Bridge to PhD
Sample written assessment

The instructions for the assessment are as follows.

- You have up to 4.5 hours to complete any 8 of the following 12 problems.
- You may use any textbook references for assistance.
- You may not use the internet or receive help from anyone else.
- You will submit your responses by email when you are finished (preferably with a single pdf file).

Name

1. Show there is an $x \in [0, 1]$ for which

$$\cos(x) = x.$$

2. Compute the limit

$$\lim_{n\to\infty} n(2^{1/n} - 1).$$

3. Does the integral

$$\int_0^\infty e^{-x} \sin(x) dx$$

converge? If so, how would you evaluate it?

4. Find all x, y, z which solve the equations

$$\begin{cases} 2x + y - z = 8 \\ -3x - y + 2z = -11 \\ -2x + y + 2z = -3. \end{cases}$$

5. Find the formula for the linear transformation of $\mathbb{R}^2$ which is reflection across the line $y = \frac{1}{2}x$.

6. The space V of cubic polynomials on $[-1,1]$ with real coefficients has a natural basis $\{1, x, x^2, x^3\}$ and inner product

$$\langle p, q \rangle = \int_{-1}^{1} p(x)q(x)dx \quad (p, q \in V).$$

Find an orthogonal basis $\{p_0, p_2, p_3, p_3\}$ for V, where "orthogonal" is determined by the above inner product.

7. Suppose $(x_k)_{k\in\mathbb{N}}$ is a sequence of real numbers and for each $\epsilon > 0$ there is $N \in \mathbb{N}$ such that

$$\sum_{k=n+1}^{m} |x_k| < \epsilon$$

for $N \leq n < m$. Show that the infinite sum

$$\sum_{k=1}^{\infty} x_k$$

converges.

8. Verify

$$\ln(1+x) \geq x - \frac{1}{2}x^2$$

for $x \geq 0$.

9. Suppose $g : \mathbb{R}^2 \to \mathbb{R}$ is a smooth function of the (x, y) variables which satisfies

$$\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} < 0.$$

Show that g cannot have a local minimum at any $(x, y) \in \mathbb{R}^2$.

10. Suppose

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 4 & 3 & 2 & 7 & 6 & 1 & 5 \end{pmatrix}$$

is a permutation of $\{1, \ldots, 8\}$. That is, $\sigma(1) = 8$, $\sigma(2) = 4$ and so on.

(i) Write σ as a product of transpositions.

(ii) What is the order of σ?

11. Suppose $a, b \in \mathbb{N}$ satisfy

$$ax + by = 1$$

for some $x, y \in \mathbb{Z}$. What conclusions can we make on a, b?

12. Suppose R is a integral domain and that there is a smallest positive integer n such

$$\underbrace{1 + \cdots + 1}_{n \text{ times}} = 0.$$

Here 0 and 1 are the respective additive and multiplicative identities of R. Show that n is prime.