j}(y,\tau) - X^{k_j}(z,\tau))| \leq 12Da^2e^{2b\tau^2}(1 + y^2 + z^2) \tag{4.16}$$

for each $j \in \mathbb{N}$, $y, z \in \mathbb{R}$, $\tau \geq 0$. Note that

$$\lim_{j\to\infty} W(X^{k_j}(y,\tau) - X^{k_j}(z,\tau)) = W(X(y,\tau) - X(z,\tau))$$

locally uniformly for $y, z \in \mathbb{R}$ for each $\tau \geq 0$. Also observe that $\rho_0^{k_j} \times \rho_0^{k_j} \to \rho_0 \times \rho_0$ narrowly in $\mathcal{P}(\mathbb{R}^2)$, and

$$\lim_{j\to\infty} \iint_{\mathbb{R}^2} (y^2 + z^2) d\rho_0^{k_j}(y) d\rho_0^{k_j}(z) = \iint_{\mathbb{R}^2} (y^2 + z^2) d\rho_0(y) d\rho_0(z),$$

which implies that $(y, z) \mapsto y^2 + z^2$ is uniformly integrable with respect to $(\rho_0^{k_j} \times \rho_0^{k_j})_{j\in\mathbb{N}}$. It follows from Lemma 2.1 that (4.14) holds for each $\tau \geq 0$.

In view of estimate (4.16), we can also apply a standard variant of dominated convergence to find

$$\lim_{j\to\infty} \int_s^t \int_{\mathbb{R}} \int_{\mathbb{R}} W(X^{k_j}(y,\tau) - X^{k_j}(z,\tau)) d\rho_0^{k_j}(y) d\rho_0^{k_j}(z) d\tau$$
$$= \int_s^t \int_{\mathbb{R}} \int_{\mathbb{R}} W(X(y,\tau) - X(z,\tau)) d\rho_0(y) d\rho_0(z) d\tau$$

for each $0 \leq s \leq t$. Combining with (4.13) gives

$$\lim_{j\to\infty} \int_s^t E^j(\tau) d\tau = \int_s^t E(\tau) d\tau \tag{4.17}$$

for each $0 \leq s \leq t$. As $E^j : [0,\infty) \to [0,\infty)$ is uniformly bounded and nonincreasing, $E^j(t)$ also converges for each $t \geq 0$ by Helly's selection theorem (Lemma 3.3.3 in [1]). By (4.17), $\lim_{j\to\infty} E^j(t) = E(t)$ for almost every $t \in [0,\infty)$. We then conclude that for almost every $t, s \in [0,\infty)$ with $t \geq s$

$$E(t) = \lim_{j\to\infty} E^j(t) \leq \lim_{j\to\infty} E^j(s) = E(s).$$

□

4.2. **Solution of the pressureless Euler equations.** We are now in position to establish the existence of a weak solution of the pressureless Euler equations (1.1) which satisfy the initial conditions (1.2). These types of solutions are defined as follows.

Definition 4.3. A narrowly continuous $\rho : [0,\infty) \to \mathcal{P}(\mathbb{R}^2); t \mapsto \rho_t$ and a Borel measurable $v : \mathbb{R} \times [0,\infty) \to \mathbb{R}$ is a *weak solution pair of the pressureless Euler equations* (1.1) which satisfies the initial conditions (1.2) if the following hold:

(i) For each $T > 0$,

$$\int_0^T \left\{ \int_{\mathbb{R}} v^2 d\rho_t + \iint_{\mathbb{R}^2} |W'(x-y)| d\rho_t(x) d\rho_t(y) \right\} dt < \infty.$$

(ii) For each $\phi \in C_c^\infty(\mathbb{R}\times[0,\infty))$,

$$\int_0^\infty\int_{\mathbb{R}}(\partial_t\phi + v\partial_x\phi)d\rho_t dt + \int_{\mathbb{R}}\phi(\cdot,0)d\rho_0 = 0.$$

(iii) For each $\phi \in C_c^\infty(\mathbb{R}\times[0,\infty))$,

$$\int_0^\infty\int_{\mathbb{R}}(v\partial_t\phi + v^2\partial_x\phi)d\rho_t dt + \int_{\mathbb{R}}\phi(\cdot,0)v_0 d\rho_0 = \int_0^\infty\int_{\mathbb{R}}\phi(W'*\rho_t)d\rho_t dt.$$

Corollary 4.4. *There exists a weak solution pair ρ and v of the pressureless Euler equations* (1.1) *which satisfies the initial conditions* (1.2)*. Moreover, this solution pair has the following two properties:*

(i) *For almost every $0 \le s \le t$,*

$$\int_{\mathbb{R}}\frac{1}{2}v(x,t)^2 d\rho_t(x) + \iint_{\mathbb{R}^2}\frac{1}{2}W(x-y)d\rho_t(x)d\rho_t(y)$$
$$\le \int_{\mathbb{R}}\frac{1}{2}v(x,s)^2 d\rho_s(x) + \iint_{\mathbb{R}^2}\frac{1}{2}W(x-y)d\rho_s(x)d\rho_s(y).$$

(ii) *For almost every $t>0$ and ρ_t almost every $x, y \in \mathbb{R}$,*

$$(v(x,t)-v(y,t))(x-y) \le \frac{\sqrt{c}}{\tanh(\sqrt{c}t)}(x-y)^2.$$

Proof. 1. Specifying v: Let X denote the solution of the pressureless Euler flow equation (1.3) which satisfies (1.4) as described in Theorem 1.1 and define

$$\nu(S) := \int_0^\infty\int_{\mathbb{R}}\left[v_0 - \int_0^t (W'*\rho_s)(X(s))ds\right]\chi_S(X(t),t)d\rho_0 dt$$

for Borel $S \subset \mathbb{R}\times[0,\infty)$. This clearly defines a signed Borel measure on $\mathbb{R}\times[0,\infty)$, and it is not hard to check that ν is sigma finite. Let us also set

$$\mu(S) := \int_0^\infty\int_{\mathbb{R}}\chi_S d\rho_t dt = \int_0^\infty\int_{\mathbb{R}}\chi_S(X(t),t)d\rho_0 dt$$

for Borel $S \subset \mathbb{R}\times[0,\infty)$. It is clear that μ it is also sigma finite and that ν is absolutely continuous with respect to μ.

By the Radon-Nikodým theorem, there is a Borel measurable $v : \mathbb{R}\times[0,\infty)\to\mathbb{R}$ such that

$$\nu(S) = \int_S v d\mu.$$

Note in particular that

$$\begin{aligned}\int_0^\infty\int_{\mathbb{R}}\phi d\nu &= \int_0^\infty\int_{\mathbb{R}}\phi(X(t),t)\left[v_0 - \int_0^t (W'*\rho_s)(X(s))ds\right]d\rho_0 dt\\ &= \int_0^\infty\int_{\mathbb{R}}\phi(X(t),t)\dot{X}(t)d\rho_0 dt\\ &= \int_0^\infty\int_{\mathbb{R}}\phi v d\mu\\ &= \int_0^\infty\int_{\mathbb{R}}\phi(X(t),t)v(X(t),t)d\rho_0 dt\end{aligned}$$

for each continuous $\phi : \mathbb{R} \times [0, \infty) \to \mathbb{R}$ with compact support. It follows that

$$\dot{X}(t) = v(X(t), t) = \mathbb{E}_{\rho_0}\left[v_0 - \int_0^t (W' * \rho_s)(X(s))ds \middle| X(t)\right] \quad \text{a.e. } t > 0$$

ρ_0 almost everywhere.

2. Integrability: Since $X : [0, \infty) \to L^2(\rho_0)$ is locally Lipschitz,

$$\int_0^T \int_{\mathbb{R}} v^2 d\rho_t dt = \int_0^T \int_{\mathbb{R}} \dot{X}(t)^2 d\rho_0 dt < \infty$$

for each $T > 0$. Lipschitz continuity also implies that $\int_{\mathbb{R}} |X(t)| d\rho_0$ is bounded on $[0, T]$, so

$$\begin{aligned}
&\int_0^T \iint_{\mathbb{R}^2} |W'(x-y)| d\rho_t(x) d\rho_t(y) dt \\
&\quad = \int_0^T \iint_{\mathbb{R}^2} |W'(X(w,t) - X(z,t))| d\rho_0(w) d\rho_0(z) dt \\
&\quad \le C \int_0^T \iint_{\mathbb{R}^2} (|X(w,t) - X(z,t)| + 1) d\rho_0(w) d\rho_0(z) dt \\
&\quad \le C \int_0^T \iint_{\mathbb{R}^2} (|X(w,t)| + |X(z,t)| + 1) d\rho_0(w) d\rho_0(z) dt \\
&\quad \le 2C \left(\int_0^T \int_{\mathbb{R}} |X(t)| d\rho_0 dt + 1 \right) < \infty.
\end{aligned}$$

Thus, ρ and v satisfy part (i) of Definition 4.3.

3. Weak solution property: Suppose $\phi \in C_c^\infty(\mathbb{R} \times [0, \infty))$

$$\begin{aligned}
&\int_0^\infty \int_{\mathbb{R}} (\partial_t \phi + v \partial_x \phi)\, d\rho_t dt \\
&\quad = \int_0^\infty \int_{\mathbb{R}} (\partial_t \phi(X(t), t) + v(X(t), t) \partial_x \phi(X(t), t))\, d\rho_0 dt \\
&\quad = \int_0^\infty \int_{\mathbb{R}} \left(\partial_t \phi(X(t), t) + \dot{X}(t) \partial_x \phi(X(t), t) \right) d\rho_0 dt \\
&\quad = \int_0^\infty \int_{\mathbb{R}} \frac{d}{dt} \phi(X(t), t) d\rho_0 dt \\
&\quad = \int_{\mathbb{R}} \int_0^\infty \frac{d}{dt} \phi(X(t), t) dt d\rho_0 \\
&\quad = -\int_{\mathbb{R}} \phi(X(0), 0) d\rho_0 \\
&\quad = -\int_{\mathbb{R}} \phi(\cdot, 0) d\rho_0.
\end{aligned}$$

This proves part (ii) of Definition 4.3. As for part (iii) of that definition,

$$
\begin{aligned}
&\int_0^\infty \int_{\mathbb{R}} (\partial_t \phi + v\partial_x \phi)\, v d\rho_t dt \\
&= \int_0^\infty \int_{\mathbb{R}} (\partial_t \phi(X(t),t) + v(X(t),t)\partial_x \phi(X(t),t))\, v(X(t),t) d\rho_0 dt \\
&= \int_0^\infty \int_{\mathbb{R}} (\partial_t \phi(X(t),t) + v(X(t),t)\partial_x \phi(X(t),t))\, \dot{X}(t) d\rho_0 dt \\
&= \int_0^\infty \int_{\mathbb{R}} \frac{d}{dt}\phi(X(t),t)\dot{X}(t) d\rho_0 dt \\
&= \int_0^\infty \int_{\mathbb{R}} \frac{d}{dt}\phi(X(t),t)\left[v_0 - \int_0^t (W' * \rho_s)(X(s))ds\right] d\rho_0 dt \\
&= \int_{\mathbb{R}} \int_0^\infty \frac{d}{dt}\phi(X(t),t)\left[v_0 - \int_0^t (W' * \rho_s)(X(s))ds\right] dt d\rho_0 \\
&= \int_0^\infty \int_{\mathbb{R}} \phi(X(t),t)(W' * \rho_t)(X(t)) d\rho_0 dt - \int_{\mathbb{R}} \phi(X(0),0) v_0 d\rho_0 \\
&= \int_0^\infty \int_{\mathbb{R}} \phi(W' * \rho_t) d\rho_t dt - \int_{\mathbb{R}} \phi(\cdot,0) v_0 d\rho_0.
\end{aligned}
$$

4. Nonincreasing energy and entropy inequality: Since

$$
\begin{aligned}
\mathcal{E}(\tau) &:= \int_{\mathbb{R}} \frac{1}{2} v(x,\tau)^2 d\rho_\tau(x) + \iint_{\mathbb{R}^2} \frac{1}{2} W(x-y) d\rho_\tau(x) d\rho_\tau(y) \\
&= \int_{\mathbb{R}} \frac{1}{2} v(X(\tau),\tau)^2 d\rho_0 + \iint_{\mathbb{R}^2} \frac{1}{2} W(X(y,\tau) - X(z,\tau)) d\rho_0(y) d\rho_0(z) \\
&= \int_{\mathbb{R}} \frac{1}{2} \dot{X}(\tau)^2 d\rho_0 + \iint_{\mathbb{R}^2} \frac{1}{2} W(X(y,\tau) - X(z,\tau)) d\rho_0(y) d\rho_0(z)
\end{aligned}
$$

for almost every $\tau \ge 0$, $\mathcal{E}(t) \le \mathcal{E}(s)$ for almost every $t, s \in [0,\infty)$ with $s \le t$. Here, of course, we are employing conclusion (i) of Theorem 1.1.

Recall that $[0,\infty) \ni t \mapsto X(y,t)$ is absolutely continuous on any compact interval within $[0,\infty)$ for ρ_0 almost every $y \in \mathbb{R}$. Let us denote this set of y as $Q \subset \mathbb{R}$, and we emphasize that Q is ρ_0 measurable and $\rho_0(Q) = 1$. By conclusion (iii) of Theorem 1.1,

$$
\begin{aligned}
0 &\ge \frac{d}{dt} \frac{(X(y,t) - X(z,t))^2}{\sinh(\sqrt{c}t)^2} \\
&= \frac{2(X(y,t) - X(z,t))(\partial_t X(y,t) - \partial_t X(z,t))}{\sinh(\sqrt{c}t)^2} \\
&\quad - 2\frac{\sqrt{c}\cosh(\sqrt{c}t)}{\sinh(\sqrt{c}t)^3}(X(y,t) - X(z,t))^2 \\
&= \frac{2}{\sinh(\sqrt{c}t)^2}\Big[(X(y,t) - X(z,t))(v(X(y,t),t) - v(X(z,t),t)) \\
&\qquad\qquad - \frac{\sqrt{c}}{\tanh(\sqrt{c}t)}(X(y,t) - X(z,t))^2\Big]
\end{aligned}
$$

for Lebesgue almost every $t > 0$ and $y, z \in Q$.

As a result, we have proved part (ii) of this assertion for x, y belonging to the image of Q under $X(t)$,

$$X(t)(Q) = \{X(y,t) \in \mathbb{R} : y \in Q\}$$

for almost every $t > 0$. Without loss of generality, we may suppose Q is a countable union of closed sets (Theorem 1.19 of [11]). By part (ii) of Theorem 1.1, we may as well assume $X(t)$ is continuous on $\mathbb{R}$. It follows that $X(t)(Q)$ is Borel measurable (Proposition A.1 in [16]). As $Q \subset X(t)^{-1}[X(t)(Q)]$,

$$\rho_t(X(t)(Q)) = \rho_0(X(t)^{-1}[X(t)(Q)]) \geq \rho_0(Q) = 1.$$

We conclude that part (ii) of this assertion holds for Lebesgue almost every $t > 0$ and x, y belonging to a Borel set of full measure for ρ_t. □

5. Euler-Poisson equations in 1D

For the remainder of this paper, we will assume

$$W(x) = |x|, \quad x \in \mathbb{R}.$$

As W is not continuously differentiable, we will also consider the closely related interaction potential

$$W_\epsilon(x) = (x^2 + \epsilon^2)^{1/2}, \quad x \in \mathbb{R},$$

for $\epsilon > 0$ and small. Clearly,

$$W(x) \leq W_\epsilon(x) \leq W(x) + \epsilon, \quad x \in \mathbb{R}. \tag{5.1}$$

Also observe that W_ϵ is convex, even, and continuously differentiable with

$$|W_\epsilon'(x)| \leq 1.$$

Therefore, there is a locally Lipschitz continuous $X^\epsilon : [0,\infty) \to L^2(\rho_0)$ which satisfies

$$\begin{cases} \dot{X}^\epsilon(t) = \mathbb{E}_{\rho_0}\left[v_0 - \int_0^t (W_\epsilon' * \rho_s^\epsilon)(X^\epsilon(s))ds \middle| X^\epsilon(t)\right] & \text{a.e. } t \geq 0, \\ X^\epsilon(0) = \mathrm{id}_\mathbb{R} \end{cases} \tag{5.2}$$

for each $\epsilon > 0$. Here

$$\rho_t^\epsilon := X^\epsilon(t)_{\#}\rho_0, \quad t \geq 0.$$

We will argue below that there is a sequence of positive numbers $\epsilon_k \to 0$ and $(X^{\epsilon_k})_{k\in\mathbb{N}}$ which converges in a strong sense to a solution X of the Euler-Poisson flow equation (1.10) which satisfies the initial condition (1.4). We will then make a final remark on the existence of weak solution pairs to the Euler-Poisson system (1.9).

5.1. **A strongly convergent subsequence.**

Let us begin by recalling a few facts we have already established for X^ϵ.

Lipschitz continuity in time. In view of Theorem 1.1, X^ϵ satisfies

$$\begin{aligned}\int_{\mathbb{R}}\frac{1}{2}(\dot{X}^\epsilon(t))^2d\rho_0 &\le \int_{\mathbb{R}}\frac{1}{2}(\dot{X}^\epsilon(t))^2d\rho_0+\iint_{\mathbb{R}^2}\frac{1}{2}W_\epsilon(X^\epsilon(y,t)-X^\epsilon(z,t))d\rho_0(y)d\rho_0(z)\\ &\le \int_{\mathbb{R}}\frac{1}{2}v_0^2d\rho_0+\iint_{\mathbb{R}^2}\frac{1}{2}W_\epsilon(y-z)d\rho_0(y)d\rho_0(z)\\ &\le \int_{\mathbb{R}}\frac{1}{2}v_0^2d\rho_0+\iint_{\mathbb{R}^2}\frac{1}{2}(|y-z|+\epsilon)d\rho_0(y)d\rho_0(z)\\ &= \int_{\mathbb{R}}\frac{1}{2}v_0^2d\rho_0+\iint_{\mathbb{R}^2}\frac{1}{2}|y-z|d\rho_0(y)d\rho_0(z)+\frac{1}{2}\epsilon\end{aligned}$$

for almost every $t\ge 0$. Therefore, $X^\epsilon:[0,\infty)\to L^2(\rho_0)$ is uniformly Lipschitz continuous.

Uniform spatial continuity. Theorem 1.1 also gives

$$0\le X^\epsilon(y,t)-X^\epsilon(z,t)\le y-z+t\int_z^y|v_0'(x)|dx$$

for each $y,z\in\operatorname{supp}(\rho_0)$ with $y\ge z$ and for each $t\ge 0$. As a result, we may well suppose $X^\epsilon(t):\mathbb{R}\to\mathbb{R}$ is uniformly continuous and satisfies

$$|X^\epsilon(y,t)-X^\epsilon(z,t)|\le|y-z|+t\omega(|y-z|)$$

for every $y,z\in\mathbb{R}$ and $t\ge 0$. Here ω is the modulus of continuity defined in (3.17).

Quantitative stickiness. For $0<s\le t$, there is a function $f^\epsilon_{t,s}:\mathbb{R}\to\mathbb{R}$ which satisfies

$$|f^\epsilon_{t,s}(x)-f^\epsilon_{t,s}(y)|\le\frac{t}{s}|x-y|,\quad x,y\in\mathbb{R}, \tag{5.3}$$

such that

$$X^\epsilon(y,t)=f^\epsilon_{t,s}(X^\epsilon(y,s)) \tag{5.4}$$

for $y\in\operatorname{supp}(\rho_0)$. In particular,

$$\frac{1}{t}|X^\epsilon(y,t)-X^\epsilon(z,t)|\le\frac{1}{s}|X^\epsilon(y,s)-X^\epsilon(z,s)|$$

for $y,z\in\operatorname{supp}(\rho_0)$ and $0<s\le t$.

We can use this Lipschitz continuity in time, uniform spatial continuity, and quantitative stickiness of $(X^\epsilon)_{\epsilon>0}$, along with the arguments we used to prove Proposition 4.1 and Corollary 4.2, to establish the following assertion.

Proposition 5.1. *There exist a sequence of positive numbers $\epsilon_k\to 0$ and a Lipschitz $X:[0,\infty)\to L^2(\rho_0)$ such that for each $t\ge 0$,*

$$X^{\epsilon_k}(t)\to X(t)$$

locally uniformly on $\mathbb{R}$ and

$$\lim_{j\to\infty}\int_{\mathbb{R}}h(\mathrm{id}_{\mathbb{R}},X^{\epsilon_k}(t))d\rho_0=\int_{\mathbb{R}}h(\mathrm{id}_{\mathbb{R}},X(t))d\rho_0$$

for continuous $h:\mathbb{R}^2\to\mathbb{R}$ with

$$\sup_{(x,y)\in\mathbb{R}^2}\frac{|h(x,y)|}{1+x^2+y^2}<\infty.$$

In addition, X has the following properties:

(i) *For $t \geq 0$ and $y, z \in \operatorname{supp}(\rho_0)$ with $y \leq z$,*

$$0 \leq X(z,t) - X(y,t) \leq z - y + t \int_y^z |v_0'(x)| dx.$$

(ii) *For each $0 < s \leq t$ and $y, z \in \operatorname{supp}(\rho_0)$,*

$$\frac{1}{t}|X(y,t) - X(z,t)| \leq \frac{1}{s}|X(y,s) - X(z,s)|.$$

(iii) *For each $0 < s \leq t$, $(f_{t,s}^{\epsilon_k})_{k\in\mathbb{N}}$ has a subsequence which converges locally uniformly to a function $f_{t,s} : \mathbb{R} \to \mathbb{R}$ which satisfies the Lipschitz condition* (5.3) *and*

$$X(y,t) = f_{t,s}(X(y,s))$$

for $y \in \operatorname{supp}(\rho_0)$.

(iv) *For almost every $t > 0$, there is a Borel function $u : \mathbb{R} \to \mathbb{R}$ for which*

$$\dot{X}(t) = u(X(t))$$

for ρ_0 almost everywhere.

We also have the following immediate corollary which can be proved the same way that we justified (4.9).

Corollary 5.2. *Suppose $g : \mathbb{R} \to \mathbb{R}$ is continuous and*

$$\sup_{x\in\mathbb{R}} \frac{|g(x)|}{1+|x|} < \infty.$$

Then

$$\lim_{k\to\infty} \int_s^t \int_{\mathbb{R}} \dot{X}^{\epsilon_k}(\tau) g(X^{\epsilon_k}(\tau)) d\rho_0 d\tau = \int_s^t \int_{\mathbb{R}} \dot{X}(\tau) g(X(\tau)) d\rho_0 d\tau$$

for $0 \leq s \leq t$.

This is as far as we can go with the convergence arguments we used to establish Theorem 1.1. We will need to identify another mechanism which will allow us to pass to the limit in the term with W_ϵ' in equation (5.2). This will be the topic of the following subsection.

5.2. **A convergence lemma.** Let us recall the definition of sgn:

$$\operatorname{sgn}(x) = \begin{cases} 1, & x > 0, \\ 0, & x = 0, \\ -1, & x < 0. \end{cases}$$

We will also fix a sequence $(\mu^k)_{k\in\mathbb{N}} \subset \mathcal{P}(\mathbb{R}^2)$ which converges narrowly to $\mu \in \mathcal{P}(\mathbb{R}^2)$ and additionally satisfies

$$\lim_{k\to\infty} \int_{\mathbb{R}} x^2 d\mu^k(x) = \int_{\mathbb{R}} x^2 d\mu(x). \tag{5.5}$$

The central assertion of this subsection is as follows.

Lemma 5.3. *Suppose $g : \mathbb{R} \to \mathbb{R}$ is continuous and*

$$\sup_{x\in\mathbb{R}} \frac{|g(x)|}{1+|x|} < \infty. \tag{5.6}$$

Then

$$\lim_{k\to\infty} \iint_{\mathbb{R}^2} W'_{\epsilon_k}(x-y)g(x)d\mu^k(x)d\mu^k(y) = \iint_{\mathbb{R}^2} \operatorname{sgn}(x-y)g(x)d\mu(x)d\mu(y).$$

Remark 5.4. Here and below, $(\epsilon_k)_{k\in\mathbb{N}}$ is the sequence obtained in Proposition 5.1.

We will first verify an elementary observation, which is ultimately due to the convexity of the absolute value function. In particular, we will employ

$$|y| \ge |x| + \operatorname{sgn}(x)(y-x) \tag{5.7}$$

for each $x, y \in \mathbb{R}$.

Lemma 5.5. *The following are equivalent for $\xi \in L^2(\mu)$:*

(i) *For μ almost every $x \in \mathbb{R}$,*

$$\xi(x) = \operatorname{sgn} *\mu(x).$$

(ii) *For each continuous $g : \mathbb{R} \to \mathbb{R}$ which satisfies* (5.6),

$$\iint_{\mathbb{R}^2} \frac{1}{2}|x-y+(g(x)-g(y))|d\mu(x)d\mu(y) \ge \iint_{\mathbb{R}^2} \frac{1}{2}|x-y|d\mu(x)d\mu(y) + \int_{\mathbb{R}} \xi g d\mu.$$

Proof. Suppose $g : \mathbb{R} \to \mathbb{R}$ is a continuous function which grows at most linearly as $|x| \to \infty$.

(i) $\Longrightarrow$ (ii) Employing (5.7) and noting that sgn is odd gives

$$\begin{aligned}
&\iint_{\mathbb{R}^2} \frac{1}{2}|x-y+(g(x)-g(y))|d\mu(x)d\mu(y)\\
&\quad\ge \iint_{\mathbb{R}^2} \frac{1}{2}|x-y|d\mu(x)d\mu(y) + \frac{1}{2}\iint_{\mathbb{R}^2} \operatorname{sgn}(x-y)(g(x)-g(y))d\mu(x)d\mu(y)\\
&\quad= \iint_{\mathbb{R}^2} \frac{1}{2}|x-y|d\mu(x)d\mu(y) + \frac{1}{2}\iint_{\mathbb{R}^2} \operatorname{sgn}(x-y)g(x)d\mu(x)d\mu(y)\\
&\qquad\qquad - \frac{1}{2}\iint_{\mathbb{R}^2} \operatorname{sgn}(x-y)g(y)d\mu(x)d\mu(y)\\
&\quad= \iint_{\mathbb{R}^2} \frac{1}{2}|x-y|d\mu(x)d\mu(y) + \frac{1}{2}\iint_{\mathbb{R}^2} \operatorname{sgn}(x-y)g(x)d\mu(x)d\mu(y)\\
&\qquad\qquad + \frac{1}{2}\iint_{\mathbb{R}^2} \operatorname{sgn}(y-x)g(y)d\mu(x)d\mu(y)\\
&\quad= \iint_{\mathbb{R}^2} \frac{1}{2}|x-y|d\mu(x)d\mu(y) + \int_{\mathbb{R}}\left(\int_{\mathbb{R}} \operatorname{sgn}(x-y)d\mu(y)\right)g(y)d\mu(x)\\
&\quad= \iint_{\mathbb{R}^2} \frac{1}{2}|x-y|d\mu(x)d\mu(y) + \int_{\mathbb{R}} (\operatorname{sgn} * \mu)g d\mu.
\end{aligned}$$

(ii) $\Longrightarrow$ (i) By assumption,

$$\frac{1}{2}\int_{\mathbb{R}} \frac{|x-y+t(g(x)-g(y))|-|x-y|}{t} d\mu(x)d\mu(y) \ge \int_{\mathbb{R}} g\xi d\mu. \tag{5.8}$$

Also notice

$$\left|\frac{|x-y+t(g(x)-g(y))|-|x-y|}{t}\right| \le |g(x)-g(y)|$$

and

$$\lim_{t\to 0^+}\frac{|x-y+t(g(x)-g(y))|-|x-y|}{t}=\operatorname{sgn}(x-y)(g(x)-g(y)).$$

for $x, y \in \mathbb{R}$. By dominated convergence, we can send $t \to 0^+$ in (5.8) to find

$$\int_{\mathbb{R}} g\xi d\mu \le \frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} \operatorname{sgn}(x-y)(g(x)-g(y))d\mu(x)d\mu(y)=\int_{\mathbb{R}}(\operatorname{sgn}*\mu)gd\mu.$$

Replacing g with $-g$ gives

$$\int_{\mathbb{R}} g(\xi-\operatorname{sgn}*\mu)d\mu=0.$$

As g is arbitrary, $\xi - \operatorname{sgn}*\mu$ vanishes μ almost everywhere. □

Proof of Lemma 5.3. Using the same method to prove (i) $\Longrightarrow$ (ii) in Lemma 5.5, we find

$$\iint_{\mathbb{R}^2}\frac{1}{2}W_{\epsilon_k}(x-y+(g(x)-g(y)))d\mu^k(x)d\mu^k(y)$$
$$\ge \iint_{\mathbb{R}^2}\frac{1}{2}W_{\epsilon_k}(x-y)d\mu^k(x)d\mu^k(y)+\int_{\mathbb{R}}(W'_{\epsilon_k}*\mu^k)gd\mu^k$$

for each continuous and at most linearly growing $g : \mathbb{R} \to \mathbb{R}$. And by (5.1),

$$\frac{1}{2}\epsilon_k+\iint_{\mathbb{R}^2}\frac{1}{2}|x-y+(g(x)-g(y))|d\mu^k(x)d\mu^k(y)$$
$$\ge \iint_{\mathbb{R}^2}\frac{1}{2}|x-y|d\mu^k(x)d\mu^k(y)+\int_{\mathbb{R}}(W'_{\epsilon_k}*\mu^k)gd\mu^k. \tag{5.9}$$

Since $|W'_{\epsilon_k}| \le 1$,

$$|(W'_{\epsilon_k}*\mu^k)(z)|\le 1, \quad z\in\mathbb{R}.$$

Combining this fact with (5.5) provides a subsequence $(W'_{\epsilon_{k_j}} * \mu^{k_j})_{j\in\mathbb{N}}$ and $\xi \in L^2(\mu)$ such that

$$\lim_{j\to\infty}\int_{\mathbb{R}}(W'_{\epsilon_{k_j}}*\mu^{k_j})gd\mu^{k_j}=\int_{\mathbb{R}}\xi g d\mu$$

for each continuous and at most linearly growing $g : \mathbb{R} \to \mathbb{R}$ (Theorem 5.4.4 of [1]). Sending $k = k_j \to \infty$ in (5.9) gives

$$\iint_{\mathbb{R}^2}\frac{1}{2}|x-y+(g(x)-g(y))|d\mu(x)d\mu(y)\ge \iint_{\mathbb{R}^2}\frac{1}{2}|x-y|d\mu(x)d\mu(y)+\int_{\mathbb{R}}\xi g d\mu.$$

Lemma 5.5 implies

$$\xi=\operatorname{sgn}*\mu.$$

Since this limit is independent of the subsequence,

$$\lim_{k\to\infty}\iint_{\mathbb{R}^2}W'_{\epsilon_k}(x-y)g(x)d\mu^k(x)d\mu^k(y)=\lim_{k\to\infty}\int_{\mathbb{R}}(W'_{\epsilon_k}*\mu^k)gd\mu^k$$
$$=\int_{\mathbb{R}}(\operatorname{sgn}*\mu)gd\mu$$
$$=\iint_{\mathbb{R}^2}\operatorname{sgn}(x-y)g(x)d\mu(x)d\mu(y)$$

for each continuous $g : \mathbb{R} \to \mathbb{R}$ which satisfies (5.3). □

We will actually need a minor refinement of Lemma 5.5 in our proof of Theorem 1.2.

Corollary 5.6. *Suppose* $(g^k)_{k\in\mathbb{N}}$ *is a sequence of continuous functions on* $\mathbb{R}$ *which satisfies*

$$\sup_{x\in\mathbb{R}}\frac{|g^k(x)|}{1+|x|}\le C$$

for some C *and which converges locally uniformly to* $g:\mathbb{R}\to\mathbb{R}$. *Then*

$$\lim_{k\to\infty}\iint_{\mathbb{R}^2}W'_{\epsilon_k}(x-y)g^k(x)d\mu^k(x)d\mu^k(y)=\iint_{\mathbb{R}^2}\operatorname{sgn}(x-y)g(x)d\mu(x)d\mu(y).$$

Proof. As $g^k(x)\to g(x)$ for each $x\in\mathbb{R}$, we also have

$$\sup_{x\in\mathbb{R}}\frac{|g(x)|}{1+|x|}\le C.$$

Fix $\delta>0$ and choose a compact interval $K_\delta\subset\mathbb{R}$ such that

$$\int_{\mathbb{R}\setminus K_\delta}(1+|x|)d\mu^k(x)\le\frac{\delta}{2C}\tag{5.10}$$

for $k\in\mathbb{N}$; such an interval exists as $1+|x|$ is uniformly integrable with respect to $(\mu^k)_{k\in\mathbb{N}}$ by assumption (5.5). In view of Lemma 5.3.

$$\begin{aligned}
&\iint_{\mathbb{R}^2}W'_{\epsilon_k}(x-y)g^k(x)d\mu^k(x)d\mu^k(y)\\
&\qquad=\int_{\mathbb{R}}(W'_{\epsilon_k}*\mu^k)g^k d\mu^k\\
&\qquad=\int_{\mathbb{R}}(W'_{\epsilon_k}*\mu^k)g d\mu^k+\int_{\mathbb{R}}(W'_{\epsilon_k}*\mu^k)(g^k-g)d\mu^k\\
&\qquad=\int_{\mathbb{R}}(\operatorname{sgn}*\mu)g d\mu+o(1)+\int_{\mathbb{R}}(W'_{\epsilon_k}*\mu^k)(g^k-g)d\mu^k\\
&\qquad=\int_{\mathbb{R}}(\operatorname{sgn}*\mu)g d\mu+o(1)+\int_{K_\delta}(W'_{\epsilon_k}*\mu^k)(g^k-g)d\mu^k\\
&\qquad\qquad+\int_{\mathbb{R}\setminus K_\delta}(W'_{\epsilon_k}*\mu^k)(g^k-g)d\mu^k\\
&=\iint_{\mathbb{R}^2}\operatorname{sgn}(x-y)g(x)d\mu(x)d\mu(y)+o(1)\\
&\qquad+\int_{K_\delta}(W'_{\epsilon_k}*\mu^k)(g^k-g)d\mu^k+\int_{\mathbb{R}\setminus K_\delta}(W'_{\epsilon_k}*\mu^k)(g^k-g)d\mu^k
\end{aligned}$$

as $k\to\infty$.

Observe

$$\left|\int_{K_\delta}(W'_{\epsilon_k}*\mu^k)(g^k-g)d\mu^k\right|\le\max_{K_\delta}|g^k-g|.$$

And by (5.10),

$$\left|\int_{\mathbb{R}\setminus K_\delta}(W'_{\epsilon_k}*\mu^k)(g^k-g)d\mu^k\right|\le 2C\int_{\mathbb{R}\setminus K_\delta}(1+|x|)d\mu^k(x)\le\delta.$$

As a result,

$$\limsup_{k\to\infty}\left|\iint_{\mathbb{R}^2} W'_{\epsilon_k}(x-y)g^k(x)d\mu^k(x)d\mu^k(y) - \iint_{\mathbb{R}^2}\operatorname{sgn}(x-y)g(x)d\mu(x)d\mu(y)\right| \le \delta.$$

The claim follows as $\delta > 0$ was arbitrarily chosen. □

5.3. **Solution of the flow equation.** This subsection is dedicated to the proof of Theorem 1.2. Here, we will show that the mapping X obtained in Proposition 5.1 is a solution flow equation (1.10) which has all of the required properties. First note that since $X^{\epsilon_k}(0) = \mathrm{id}_{\mathbb{R}}$ and $X^{\epsilon_k}(0) \to X(0)$ in $L^2(\rho_0)$ as $k \to \infty$, then $X(0) = \mathrm{id}_{\mathbb{R}}$. Next we claim that X satisfies flow equation (1.10). It suffices to let $0 \le s \le t$, fix a continuous $h : \mathbb{R} \to \mathbb{R}$ which satisfies

$$\sup_{x\in\mathbb{R}} \frac{|h(x)|}{1+|x|} \le C,$$

and show

(5.11)
$$\int_s^t \int_{\mathbb{R}} \dot{X}(\tau)h(X(\tau))d\rho_0 d\tau = \int_s^t \int_{\mathbb{R}} \left[v_0 - \int_0^\tau (\operatorname{sgn} * \rho_\xi)(X(\xi))d\xi\right] h(X(t))d\rho_0 d\tau.$$

Once we establish this identity, parts (i), (ii), and (iii) of Theorem 1.2 would follow by minor variations of the arguments we gave in our proof of Theorem 1.1.

To this end, we recall that for each $k \in N$,

$$\begin{aligned}\int_s^t \int_{\mathbb{R}} & \dot{X}^{\epsilon_k}(\tau)h(X^{\epsilon_k}(\tau))d\rho_0 d\tau \\ &= \int_s^t \int_{\mathbb{R}} \left[v_0 - \int_0^\tau (W'_{\epsilon_k} * \rho^{\epsilon_k}_\xi)(X^{\epsilon_k}(\xi))d\xi\right] h(X^{\epsilon_k}(\tau))d\rho_0 d\tau.\end{aligned} \tag{5.12}$$

Moreover, Proposition 5.1 implies

$$\rho_t = X(t)_{\#}\rho_0 = \lim_{k\to\infty} X^{\epsilon_k}(t)_{\#}\rho_0 = \lim_{k\to\infty}\rho^{\epsilon_k}_t \tag{5.13}$$

narrowly and

$$\lim_{k\to\infty}\int_{\mathbb{R}} x^2 d\rho^{\epsilon_k}_t(x) = \lim_{k\to\infty}\int_{\mathbb{R}} (X^{\epsilon_k}(t))^2 d\rho_0 = \int_{\mathbb{R}} (X(t))^2 d\rho_0 = \int_{\mathbb{R}} x^2 d\rho_t(x) \tag{5.14}$$

for each $t \ge 0$.

Proposition 5.1 also can be used to show

$$\lim_{k\to\infty}\int_s^t \int_{\mathbb{R}} v_0 h(X^{\epsilon_k}(\tau))d\rho_0 d\tau = \int_s^t \int_{\mathbb{R}} v_0 h(X(t))d\rho_0 d\tau.$$

Furthermore,

$$\lim_{k\to\infty}\int_s^t \int_{\mathbb{R}} \dot{X}^{\epsilon_k}(\tau)h(X^{\epsilon_k}(\tau))d\rho_0 d\tau = \int_s^t \int_{\mathbb{R}} \dot{X}(\tau)h(X(\tau))d\rho_0 d\tau$$

as noted in Corollary 5.2. As a result, we are left to justify the limit

$$\lim_{k\to\infty}\int_s^t\int_{\mathbb{R}}\left(\int_0^\tau (W'_{\epsilon_k}*\rho^{\epsilon_k}_\xi)(X^{\epsilon_k}(\xi))d\xi\right)h(X^{\epsilon_k}(\tau))d\rho_0 d\tau \\ =\int_s^t\int_{\mathbb{R}}\left(\int_0^\tau (\mathrm{sgn}*\rho_\xi)(X(\xi))d\xi\right)h(X(\tau))d\rho_0 d\tau. \tag{5.15}$$

Then we would be able to send $k\to\infty$ in (5.13) to conclude (5.11).

So we will now focus on establishing (5.15). Observe

$$\int_s^t\int_{\mathbb{R}}\left(\int_0^\tau (W'_{\epsilon_k}*\rho^{\epsilon_k}_\xi)(X^{\epsilon_k}(\xi))d\xi\right)h(X^{\epsilon_k}(\tau))d\rho_0 d\tau \\ =\int_s^t\int_0^\tau\left[\int_{\mathbb{R}}(W'_{\epsilon_k}*\rho^{\epsilon_k}_\xi)(X^{\epsilon_k}(\xi))h(X^{\epsilon_k}(\tau))d\rho_0\right]d\xi d\tau.$$

Since W'_{ϵ_k} is uniformly bounded and h grows at most linearly, we just need to show

$$\lim_{k\to\infty}\int_{\mathbb{R}}(W'_{\epsilon_k}*\rho^{\epsilon_k}_\xi)(X^{\epsilon_k}(\xi))h(X^{\epsilon_k}(\tau))d\rho_0=\int_{\mathbb{R}}(\mathrm{sgn}*\rho_\xi)(X(\xi))h(X(\tau))d\rho_0 \tag{5.16}$$

for each $\xi,\tau>0$ with $\xi\le\tau$. For if (5.16) holds, (5.15) would follow from a simple application of the dominated convergence theorem.

In view of (5.4),

$$\begin{aligned}\int_{\mathbb{R}}(W'_{\epsilon_k}*\rho^{\epsilon_k}_\xi)(X^{\epsilon_k}(\xi))h(X^{\epsilon_k}(\tau))d\rho_0&=\int_{\mathbb{R}}(W'_{\epsilon_k}*\rho^{\epsilon_k}_\xi)(X^{\epsilon_k}(\xi))\ h\circ f^{\epsilon_k}_{\tau,\xi}(X^{\epsilon_k}(\xi))d\rho_0\\ &=\int_{\mathbb{R}}(W'_{\epsilon_k}*\rho^{\epsilon_k}_\xi)\ h\circ f^{\epsilon_k}_{\tau,\xi}d\rho^{\epsilon_k}_\xi\\ &=\iint_{\mathbb{R}^2}W'_{\epsilon_k}(x-y)\ h\circ f^{\epsilon_k}_{\tau,\xi}(y)d\rho^{\epsilon_k}_\xi(x)d\rho^{\epsilon_k}_\xi(y).\end{aligned}$$

By part (iii) of Proposition 5.1, $f^{\epsilon_k}_{\tau,\xi}\to f_{\tau,\xi}$ locally uniformly on $\mathbb{R}$ (up to a subsequence that we will not relabel) and

$$X(\tau)=f_{\tau,\xi}(X(\xi))$$

ρ_0 almost everywhere. It follows that $h\circ f^{\epsilon_k}_{\tau,\xi}$ converges locally uniformly to $h\circ f_{\tau,\xi}$. We also have the limits (5.13) and (5.14) for each $t=\xi$. We can then apply Corollary 5.3 once we know $h\circ f^{\epsilon_k}_{\tau,\xi}(y)$ grows at most linearly in $|y|$ in a uniform way.

Fix $z_0\in\mathrm{supp}(\rho_0)$ and observe

$$\begin{aligned}|h\circ f^{\epsilon_k}_{\tau,\xi}(y)|&\le C(1+|f^{\epsilon_k}_{\tau,\xi}(y)|)\\ &\le C\left(1+|f^{\epsilon_k}_{\tau,\xi}(y)-f^{\epsilon_k}_{\tau,\xi}(X^{\epsilon_k}(z_0,\xi))|+|f^{\epsilon_k}_{\tau,\xi}(X^{\epsilon_k}(z_0,\xi))|\right)\\ &\le C\left(1+\frac{\tau}{\xi}|y-X^{\epsilon_k}(z_0,\xi)|+|X^{\epsilon_k}(z_0,\tau)|\right)\end{aligned}$$

for all $y\in\mathbb{R}$. Since $(X^{\epsilon_k}(z_0,\xi),X^{\epsilon_k}(z_0,\tau))\to(X(z_0,\xi),X(z_0,\tau))$ as $k\to\infty$, it must be that

$$\sup_{k\in\mathbb{N}}\left\{\sup_{y\in\mathbb{R}}\frac{|h\circ f^{\epsilon_k}_{\tau,\xi}(y)|}{1+|y|}\right\}<\infty.$$

Corollary 5.3 then gives

$$\begin{aligned}
&\lim_{k\to\infty}\int_{\mathbb{R}}(W'_{\epsilon_k}*\rho_\xi^{\epsilon_k})(X^{\epsilon_k}(\xi))h(X^{\epsilon_k}(\tau))d\rho_0\\
&\quad=\lim_{k\to\infty}\iint_{\mathbb{R}^2}W'_{\epsilon_k}(x-y)\; h\circ f^{\epsilon_k}_{\tau,\xi}(y)d\rho_\xi^{\epsilon_k}(x)d\rho_\xi^{\epsilon_k}(y)\\
&\quad=\iint_{\mathbb{R}^2}\operatorname{sgn}(x-y)\; h\circ f_{\tau,\xi}(y)d\rho_\xi(x)d\rho_\xi(y)\\
&\quad=\int_{\mathbb{R}}(\operatorname{sgn}*\rho_\xi)(X(\xi))\; h\circ f_{\tau,\xi}(X(\xi))d\rho_0\\
&\quad=\int_{\mathbb{R}}(\operatorname{sgn}*\rho_\xi)(X(\xi))\; h(X(\tau))d\rho_0.
\end{aligned}$$

We conclude (5.16) and in turn that X is a solution of the flow equation (1.10).

5.4. **Solving the Euler-Poisson equations.** Weak solution pairs of the Euler-Poisson system (1.9) which satisfy given initial conditions (1.2) are defined as follows.

Definition 5.7. A narrowly continuous $\rho:[0,\infty)\to\mathcal{P}(\mathbb{R}^2); t\mapsto\rho_t$ and a Borel measurable $v:\mathbb{R}\times[0,\infty)\to\mathbb{R}$ is a *weak solution pair of the Euler-Poisson equations* (1.9) which satisfies the initial conditions (1.2) if the following hold:

(i) For each $T>0$,

$$\int_0^T\int_{\mathbb{R}}v^2d\rho_tdt<\infty.$$

(ii) For each $\phi\in C_c^\infty(\mathbb{R}\times[0,\infty))$,

$$\int_0^\infty\int_{\mathbb{R}}(\partial_t\phi+v\partial_x\phi)d\rho_tdt+\int_{\mathbb{R}}\phi(\cdot,0)d\rho_0=0.$$

(iii) For each $\phi\in C_c^\infty(\mathbb{R}\times[0,\infty))$,

$$\int_0^\infty\int_{\mathbb{R}}(v\partial_t\phi+v^2\partial_x\phi)d\rho_tdt+\int_{\mathbb{R}}\phi(\cdot,0)v_0d\rho_0=\int_0^\infty\int_{\mathbb{R}}\phi(\operatorname{sgn}*\rho_t)d\rho_tdt.$$

Employing the same method used to prove Corollary 4.4 from Theorem 1.1, we have the subsequent corollary to Theorem 1.2.

Corollary 5.8. *There exists a weak solution pair ρ and v of the Euler-Poisson equations* (1.9) *which satisfies the initial conditions* (1.2). *Moreover, this solution pair additionally has the following features:*

(i) *For almost every $t,s\ge 0$ with $0\le s\le t$,*

$$\begin{aligned}
&\int_{\mathbb{R}}\frac{1}{2}v(x,t)^2d\rho_t(x)+\iint_{\mathbb{R}^2}\frac{1}{2}|x-y|d\rho_t(x)d\rho_t(y)\\
&\qquad\le\int_{\mathbb{R}}\frac{1}{2}v(x,s)^2d\rho_s(x)+\iint_{\mathbb{R}^2}\frac{1}{2}|x-y|d\rho_s(x)d\rho_s(y).
\end{aligned}$$

(ii) *For almost every $t>0$ and ρ_t almost every $x,y\in\mathbb{R}$,*

$$(v(x,t)-v(y,t))(x-y)\le\frac{1}{t}(x-y)^2.$$

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