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ON THE SYMMETRY AND MONOTONICITY OF MORREY EXTREMALS

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Abstract. We employ Clarkson's inequality to deduce that each extremal of Morrey's inequality is axially symmetric and is antisymmetric with respect to reflection about a plane orthogonal to its axis of symmetry. We also use symmetrization methods to show that each extremal is monotone in the distance from its axis and in the direction of its axis when restricted to spheres centered at the intersection of its axis and its antisymmetry plane.

1. Introduction.

Sobolev's inequality asserts for each $p \in (1, n)$, there is a constant C such that

$$\left(\int_{\mathbb{R}^n} |u|^{p^*} dx\right)^{1/p^*} \leq C \left(\int_{\mathbb{R}^n} |Du|^p dx\right)^{1/p} \tag{1.1}$$

for each $u : \mathbb{R}^n \to \mathbb{R}$ whose first partial derivatives belong to $L^p(\mathbb{R}^n)$ and which decays fast enough at infinity. Here

$$p^* = \frac{np}{n-p}.$$

Employing rearrangement methods, Talenti found the smallest constant $C = C^*$ for which (1.1) holds [30]. Talenti also found the *Sobolev extremals* or functions for which equality holds in (1.1); up to scaling, dilating, and translating, they are given by

$$u(x) = \frac{1}{(1 + |x|^{\frac{p}{p-1}})^{n/p-1}} \quad (x \in \mathbb{R}^n).$$

Notice that u is radially symmetric and monotone in the distance from the origin.

Talenti's work on Sobolev's inequality stimulated a lot of interest within the mathematics community. These results were extended by Aubin for applications in Riemannian geometry [1, 2]. Moreover, these ideas led mathematicians to employ rearrangement methods [9, 29, 31, 32], to seek best constants [16, 21, 29], and to explore the role of symmetry in various functional inequalities [8, 13, 17, 22]. In recent years, researchers have also been using new techniques such as optimal transport to pursue these types of results [3, 12, 23]. Additionally, a lot of work has been done to quantify these assertions via stability estimates [4, 7, 10, 25, 27, 28].

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However, much less is known about the equality case of the corresponding inequality for $p \in (n, \infty)$. In this setting, Morrey showed there is C such that

$$\sup_{x \neq y} \left\{ \frac{|u(x) - u(y)|}{|x - y|^{1-n/p}} \right\} \leq C \left(\int_{\mathbb{R}^n} |Du|^p dx \right)^{1/p} \tag{1.2}$$

for $u \in \mathcal{D}^{1,p}(\mathbb{R}^n)$ [24]; here $\mathcal{D}^{1,p}(\mathbb{R}^n)$ is the space of weakly differentiable $u : \mathbb{R}^n \to \mathbb{R}$ for which

$$u_{x_1}, \ldots, u_{x_n} \in L^p(\mathbb{R}^n).$$

This is known as Morrey's inequality. We note that since $p > n$, each $u \in \mathcal{D}^{1,p}(\mathbb{R}^n)$ agrees almost everywhere with a function that is Hölder continuous with exponent $1 - n/p$. Without loss of generality, we may then consider $\mathcal{D}^{1,p}(\mathbb{R}^n)$ as a subset of the continuous functions on $\mathbb{R}^n$ and identify each $u \in \mathcal{D}^{1,p}(\mathbb{R}^n)$ with its Hölder continuous representative.

In recent work, we showed that there is a smallest $C = C_* > 0$ so that Morrey's inequality holds with C_* and that there exist nonconstant functions for which equality is attained in (1.2) [19]. We will call these functions *Morrey extremals*. We also verified that after appropriately rotating, scaling, dilating, and translating a Morrey extremal u, it satisfies

$$\sup_{x \neq y} \left\{ \frac{|u(x) - u(y)|}{|x - y|^{1-n/p}} \right\} = \frac{|u(e_n) - u(-e_n)|}{|e_n - (-e_n)|^{1-n/p}} \tag{1.3}$$

with

$$u(e_n) = 1 \text{ and } u(-e_n) = -1. \tag{1.4}$$

Here $e_n \in \mathbb{R}^n$ is the point with 1 in its nth coordinate and 0 otherwise. Furthermore, the PDE

$$-\Delta_p u = c(\delta_{e_n} - \delta_{-e_n}) \tag{1.5}$$

holds weakly in $\mathbb{R}^n$ for a constant $c > 0$.

While C_* and the corresponding Morrey extremals are not explicitly known, many qualitative properties of these functions have been identified. In particular, Morrey extremals which satisfy (1.3) and (1.4) are known to be unique, axially symmetric about the x_n-axis and antisymmetric about the $x_n = 0$ plane ([19], Section 3 and 6). We established this in our previous work by relying on a uniqueness property of solutions of (1.5). In this paper, we will verify the following theorem as a consequence of Clarkson's inequality.

Theorem 1.1. *Suppose $p > n$, $n \geq 2$, and $u \in \mathcal{D}^{1,p}(\mathbb{R}^n)$ is the Morrey extremal which satisfies (1.3) and (1.4). Then*

$$u(Ox) = u(x), \quad x \in \mathbb{R}^n \tag{1.6}$$

for each orthogonal transformation $O : \mathbb{R}^n \to \mathbb{R}^n$ with $Oe_n = e_n$. Moreover,

$$u(Tx) = -u(x), \quad x \in \mathbb{R}^n \tag{1.7}$$

where $Tx = x - 2x_n e_n$.

In addition to the symmetries listed above, the Morrey extremal u which satisfies (1.3) and (1.4) has some interesting monotonicity features. The first feature is that u is either nonincreasing or nondecreasing in the distance from the x_n-axis. The key here is that in addition to being axially symmetric u is positive and quasiconcave when restricted to the half space $x_n > 0$.

Theorem 1.2. *Assume $p > n$, $n \geq 2$, and $u \in \mathcal{D}^{1,p}(\mathbb{R}^n)$ is the Morrey extremal which satisfies (1.3) and (1.4). If $x^1, x^2 \in \mathbb{R}^n$ with*

$$x_n^1 = x_n^2 \geq 0 \ \ \textit{and} \ \ |x^2| \leq |x^1|,$$

or if

$$x_n^1 = x_n^2 \leq 0 \ \ \textit{and} \ \ |x^1| \leq |x^2|,$$

then

$$u(x^1) \leq u(x^2).$$

The second monotonicity feature is that u is nondecreasing in the x_n variable when restricted to each sphere centered at the origin. We will use symmetrization methods to prove this and employ a certain Pólya-Szegö inequality. In particular, we will verify more generally that u^+ and u^- are equal to their cap rearrangements as defined in Section 5.

Theorem 1.3. *Assume $p > n$, $n \geq 2$, and $u \in \mathcal{D}^{1,p}(\mathbb{R}^n)$ is the Morrey extremal which satisfies (1.3) and (1.4). If $x^1, x^2 \in \mathbb{R}^n$ with*

$$|x^1| = |x^2| \ \ \textit{and} \ \ x_n^1 \leq x_n^2,$$

then

$$u(x^1) \leq u(x^2).$$

In what follows, we will prove Theorems 1.1, 1.2, 1.3 in sections 2, 4 and 5, respectively. In addition, we will take a detour to verify the axial symmetry of Morrey extremals using the "axial average" and "axial sweep" transformations presented in section 3. In the appendix, we'll also prove a useful approximation result for functions in $\mathcal{D}^{1,p}(\mathbb{R}^n)$ with $p > n$. Finally, we would like to thank Eric Carlen, Elliott Lieb, and Peter McGrath for their advice and insightful discussions related to this work.

2. Axial symmetry and reflectional antisymmetry.

For the remainder of this note, we will suppose

$$p > n \quad \text{and} \quad n \geq 2.$$

We will also use the notation

$$[v]_{1-n/p} := \sup_{x \neq y} \left\{ \frac{|v(x) - v(y)|}{|x - y|^{1-n/p}} \right\} \tag{2.1}$$

for the $1 - n/p$ Hölder seminorm of v. This will allow us to write the sharp form of Morrey's inequality (1.2) a bit more concisely as

$$[v]_{1-n/p} \leq C_* \left(\int_{\mathbb{R}^n} |Dv|^p dx \right)^{1/p}.$$

Let us now recall the elementary inequality: for $a, b \in \mathbb{R}^n$,

$$\left| \frac{a+b}{2} \right|^p + \left| \frac{a-b}{2} \right|^p \leq \frac{1}{2}|a|^p + \frac{1}{2}|b|^p. \tag{2.2}$$

This type of inequality was studied by Clarkson [11] in connection with uniformly convex spaces, and it immediately implies

$$\int_{\mathbb{R}^n} \left| \frac{Dv + Dw}{2} \right|^p dx + \int_{\mathbb{R}^n} \left| \frac{Dv - Dw}{2} \right|^p dx \leq \frac{1}{2} \int_{\mathbb{R}^n} |Dv|^p dx + \frac{1}{2} \int_{\mathbb{R}^n} |Dw|^p dx \tag{2.3}$$

for $v, w \in \mathcal{D}^{1,p}(\mathbb{R}^n)$. A direct consequence is as follows.

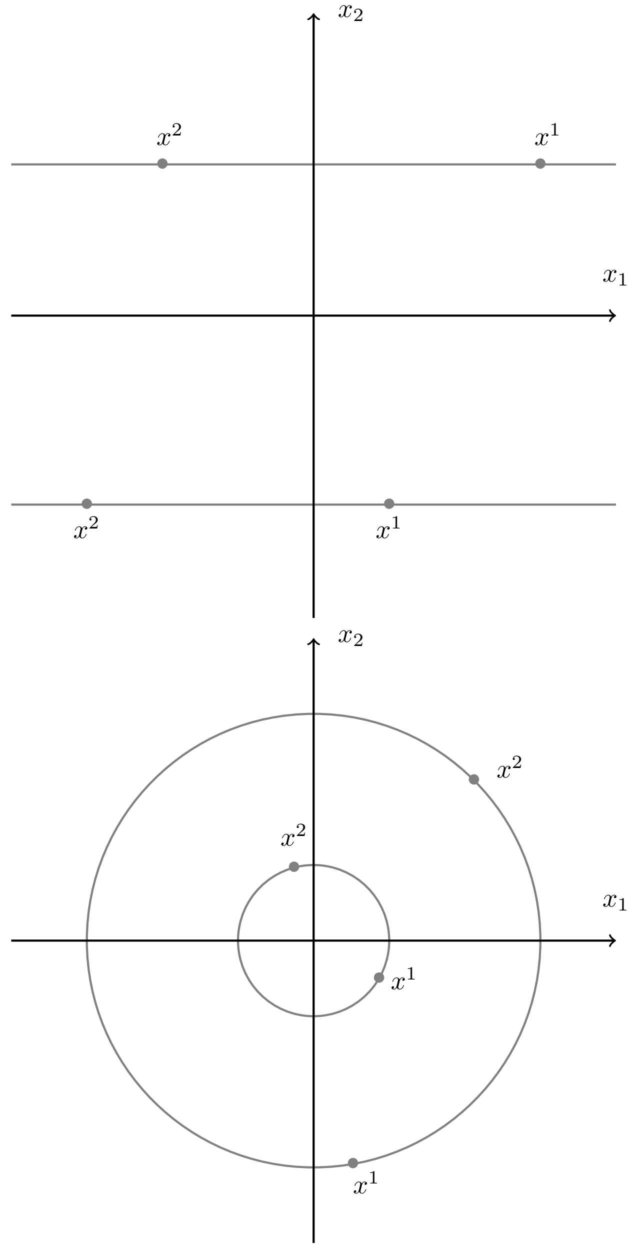

FIGURE 1. These diagrams illustrate the monotonicity properties of the Morrey extremal satisfying (1.3) and (1.4) for $n = 2$. Theorems 1.2 and 1.3 respectively assert that $u(x^1) \le u(x^2)$ for $x^1, x^2 \in \mathbb{R}^2$ which are ordered as on the horizontal lines in the top diagram and as on each circle in the bottom diagram.

Lemma 2.1. *Suppose u is a Morrey extremal which satisfies (1.3) and (1.4). Further assume $v \in \mathcal{D}^{1,p}(\mathbb{R}^n)$ satisfies*

$$v(e_n) = 1 \text{ and } v(-e_n) = -1 \tag{2.4}$$

and

$$\int_{\mathbb{R}^n} |Dv|^p dx \leq \int_{\mathbb{R}^n} |Du|^p dx.$$

Then $u \equiv v$.

Proof. Define

$$w := \frac{u+v}{2}.$$

Our first assumption on v gives

$$w(e_n) - w(-e_n) = \frac{v(e_n) - v(-e_n)}{2} + \frac{u(e_n) - u(-e_n)}{2} = u(e_n) - u(-e_n).$$

It follows that

$$[w]_{1-n/p} \geq \frac{|u(e_n) - u(-e_n)|}{|e_n - (-e_n)|^{1-n/p}} = [u]_{1-n/p}.$$

Inequality (2.3) and our second assumption on v imply

$$\begin{aligned}&\int_{\mathbb{R}^n} |Dw|^p dx + \int_{\mathbb{R}^n} \left|\frac{Du - Dv}{2}\right|^p dx \\ \leq& \frac{1}{2}\int_{\mathbb{R}^n} |Du|^p dx + \frac{1}{2}\int_{\mathbb{R}^n} |Dv|^p dx \leq \int_{\mathbb{R}^n} |Du|^p dx.\end{aligned} \tag{2.5}$$

In particular, if $Du \not\equiv Dv$ then

$$[u]_{1-n/p} \leq [w]_{1-n/p} \leq C_* \left(\int_{\mathbb{R}^n} |Dw|^p dx\right)^{1/p} < C_* \left(\int_{\mathbb{R}^n} |Du|^p dx\right)^{1/p}.$$

However, this would contradict our hypothesis that u is an extremal. Consequently, there is a constant c such that $v(x) = u(x) + c$ for all $x \in \mathbb{R}^n$. Choosing $x = e_n$ gives

$$u(e_n) = v(e_n) = u(e_n) + c.$$

That is, $c = 0$. □

Proof of Theorem 1.1. Let $O : \mathbb{R}^n \to \mathbb{R}^n$ be an orthogonal transformation which satisfies $Oe_n = e_n$ and set

$$v(x) = u(Ox), \quad x \in \mathbb{R}^n.$$

Then $v(e_n) = u(e_n) = 1$ and $v(-e_n) = u(O(-e_n)) = u(-e_n) = -1$. Moreover,

$$\int_{\mathbb{R}^n} |Dv|^p dx = \int_{\mathbb{R}^n} |Du|^p dx$$

by the change of variables theorem ([15], Theorem 2.44). In view of Lemma 2.1, $v(x) = u(x)$ for all $x \in \mathbb{R}^n$.

Now set

$$w(x) = -u\,(Tx)$$

for $x \in \mathbb{R}^n$, where $Tx = x - 2x_n e_n$. As $T(e_n) = -e_n$,

$$w(e_n) = 1 \text{ and } w(-e_n) = -1. \tag{2.6}$$

Furthermore, since T is an orthogonal transformation of $\mathbb{R}^n$, we can apply the change of variables theorem again to conclude

$$\int_{\mathbb{R}^n} |Dw|^p dx = \int_{\mathbb{R}^n} |Du|^p dx.$$

Lemma 2.1 then implies $w(x) = u(x)$ for all $x \in \mathbb{R}^n$. □

3. **Alternative proofs of axial symmetry.** In this section, we will use two transformations of $\mathcal{D}^{1,p}(\mathbb{R}^n)$ functions which result in functions which are axially symmetric with respect to the x_n-axis. To this end, it will be convenient for us to use the variables

$$x = (y, x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$$

and consider each $u \in \mathcal{D}^{1,p}(\mathbb{R}^n)$ as the function

$$(y, x_n) \mapsto u(y, x_n).$$

For $n \geq 3$ and $u \in \mathcal{D}^{1,p}(\mathbb{R}^n)$, we will also set

$$D_r u := \frac{D_y u \cdot y}{|y|^2} y \quad \text{and} \quad D_{\mathbb{S}^{n-2}} u := D_y u - D_r u.$$

This allows us to express the gradient of u as

$$Du = (D_r u + D_{\mathbb{S}^{n-2}} u, \partial_{x_n} u).$$

When $n = 2$, we will write

$$D_{\mathbb{S}^0} u(y, x_2) := \frac{\partial_y u(y, x_2) + \partial_y u(-y, x_2)}{2}$$

to stay consistent with our considerations below for $n \geq 3$.

In what follows, we'll also make use of the fact that each $u \in \mathcal{D}^{1,p}(\mathbb{R}^n)$ can be approximated by smooth functions. That is, for each $u \in \mathcal{D}^{1,p}(\mathbb{R}^n)$, there is $(u_k)_{k\in\mathbb{N}} \subset C^\infty(\mathbb{R}^n) \cap \mathcal{D}^{1,p}(\mathbb{R}^n)$ such that

$$\begin{cases} u_k \to u \text{ uniformly on } \mathbb{R}^n \\ Du_k \to Du \text{ in } L^p(\mathbb{R}^n; \mathbb{R}^n) \end{cases}$$

as $k \to \infty$. This assertion likely follows from a general approximation theorem. Nevertheless, we have written a short proof of this fact in Proposition 3 of the appendix.

3.1. **Axial average.** For a given $u \in \mathcal{D}^{1,p}(\mathbb{R}^n)$, set

$$u^*(y, x_n) := \begin{cases} \displaystyle⨍_{|z|=|y|} u(z, x_n) d\sigma(z), & |y| > 0 \\ u(0, x_n), & |y| = 0 \end{cases}$$

as its *axial average*. Here σ is $n-2$ dimensional Hausdorff measure. It is immediate from this definition that u^* is axially symmetric with respect to the x_n-axis. We'll establish a Hardy type inequality and then a Pólya-Szegö type inequality involving u and u^*.

Lemma 3.1. *There is a constant C such that*

$$\int_{\mathbb{R}^n} \frac{|u - u^*|^p}{|x - x_n e_n|^p} dx \leq C \int_{\mathbb{R}^n} |D_{\mathbb{S}^{n-2}} u|^p dx \tag{3.1}$$

for each $u \in \mathcal{D}^{1,p}(\mathbb{R}^n)$.

Proof. First assume $n \geq 3$ and $u \in C^\infty(\mathbb{R}^n) \cap \mathcal{D}^{1,p}(\mathbb{R}^n)$. Recall Poincaré's inequality on $\mathbb{S}^{n-2}$: there is a constant C such that

$$\int_{\mathbb{S}^{n-2}} \left| v - \fint_{\mathbb{S}^{n-2}} v d\sigma \right|^p d\sigma \leq C \int_{\mathbb{S}^{n-2}} |D_{\mathbb{S}^{n-2}} v|^p d\sigma \tag{3.2}$$

for each $v \in C^\infty(\mathbb{S}^{n-2})$. This inequality can be proved by a minor variation of Theorem 2.10 in [18]. Substituting $v(\xi) = u(r\xi, x_n)$ gives

$$\int_{|y|=r} \left| u(y, x_n) - \fint_{|z|=r} u(z, x_n) d\sigma(z) \right|^p d\sigma(y) \leq C r^p \int_{|y|=r} |D_{\mathbb{S}^{n-2}} u(y, x_n)|^p d\sigma(y)$$

for each $r > 0$. That is,

$$\int_{|y|=r} \frac{|u(y, x_n) - u^*(y, x_n)|^p}{r^p} d\sigma(y) \leq C \int_{|y|=r} |D_{\mathbb{S}^{n-2}} u(y, x_n)|^p d\sigma(y).$$

Integrating this inequality over $(r, x_n) \in [0, R] \times [-L, L]$ leads to

$$\begin{aligned} &\int_{B_R \times [-L,L]} \frac{|u - u^*|^p}{|x - x_n e_n|^p} dx \\ =& \int_{-L}^{L} \int_0^R \int_{|y|=r} \frac{|u(y, x_n) - u^*(y, x_n)|^p}{|y|^p} d\sigma(y) dr dx_n \\ \leq& C \int_{-L}^{L} \int_0^R \int_{|y|=r} |D_{\mathbb{S}^{n-2}} u(y, x_n)|^p d\sigma(y) dr dx_n \\ =& C \int_{B_R \times [-L,L]} |D_{\mathbb{S}^{n-2}} u|^p dx \leq C \int_{\mathbb{R}^n} |D_{\mathbb{S}^{n-2}} u|^p dx. \end{aligned} \tag{3.3}$$

Here $B_R := B_R(0) \subset \mathbb{R}^{n-1}$. Using Proposition 3, it is routine to show (3.3) holds for each $u \in \mathcal{D}^{1,p}(\mathbb{R}^n)$. We then conclude (3.1) by sending $L, R \to \infty$.

Now suppose $n = 2$. As $D_{\mathbb{S}^0} u \in L^p(\mathbb{R}^2)$,

$$\int_{\mathbb{R}^2} |D_{\mathbb{S}^0} u|^p dx = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |D_{\mathbb{S}^0} u(y, x_2)|^p dy \right) dx_2 < \infty.$$

Consequently,

$$\int_{\mathbb{R}} |D_{\mathbb{S}^0} u(y, x_2)|^p dy < \infty$$

for almost every $x_2 \in \mathbb{R}$. For any such x_2, we can apply Hardy's inequality

$$\int_{\mathbb{R}} \frac{|f(y)|^p}{|y|^p} dy \leq c_p \int_{\mathbb{R}} |f'(y)|^p dy \tag{3.4}$$

to find

$$\begin{aligned} &\int_{\mathbb{R}} \frac{|u(y, x_2) - u^*(y, x_2)|^p}{|y|^p} dy = \int_{\mathbb{R}} \frac{|(u(y, x_2) - u(-y, x_2))/2|^p}{|y|^p} dy \\ \leq& c_p \int_{\mathbb{R}} |D_{\mathbb{S}^0} u(y, x_2)|^p dy. \end{aligned} \tag{3.5}$$

The inequality (3.1) now follows from integrating over x_2. □

Proposition 1. *For all $u \in \mathcal{D}^{1,p}(\mathbb{R}^n)$,*

$$\int_{\mathbb{R}^n} |Du^*|^p dx \leq \int_{\mathbb{R}^n} |Du|^p dx. \tag{3.6}$$

Equality holds if and only if $u = u^*$.

Proof. Let us first assume $n \geq 3$ and that $u \in C^\infty(\mathbb{R}^n) \cap \mathcal{D}^{1,p}(\mathbb{R}^n)$. Direct computation gives

$$Du^*(y,x_n) = \fint_{|z|=r} \left(\left[D_y u(z,x_n) \cdot \frac{z}{r} \right] \frac{y}{r}, \partial_{x_n} u(z,x_n) \right) d\sigma(z) \tag{3.7}$$

for $r = |y| > 0$; and by Jensen's inequality,

$$\begin{aligned} |Du^*(y,x_n)|^p &\leq \fint_{|z|=r} \left| \left(\left[D_y u(z,x_n) \cdot \frac{z}{r} \right] \frac{y}{r}, \partial_{x_n} u(z,x_n) \right) \right|^p d\sigma(z) \\ &= \fint_{|z|=r} |(D_r u(z,x_n), \partial_{x_n} u(z,x_n)|^p d\sigma(z). \end{aligned} \tag{3.8}$$

It follows that

$$\begin{aligned} &\int_{\mathbb{R}^n} |Du^*|^p dx \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} |Du^*(y,x_n)|^p dy dx_n = \int_{\mathbb{R}} \int_0^\infty \left(\int_{|y|=r} |Du^*(y,x_n)|^p d\sigma(y) \right) dr dx_n \\ &\leq \int_{\mathbb{R}} \int_0^\infty \int_{|y|=r} \left(\fint_{|z|=r} |(D_r u(z,x_n), \partial_{x_n} u(z,x_n))|^p d\sigma(z) \right) d\sigma(y) dr dx_n \\ &= \int_{\mathbb{R}} \int_0^\infty \int_{|z|=r} |(D_r u(z,x_n), \partial_{x_n} u(z,x_n))|^p d\sigma(z) dr dx_n \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} |(D_r u(z,x_n), \partial_{x_n} u(z,x_n))|^p dz dx_n = \int_{\mathbb{R}^n} |(D_r u, \partial_{x_n} u)|^p dx. \end{aligned} \tag{3.9}$$

Combining this inequality with Proposition 3, we deduce

$$\int_{\mathbb{R}^n} |Du^*|^p dx \leq \int_{\mathbb{R}^n} |(D_r u, \partial_{x_n} u)|^p dx \tag{3.10}$$

for each $u \in \mathcal{D}^{1,p}(\mathbb{R}^n)$. Employing the elementary inequality

$$|a|^p + |b|^p \leq (|a|^2 + |b|^2)^{p/2}, \quad a, b \in \mathbb{R}^n$$

with $a = (D_r u, u_{x_n})$ and $b = (D_{\mathbb{S}^{n-2}} u, 0)$ gives

$$|(D_r u, u_{x_n})|^p + |D_{\mathbb{S}^{n-2}} u|^p \leq |Du|^p. \tag{3.11}$$

It follows that

$$\int_{\mathbb{R}^n} |Du^*|^p dx \leq \int_{\mathbb{R}^n} |Du|^p dx - \int_{\mathbb{R}^n} |D_{\mathbb{S}^{n-2}} u|^p dx. \tag{3.12}$$

Consequently, if equality holds in (3.6),

$$\int_{\mathbb{R}^n} |D_{\mathbb{S}^{n-2}} u|^p dx = 0.$$

We can then appeal to (3.1) to find $u = u^*$.

Now suppose $n = 2$. Here

$$u^*(y,x_2) = \frac{u(y,x_2) + u(-y,x_2)}{2} \tag{3.13}$$

and

$$Du^*(y,x_2) = \frac{Du(y,x_2) + (-\partial_y u(-y,x_2), \partial_{x_2} u(-y,x_2))}{2}. \tag{3.14}$$

By Clarkson's inequality (2.3),

$$\begin{aligned}
&\int_{\mathbb{R}^2} |Du^*|^p dx \\
=&\iint_{\mathbb{R}^2} \left|\frac{Du(y,x_2)+(-\partial_y u(-y,x_2),\partial_{x_2}u(-y,x_2))}{2}\right|^p dydx_2 \\
\le&\frac{1}{2}\iint_{\mathbb{R}^2} |Du(y,x_2)|^p dydx_2 + \frac{1}{2}\iint_{\mathbb{R}^2} |(-\partial_y u(-y,x_2),\partial_{x_2}u(-y,x_2))|^p dydx_2 \\
&-\iint_{\mathbb{R}^2} \left|\frac{Du(y,x_2)-(-\partial_y u(-y,x_2),\partial_{x_2}u(-y,x_2))}{2}\right|^p dydx_2 \\
=&\frac{1}{2}\iint_{\mathbb{R}^2} |Du(y,x_2)|^p dydx_2 + \frac{1}{2}\iint_{\mathbb{R}^2} |Du(-y,x_2)|^p dydx_2 \\
&-\iint_{\mathbb{R}^2} \left|\left(\frac{\partial_y u(y,x_2)+\partial_y u(-y,x_2)}{2},\frac{\partial_{x_2}u(y,x_2)-\partial_{x_2}u(-y,x_2)}{2}\right)\right|^p dydx_2 \\
\le&\int_{\mathbb{R}^2} |Du|^p dx - \int_{\mathbb{R}^2} |D_{\mathbb{S}^0}u|^p dx.
\end{aligned}$$

If equality holds in (3.6), we can again appeal to (3.1) to find $u = u^*$. □

As $u = u^*$ along the x_n-axis, the following corollary follows directly from Lemma 2.1 and inequality (3.6).

Corollary 1. *Suppose $u \in \mathcal{D}^{1,p}(\mathbb{R}^n)$ is a Morrey extremal which satisfies (1.3) and (1.4). Then $u = u^*$.*

3.2. **Axial sweep.** For a given $u \in \mathcal{D}^{1,p}(\mathbb{R}^n)$, we will also consider its *axial sweep*

$$u^\zeta(y, x_n) := u(|y|\zeta, x_n) \tag{3.15}$$

with respect to a direction $\zeta \in \mathbb{S}^{n-2}$. Clearly, u^ζ is axially symmetric for each ζ. In this section, we shall prove the following assertion.

Proposition 2. *Suppose $u \in \mathcal{D}^{1,p}(\mathbb{R}^n)$. There is $\zeta \in \mathbb{S}^{n-2}$ for which*

$$\int_{\mathbb{R}^n} |Du^\zeta|^p dx \le \int_{\mathbb{R}^n} |Du|^p dx. \tag{3.16}$$

If $n \ge 3$ and u is not axially symmetric, ζ can be chosen so that this inequality is strict.

As $u = u^\zeta$ along the x_n-axis, it follows immediately from Lemma 2.1 and Proposition 2 that each extremal is axially symmetric.

Corollary 2. *Suppose $u \in \mathcal{D}^{1,p}(\mathbb{R}^n)$ is a Morrey extremal which satisfies (1.3) and (1.4). Then $u = u^\zeta$ for every $\zeta \in \mathbb{S}^{n-2}$.*

The key to proving Proposition 2 is the following inequality.

Lemma 3.2. *Suppose $u \in \mathcal{D}^{1,p}(\mathbb{R}^n)$. Then $u^\zeta \in \mathcal{D}^{1,p}(\mathbb{R}^n)$ for σ almost every $\zeta \in \mathbb{S}^{n-2}$ and*

$$\fint_{|\zeta|=1}\left(\int_{\mathbb{R}^n} |Du^\zeta|^p dx\right) d\sigma(\zeta) \le \int_{\mathbb{R}^n} |Du|^p dx. \tag{3.17}$$

If equality holds and $n \ge 3$, $u = u^\zeta$ for each $\zeta \in \mathbb{S}^{n-2}$.

Proof. 1. First suppose $n = 2$. Note $u^{\pm 1}(y, x_2) = u(\pm|y|, x_2)$. Direct computation shows

$$\int_{\mathbb{R}^2} |Du^1|^p dx = 2\int_{\mathbb{R}}\int_0^\infty |Du(y,x_2)|^p dydx_2$$

and

$$\int_{\mathbb{R}^2} |Du^{-1}|^p dx = 2\int_{\mathbb{R}}\int_{-\infty}^0 |Du(y,x_2)|^p dydx_2.$$

Therefore,

$$⨍_{|\zeta|=1}\left(\int_{\mathbb{R}^n}|Du^\zeta|^p dx\right)d\sigma(\zeta) = \int_{\mathbb{R}^2}\frac{|Du^1|^p + |Du^{-1}|^p}{2}dx = \int_{\mathbb{R}^2}|Du|^p dx. \quad (3.18)$$

2. Now let's assume $n \geq 3$ and that $u \in C^\infty(\mathbb{R}^n) \cap \mathcal{D}^{1,p}(\mathbb{R}^n)$. Note that for $r = |y| > 0$,

$$Du^\zeta(y, x_n) = \left([D_y u(r\zeta, x_n)\cdot\zeta]\,\frac{y}{r}, \partial_{x_n} u(r\zeta, x_n)\right) \quad (3.19)$$

and

$$|Du^\zeta(y,x_n)| = |(D_r u(r\zeta, x_n), \partial_{x_n} u(r\zeta, x_n))|. \quad (3.20)$$

As a result,

$$\begin{aligned} &⨍_{|\zeta|=1}|Du^\zeta(y,x_n)|^p d\sigma(\zeta)\\ &= ⨍_{|\zeta|=1}\left|(D_r u(r\zeta,x_n), \partial_{x_n}u(r\zeta,x_n))\right|^p d\sigma(\zeta)\\ &= ⨍_{|z|=r}\left|(D_r u(z,x_n), \partial_{x_n}u(z,x_n))\right|^p d\sigma(z). \end{aligned} \quad (3.21)$$

Observe that $|Du^\zeta(y,x_n)|$ is continuous on set of $(y, x_n, \zeta) \in (\mathbb{R}^{n-1}\setminus\{0\})\times\mathbb{R}\times\mathbb{S}^{n-2}$. So we can apply Fubini's theorem and conclude

$$\begin{aligned} &⨍_{|\zeta|=1}\left(\int_{\mathbb{R}^n}|Du^\zeta|^p dx\right)d\sigma(\zeta)\\ &=\int_{\mathbb{R}^n}\left(⨍_{|\zeta|=1}|Du^\zeta|^p d\sigma(\zeta)\right)dx\\ &=\int_{\mathbb{R}}\int_0^\infty\int_{|y|=r}\left(⨍_{|\zeta|=1}|Du^\zeta(y,x_n)|^p d\sigma(\zeta)\right)d\sigma(y)drdx_n\\ &=\int_{\mathbb{R}}\int_0^\infty\int_{|y|=r}\left(⨍_{|z|=r}\left|(D_r u(z,x_n),\partial_{x_n}u(z,x_n))\right|^p d\sigma(z)\right)d\sigma(y)drdx_n\\ &=\int_{\mathbb{R}}\int_0^\infty\int_{|z|=r}|(D_r u(z,x_n),\partial_{x_n}u(z,x_n))|^p d\sigma(z)drdx_n\\ &=\int_{\mathbb{R}}\int_{\mathbb{R}^{n-1}}|(D_r u(z,x_n),\partial_{x_n}u(z,x_n))|^p dzdx_n = \int_{\mathbb{R}^n}|(D_r u, \partial_{x_n}u)|^p dx. \end{aligned} \quad (3.22)$$

3. For $u \in \mathcal{D}^{1,p}(\mathbb{R}^n)$, we can select $(u_k)_{k\in\mathbb{N}} \subset C^\infty(\mathbb{R}^n)\cap\mathcal{D}^{1,p}(\mathbb{R}^n)$ such that $u_k \to u$ uniformly in $\mathbb{R}^n$ and $Du_k \to Du$ in $L^p(\mathbb{R}^n;\mathbb{R}^n)$ as $k\to\infty$. It is also easy to check from the definition that $u_k^\zeta \to u^\zeta$ uniformly on $\mathbb{R}^n$ for each $\zeta\in\mathbb{S}^{n-2}$. We define

$$v_k(y,x_n,\zeta) = \begin{cases} Du_k^\zeta(y,x_n), & y\neq 0\\ 0, & y = 0\end{cases}$$

for $(y, x_n, \zeta) \in \mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{S}^{n-2}$. By the estimate derived above

$$\begin{aligned}&\iint_{\mathbb{R}^n \times \mathbb{S}^{n-2}} |v_k|^p d(\mathcal{L} \times \sigma)\\&= \int_{|\zeta|=1} \left(\int_{\mathbb{R}^n} |Du_k^\zeta|^p dx\right) d\sigma(\zeta) = \sigma(\mathbb{S}^{n-2}) \int_{\mathbb{R}^n} |(D_r u_k, \partial_{x_n} u_k)|^p dx. \end{aligned} \tag{3.23}$$

Here $\mathcal{L}$ denotes Lebesgue measure on $\mathbb{R}^n$.

Since $Du_k \to Du$ in $L^p(\mathbb{R}^n; \mathbb{R}^n)$,

$$\sup_{k\in\mathbb{N}} \iint_{\mathbb{R}^n \times \mathbb{S}^{n-2}} |v_k|^p d(\mathcal{L} \times \sigma) < \infty. \tag{3.24}$$

As a result, there is a measurable $v : \mathbb{R}^n \times \mathbb{S}^{n-2} \to \mathbb{R}^n$ with $|v| \in L^p(\mathbb{R}^n \times \mathbb{S}^{n-2}; \mathcal{L} \times \sigma)$ and a subsequence $(v_{k_j})_{j\in\mathbb{N}}$ such that

$$\lim_{j\to\infty} \iint_{\mathbb{R}^n \times \mathbb{S}^{n-2}} v_{k_j} \cdot \varphi d(\mathcal{L} \times \sigma) = \iint_{\mathbb{R}^n \times \mathbb{S}^{n-2}} v \cdot \varphi d(\mathcal{L} \times \sigma) \tag{3.25}$$

for each measurable $\varphi : \mathbb{R}^n \times \mathbb{S}^{n-2} \to \mathbb{R}^n$ with $|\varphi| \in L^{p/(p-1)}(\mathbb{R}^n \times \mathbb{S}^{n-2}; \mathcal{L} \times \sigma)$.

In view of this weak convergence, we also have

$$\begin{aligned}&\iint_{\mathbb{R}^n \times \mathbb{S}^{n-2}} |v|^p d(\mathcal{L} \times \sigma)\\&\le \liminf_{j\to\infty} \iint_{\mathbb{R}^n \times \mathbb{S}^{n-2}} |v_{k_j}|^p d(\mathcal{L} \times \sigma) \le \sigma(\mathbb{S}^{n-2}) \int_{\mathbb{R}^n} |(D_r u, \partial_{x_n} u)|^p dx. \end{aligned} \tag{3.26}$$

We can apply Fubini's theorem once again to find

$$\int_{|\zeta|=1} \left(\int_{\mathbb{R}^n} |v^\zeta(x)|^p dx\right) d\sigma(\zeta) \le \sigma(\mathbb{S}^{n-2}) \int_{\mathbb{R}^n} |(D_r u, \partial_{x_n} u)|^p dx, \tag{3.27}$$

where $v^\zeta := v(\cdot, \zeta) \in L^p(\mathbb{R}^n; \mathbb{R}^n)$ for σ almost every $\zeta \in \mathbb{S}^{n-2}$ (Theorem 2.37 of [15]).

4. We claim

$$Du^\zeta = v^\zeta \text{ for } \sigma \text{ almost every } \zeta. \tag{3.28}$$

Inequality (3.17) would then follow from (3.27). To this end, we let $\phi \in C_c^1(\mathbb{R}^n; \mathbb{R}^n)$, $\eta \in C(\mathbb{S}^{n-2})$, and integrate by parts to get

$$\begin{aligned}&\int_{\mathbb{R}^n \times \mathbb{S}^{n-2}} v_k \cdot \phi\eta \, d(\mathcal{L} \times \sigma)\\&= \int_{\mathbb{S}^{n-2}} \left(\int_{\mathbb{R}^n} Du_k^\zeta \cdot \phi dx\right) \eta d\sigma(\zeta) = -\int_{\mathbb{S}^{n-2}} \left(\int_{\mathbb{R}^n} u_k^\zeta \operatorname{div}(\phi) dx\right) \eta d\sigma(\zeta).\end{aligned}$$

Since $\operatorname{div}(\phi)$ has compact support and $u_k \to u$ uniformly in $\mathbb{R}^n$, we find in the limit as $k = k_j \to \infty$ that

$$\int_{\mathbb{R}^n \times \mathbb{S}^{n-2}} v \cdot \phi\eta d(\mathcal{L} \times \sigma) = -\int_{\mathbb{S}^{n-2}} \left(\int_{\mathbb{R}^n} u^\zeta \operatorname{div}(\phi) dx\right) \eta d\sigma(\zeta).$$

That is,

$$\int_{\mathbb{S}^{n-2}} \left(\int_{\mathbb{R}^n} v^\zeta \cdot \phi dx\right) \eta d\sigma(\zeta) = -\int_{\mathbb{S}^{n-2}} \left(\int_{\mathbb{R}^n} u^\zeta \operatorname{div}(\phi) dx\right) \eta d\sigma(\zeta).$$

And as η is arbitrary

$$\int_{\mathbb{R}^n} v^\zeta \cdot \phi dx = -\int_{\mathbb{R}^n} u^\zeta \operatorname{div}(\phi) dx \tag{3.29}$$

for σ almost every $\zeta \in \mathbb{S}^{n-2}$. Since $C^1_c(\mathbb{R}^n;\mathbb{R}^n)$ equipped with the norm

$$\phi \mapsto \max\{\|\phi\|_\infty, \|D\phi\|_\infty\}$$

is separable, (3.29) holds for a subset of $\mathbb{S}^{n-2}$ with full measure that is independent of ϕ. We conclude (3.28).

5. Observe that we have established

$$\begin{aligned}&\fint_{|\zeta|=1}\left(\int_{\mathbb{R}^n}|Du^\zeta|^p dx\right)d\sigma(\zeta)\\ &\le \int_{\mathbb{R}^n}|(D_r u, \partial_{x_n} u)|^p dx \le \int_{\mathbb{R}^n}|Du|^p dx - \int_{\mathbb{R}^n}|D_{\mathbb{S}^{n-2}} u|^p dx \end{aligned} \tag{3.30}$$

for $n \ge 3$. So if equality holds in (3.17) and $n \ge 3$,

$$\int_{\mathbb{R}^n}|D_{\mathbb{S}^{n-2}} u|^p dx = 0.$$

In view of (3.1), u is axially symmetric with respect to the x_n-axis and so $u = u^\zeta$ for each $\zeta \in \mathbb{S}^{n-2}$. □

Proof of Proposition 2. In view of inequality (3.17), there is a subset $S \subset \mathbb{S}^{n-2}$ for which $\sigma(S) > 0$ and (3.16) holds for σ almost every $\zeta \in S$. When $n \ge 3$, we have the refinement (3.30) which gives a subset $S \subset \mathbb{S}^{n-2}$ of positive measure such that

$$\int_{\mathbb{R}^n}|Du^\zeta|^p dx \le \int_{\mathbb{R}^n}|Du|^p dx - \int_{\mathbb{R}^n}|D_{\mathbb{S}^{n-2}} u|^p dx$$

for σ almost every $\zeta \in S$. If u is not axially symmetric, $\int_{\mathbb{R}^n}|D_{\mathbb{S}^{n-2}} u|^p dx > 0$ and (3.16) is strict. □

4. **Monotonicity from the axis of symmetry.** This section is dedicated to proving Theorem 1.2. To this end, let us suppose u is the Morrey extremal which satisfies (1.3) and (1.4). We established in previous work that when u is restricted to the half space $x_n > 0$, it assumes values between $(0, 1]$ and is quasiconcave ([19], Sections 3 and 4). As a result, $\{u \ge c\}$ is a convex subset of the $x_n > 0$ half space for each $c \in (0, 1]$. Furthermore, we established the limit

$$\lim_{|x|\to\infty} u(x) = 0$$

in [20]. Consequently, we have that $\{u \ge c\}$ is also compact for each $c \in (0, 1]$.

Since u is axially symmetric,

$$O(\{u \ge c\}) = \{u \ge c\}$$

for all orthogonal transformations $O : \mathbb{R}^n \to \mathbb{R}^n$ such that $Oe_n = e_n$. It then follows that for $c \in (0, 1]$ and $a > 0$,

$$\{y \in \mathbb{R}^{n-1} : u(y, a) \ge c\}$$

is a closed ball in $\mathbb{R}^{n-1}$ centered at the origin whenever it is nonempty. This will be the crucial observation we use to prove Theorem 1.2.

Proof of Theorem 1.2. Suppose $y_1, y_2 \in \mathbb{R}^{n-1}$ with

$$|y_2| \le |y_1| \tag{4.1}$$

and $a > 0$. Set

$$c = u(y_1, a).$$

Note that since $a > 0$, $c \in (0,1]$. Therefore,

$$y_1 \in \{y \in \mathbb{R}^{n-1} : u(y,a) \geq c\} = \overline{B_r(0)} \subset \mathbb{R}^{n-1}$$

for some $r \geq 0$. By (4.1), $y_2 \in \overline{B_r(0)}$, as well. Consequently,

$$y_2 \in \{y \in \mathbb{R}^{n-1} : u(y,a) \geq c\}$$

and in turn

$$u(y_2,a) \geq c = u(y_1,a).$$

The assertion

$$u(y_1,a) \leq u(y_2,a)$$

when

$$|y_1| \leq |y_2| \text{ and } a < 0$$

follows similarly. Finally, the conclusion of Theorem 1.1 that $u(x - 2x_n e_n) = -u(x)$ for $x \in \mathbb{R}^n$ implies $u(y,0) = 0$ for all $y \in \mathbb{R}^{n-1}$. □

5. **Cap symmetry.** In this section, we will recall the notion of the cap symmetrization of a subset of $\mathbb{R}^n$. This leads naturally to a way to rearrange the values of a function on $\mathbb{R}^n$. It turns out that the positive and negative parts of the Morrey extremal we have been studying in this paper are invariant under this rearrangement process. A key object in our study will be the *spherical cap*

$$C_{t,\theta} := \{x \in \mathbb{R}^n : |x| = t,\ x_n > t\cos\theta\} \tag{5.1}$$

with radius $t \geq 0$ and opening angle $\theta \in [0,\pi]$. We note that for $\theta > 0$, $C_{t,\theta}$ is an open subset of the sphere ∂B_t.

The following definition of cap symmetrization is due to Sarvas [26] (see also Brock and Solynin [6]). Observe that we will also change notation and now use σ to denote $n-1$ dimensional Hausdorff measure.

Definition 5.1. Suppose $A \subset \mathbb{R}^n$ is open. The cap symmetrization of A with respect to the positive x_n-axis is the subset $A^\star \subset \mathbb{R}^n$ which satisfies

$$A^\star \cap \partial B_t = \begin{cases} \emptyset & \text{if } A \cap \partial B_t = \emptyset \\ \partial B_t & \text{if } A \cap \partial B_t = \partial B_t \\ C_{t,\theta} & \text{otherwise} \end{cases}$$

for each $t \geq 0$. If $A^\star \cap \partial B_t = C_{t,\theta}$, $\theta \in [0,\pi]$ is chosen so that

$$\sigma(A \cap \partial B_t) = \sigma(C_{t,\theta}).$$

Since $A^\star \cap \partial B_t$ is specified for each $t \geq 0$, this uniquely defines $A^\star$ when A is open. If $A \subset \mathbb{R}^n$ is closed, we define $A^\star$ as above with

$$\overline{C_{t,\theta}} = \{x \in \mathbb{R}^n : |x| = t,\ x_n \geq t\cos\theta\}$$

replacing $C_{t,\theta}$. It's plain to see that if $A \subset B$, then $A^\star \subset B^\star$. It is also not hard to deduce the implication

$$A \subset \{x_n > 0\} \Longrightarrow A^\star \subset \{x_n > 0\}. \tag{5.2}$$

Moreover, it is known that if A is open, $A^\star$ is open; and if A is closed, then $A^\star$ is closed ([26], Section 2). Furthermore, we can apply the co-area formula to conclude $A^\star$ and A have the same Lebesgue measure.

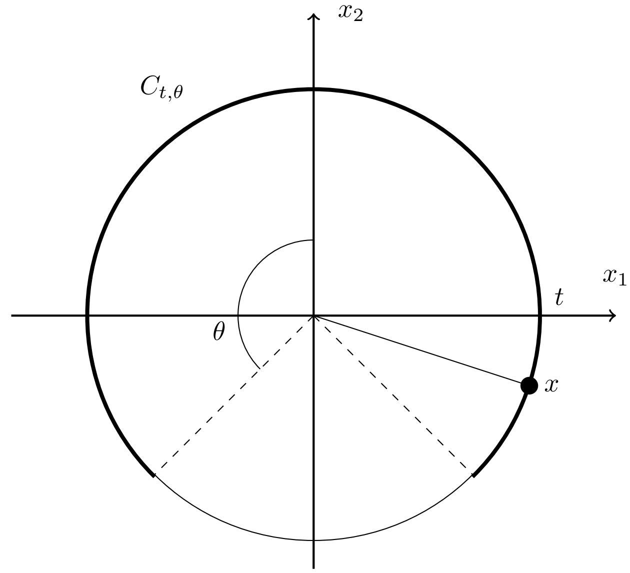

FIGURE 2. The spherical cap $C_{t,\theta}$ of radius t and opening angle θ. The cap contains all points $x \in \mathbb{R}^2$ such that $|x| = t$ and $x_2 > t\cos\theta$.

Definition 5.2. We say $v \in \mathcal{D}^{1,p}(\mathbb{R}^n)$ is *admissible* if $\{v > c\}$ has finite Lebesgue measure for each $c > \inf v$. In this case, we set

$$v^\star(x) := \sup\{c > \inf v : x \in \{v > c\}^\star\}, \quad x \in \mathbb{R}^n$$

as the cap rearrangement of v with respect to the x_n axis.

It is known that since $v \in \mathcal{D}^{1,p}(\mathbb{R}^n)$ is continuous, $v^\star$ is continuous with

$$\{v^\star > c\} = \{v > c\}^\star$$

for $c > \inf v$ ([6], Section 3). We will show that $v^\star$ is axially symmetric and also that it has the monotonicity property as described in Theorem 1.3. Then we will explain how this translates to Morrey extremals.

Lemma 5.3. *Suppose $v \in \mathcal{D}^{1,p}(\mathbb{R}^n)$ is admissible. Then $v^\star$ is axially symmetric with respect to the x_n−axis.*

Proof. Suppose $O : \mathbb{R}^n \to \mathbb{R}^n$ is an orthogonal transformation with $Oe_n = e_n$. Observe

$$O^{-1}(\partial B_t) = \partial B_t \text{ and } O^{-1}(C_{t,\theta}) = C_{t,\theta}$$

for each $t \geq 0$ and $\theta \in [0, \pi]$. As a result,

$$\begin{aligned}&\{v^\star \circ O > c\} \cap \partial B_t \\ =&O^{-1}(\{v^\star > c\}) \cap \partial B_t = O^{-1}(\{v^\star > c\}) \cap O^{-1}(\partial B_t) \\ =&O^{-1}\left(\{v^\star > c\} \cap \partial B_t\right) = \{v^\star > c\} \cap \partial B_t\end{aligned} \tag{5.3}$$

for each $t \geq 0$ and $c > \inf v^\star$. Consequently,

$$\{v^\star \circ O > c\} = \{v^\star > c\}$$

for all $c > \inf v^\star$ and in turn $v^\star \circ O = v^\star$. □

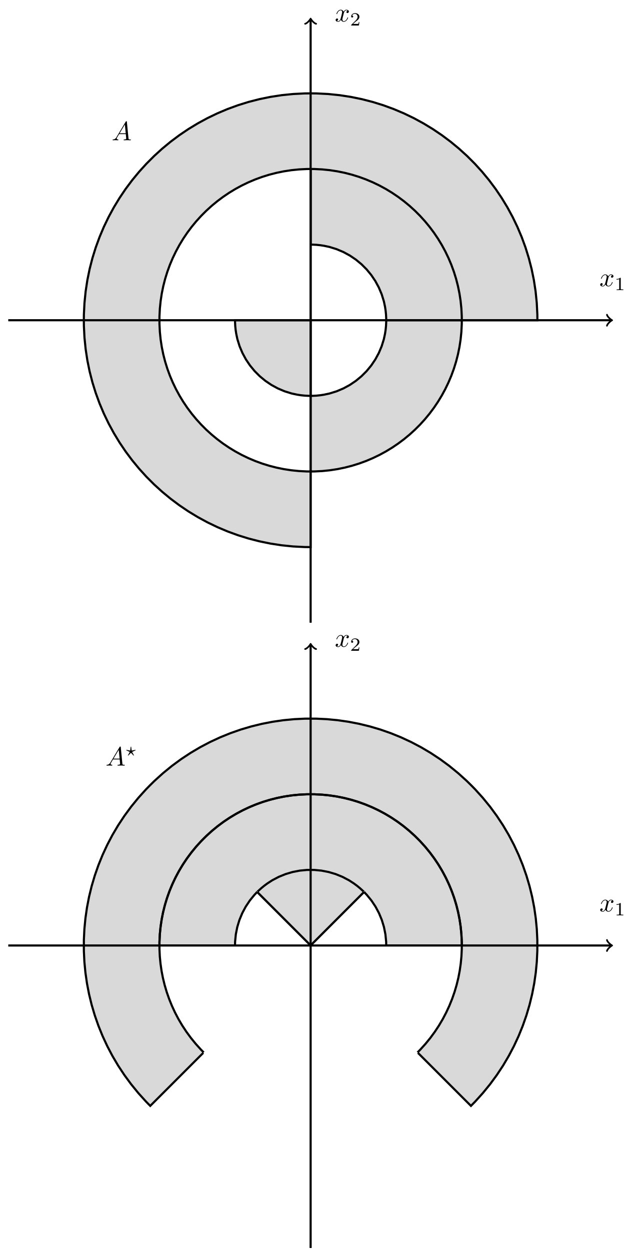

FIGURE 3. The shaded region on top represents a (closed) subset $A \subset \mathbb{R}^2$. The shaded region on bottom is $A^\star \subset \mathbb{R}^2$, the cap symmetrization of A in the direction of the positive x_2 axis.

Lemma 5.4. *Suppose $v \in \mathcal{D}^{1,p}(\mathbb{R}^n)$ is admissible. If $x^1, x^2 \in \mathbb{R}^n$ satisfies*

$$|x^1| = |x^2| \ \textit{and} \ x^1_n \leq x^2_n, \tag{5.4}$$

then

$$v^\star(x^1) \leq v^\star(x^2).$$

Proof. Set $c = v^\star(x^1)$ and $t = |x^1|$. Recall that

$$\{v^\star \ge c\} \cap \partial B_t = \{v \ge c\}^\star \cap \partial B_t = \overline{C_{t,\theta}}$$

for some $\theta \in [0, \pi]$. By (5.4), $x^2 \in \overline{C_{t,\theta}}$. Consequently,

$$v^\star(x^2) \ge c = v^\star(x^1).$$ □

Corollary 3. *Suppose $v \in \mathcal{D}^{1,p}(\mathbb{R}^n)$ is admissible. Then*

$$v^\star(e_n) = \max_{\partial B_1} v.$$

Proof. Since $\{v > \max_{\partial B_1} v\} \cap \partial B_1 = \emptyset$,

$$\{v > \max_{\partial B_1} v\}^\star \cap \partial B_1 = \{v^\star > \max_{\partial B_1} v\} \cap \partial B_1 = \emptyset.$$

It follows that

$$v^\star(e_n) \le \max_{\partial B_1} v.$$

For any $\epsilon > 0$, $\{v > \max_{\partial B_1} v - \epsilon\} \cap \partial B_1$ is nonempty and open. Therefore,

$$\{v > \max_{\partial B_1} v - \epsilon\}^\star \cap \partial B_1 = \{v^\star > \max_{\partial B_1} v - \epsilon\} \cap \partial B_1 \supset C_{1,\theta}$$

for some $\theta \in (0, \pi]$. It follows that $e_n \in \{v^\star > \max_{\partial B_1} v - \epsilon\} \cap \partial B_1$ and so

$$v^\star(e_n) > \max_{\partial B_1} v - \epsilon.$$

As $\epsilon > 0$ is arbitrary,

$$v^\star(e_n) \ge \max_{\partial B_1} v.$$ □

The last fact we will need in order to prove Theorem 1.3 is the Pólya-Szegö inequality. It states for each admissible $v \in \mathcal{D}^{1,p}(\mathbb{R}^n)$, the inequality

$$\int_{\mathbb{R}^n} |Dv^\star|^p \, dx \le \int_{\mathbb{R}^n} |Dv|^p dx \tag{5.5}$$

holds. This inequality and various other properties of cap and Steiner symmetrizations were verified by Van Schaftingen in [32].

Proof of Theorem 1.3. Define

$$v(x) := \max\{u(x), 0\}, \quad x \in \mathbb{R}^n,$$

where u is the Morrey extremal which satisfies (1.3) and (1.4). It is easy to check that $v \in \mathcal{D}^{1,p}(\mathbb{R}^n)$; and v is admissible since $\{u \ge c\}$ is compact for $c \in (0, 1]$. We recall that $u(e_n) = \sup_{\mathbb{R}^n} u = 1$, and in view of Lemma 3,

$$v^\star(e_n) = \max_{\partial B_1} u = 1.$$

Inequality (5.5) also implies

$$\int_{\mathbb{R}^n} |Dv^\star|^p \, dx \le \int_{\mathbb{R}^n} |Dv|^p \, dx = \int_{\{x_n > 0\}} |Du|^p dx.$$

By definition, $v^\star \ge 0$. We also note $\{u > c\} \subset \{x_n > 0\}$ for $c > 0$. As a result,

$$\{v^\star > c\} = \{v > c\}^\star = \{u > c\}^\star \subset \{x_n > 0\}$$

by (5.2). It follows that $v^\star|_{\{x_n \le 0\}} = 0$. Consequently,

$$\int_{\{x_n > 0\}} |Dv^\star|^p \, dx \le \int_{\{x_n > 0\}} |Du|^p dx$$

and

$$v^\star|_{\{x_n=0\}} = 0. \tag{5.6}$$

Define w as the odd extension of $v^\star|_{\{x_n\ge 0\}}$ to the half space $x_n < 0$. That is, we set

$$w(x) = \begin{cases} v^\star(x), & x_n \ge 0 \\ -v^\star(x - 2x_n e_n), & x_n < 0. \end{cases}$$

Using (5.6), it is straightforward to verify $w \in \mathcal{D}^{1,p}(\mathbb{R}^n)$. Also observe

$$w(e_n) = v^\star(e_n) = 1 \;\text{ and }\; w(-e_n) = -v^\star(e_n) = -1$$

and

$$\int_{\mathbb{R}^n} |Dw|^p\, dx = 2\int_{\{x_n>0\}} |Dv^\star|^p\, dx \le 2\int_{\{x_n>0\}} |Du|^p dx = \int_{\mathbb{R}^n} |Du|^p dx.$$

We can then employ Lemma 2.1 to conclude $u \equiv w$. In particular,

$$u|_{\{x_n\ge 0\}} = v^\star|_{\{x_n\ge 0\}}.$$

By Lemma 5.4, we also have that if $x^1, x^2 \in \mathbb{R}^n$ satisfies

$$|x^1| = |x^2| \;\text{ and }\; 0 \le x^1_n \le x^2_n, \tag{5.7}$$

then

$$u(x^1) \le u(x^2). \tag{5.8}$$

Moreover, if

$$|x^1| = |x^2| \;\text{ and }\; x^1_n \le x^2_n \le 0, \tag{5.9}$$

then

$$u(x^2 - 2x^2_n e_n) \le u(x^1 - 2x^1_n e_n)$$

by our remarks above. Since u is antisymmetric,

$$-u(x^2) \le -u(x^1)$$

which of course again gives (5.8). Since

$$u(x^1) \le 0 \le u(x^2)$$

when

$$x^1_n \le 0 \le x^2_n,$$

we conclude that (5.8) holds for all $x^1, x^2 \in \mathbb{R}^n$ satisfying

$$|x^1| = |x^2| \quad\text{and}\quad x^1_n \le x^2_n.$$

□

Appendix A. **Approximation.** This section is devoted to showing that smooth functions are "dense" in $\mathcal{D}^{1,p}(\mathbb{R}^n)$. In the following proposition, we will say that a sequence $(u_k)_{k\in\mathbb{N}} \subset \mathcal{D}^{1,p}(\mathbb{R}^n)$ converges to u in $C^{1-n/p}(\mathbb{R}^n)$ if

$$\lim_{k\to\infty} \left(\|u_k - u\|_\infty + [u_k - u]_{1-n/p}\right) = 0.$$

Note that each u_k or u need not be bounded on $\mathbb{R}^n$.

Proposition 3. *For each $u \in \mathcal{D}^{1,p}(\mathbb{R}^n)$, there is $(u_k)_{k\in\mathbb{N}} \subset C^\infty(\mathbb{R}^n) \cap \mathcal{D}^{1,p}(\mathbb{R}^n)$ such that*

$$u_k \to u$$

in $C^{1-n/p}(\mathbb{R}^n)$ and

$$Du_k \to Du$$

in $L^p(\mathbb{R}^n;\mathbb{R}^n)$ as $k \to \infty$.

Proof. Let $\eta \in C_c^\infty(\mathbb{R}^n)$ be a standard mollifier. That is, η is nonnegative with $\text{supp}(\eta) \subset B_1$ and $\int_{\mathbb{R}^n} \eta dx = 1$. We set $\eta^\epsilon(x) := \epsilon^{-n}\eta(x/\epsilon)$ and define the mollification of u as

$$u^\epsilon(x) = \int_{\mathbb{R}^n} \eta^\epsilon(y)u(x-y)dy = \int_{\mathbb{R}^n} \eta^\epsilon(x-y)u(y)dy\,, \quad x \in \mathbb{R}^n.$$

Consequently, $u^\epsilon \in C^\infty(\mathbb{R}^n)$. Also observe

$$\begin{aligned}
&|u^\epsilon(x) - u(x)| \\
=&\left|\int_{\mathbb{R}^n} \eta^\epsilon(y)(u(x-y) - u(x))dy\right| \le \int_{\mathbb{R}^n} \eta^\epsilon(y)|u(x-y) - u(x)|dy \\
\le&[u]_{1-n/p}\int_{\mathbb{R}^n} \eta^\epsilon(y)|y|^{1-n/p}dy = [u]_{1-n/p}\int_{B_\epsilon(0)} \eta^\epsilon(y)|y|^{1-n/p}dy \\
\le&[u]_{1-n/p}\epsilon^{1-n/p}
\end{aligned} \tag{A.1}$$

for $x \in \mathbb{R}^n$. Thus, $u^\epsilon \to u$ uniformly in $\mathbb{R}^n$.

We also have that

$$Du^\epsilon(x) = \int_{\mathbb{R}^n} \eta^\epsilon(x-y)Du(y)dy \tag{A.2}$$

([14], Theorem 1). And since $Du \in L^p(\mathbb{R}^n;\mathbb{R}^n)$,

$$Du^\epsilon \to Du \text{ in } L^p(\mathbb{R}^n;\mathbb{R}^n)$$

as $\epsilon \to 0^+$ ([5], Theorem 4.22). In addition, we can invoke Morrey's inequality to find

$$[u^\epsilon - u]_{1-n/p} \le C_*\left(\int_{\mathbb{R}^n} |Du^\epsilon - Du|^p dx\right)^{1/p} \to 0$$

as $\epsilon \to 0^+$. As a result,

$$u^\epsilon \to u \text{ in } C^{1-n/p}(\mathbb{R}^n)$$

as $\epsilon \to 0^+$, as well.

Suppose $\epsilon_k > 0$ with $\lim_{k\to\infty} \epsilon_k = 0$ and set $u_k := u^{\epsilon_k}$ for $k \in \mathbb{N}$. We then have that $(u_k)_{k\in\mathbb{N}} \subset C^\infty(\mathbb{R}^n) \cap \mathcal{D}^{1,p}(\mathbb{R}^n)$ satisfies the desired conclusions. □

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