Optimal Taxes on Capital in the OLG Model with Uninsurable Idiosyncratic Income Risk*

Dirk Krueger[†] Alexander Ludwig[‡] Sergio Villalvazo[§]

April 8, 2021

Abstract

We characterize the optimal linear tax on capital in an Overlapping Generations model with two period lived households facing uninsurable idiosyncratic labor income risk. The Ramsey government internalizes the general equilibrium effects of private precautionary saving on factor prices and taxes capital unless the weight on future generations in the social welfare function is sufficiently high. For logarithmic utility a complete analytical solution of the Ramsey problem exhibits an optimal aggregate saving rate that is independent of income risk, whereas the optimal time-invariant tax on capital implementing this saving rate is increasing in income risk. The optimal saving rate is constant along the transition and its sign depends on the magnitude of risk and on the Pareto weight of future generations. If the Ramsey tax rate that maximizes steady state utility is positive, then implementing this tax rate permanently induces a Pareto-improving transition even if the initial equilibrium capital stock is below the golden rule.

Keywords: Idiosyncratic Risk, Taxation of Capital, Overlapping Generations, Precautionary Saving, Pecuniary Externalities

J.E.L. classification codes: H21, H31, E21

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*We thank Chris Busch, YiLi Chien, Marcus Hagedorn, Daniel Harenberg, Hans Holter, Marek Kapička, Richard Kihlstrom, Yena Park, Catarina Reis, Victor Rios-Rull, Aleh Tsyvinski, Claus Thustrup Kreiner, six anonymous referees as well as seminar participants at various institutions for helpful comments and Leon Hütsch for excellent research assistance. Dirk Krueger thanks the NSF for continued financial support under grant SES-1757084. Alex Ludwig gratefully acknowledges financial support by the Research Center SAFE, funded by the State of Hessen initiative for research LOEWE, and by NORFACE Dynamics of Inequality across the Life-Course (TRISP) grant: 462-16-120.

[†]University of Pennsylvania, CEPR and NBER

[‡]SAFE, Goethe University Frankfurt

[§]University of Pennsylvania

1 Introduction

How should a benevolent government tax capital in a canonical production economy when households face uninsurable idiosyncratic labor income risk and engage in precautionary saving against this risk? Partial answers to this question have been given in Bewley style general equilibrium models with neoclassical production and infinitely lived consumers, starting from Aiyagari (1995)'s characterization of the optimal steady state capital income tax rate, and continuing with recent work providing characterizations of the optimal path of capital income taxes by Panousi and Reis (2015, 2019), Gottardi et al. (2015), Dyrda and Pedroni (2018), Açikgöz et al. (2018), Chen et al. (2019) and Chien and Wen (2020).

The objective of this paper is to characterize optimal linear taxes on capital in a canonical Diamond (1965) style Overlapping Generations model with uninsurable idiosyncratic labor income risk in the second period of life. Under the assumption of logarithmic utility, we provide a full analytical solution to the optimal tax problem in which the Ramsey government (Ramsey 1927), taking optimal behavior of private households as given, uses the revenues from the tax on capital to finance lump-sum transfers to households and maximizes a social welfare function with arbitrary Pareto weights on different generations.

We show that the optimal tax rate on capital is shaped by two competing forces. On one hand, uninsurable idiosyncratic income risk leads to private precautionary saving which in general equilibrium increases wages and reduces capital returns, creating a pecuniary externality by increasing the risky income component. As pointed out in Aiyagari (1995) and Davila et al. (2012), this mechanism creates a Pigouvian incentive for the Ramsey government to tax capital at a positive rate. On the other hand, a higher future capital stock associated with larger saving of the current young benefits future generations, and to the extent that the Ramsey government values these future generations, subsidizing capital accumulation through negative capital taxation allows the government to redistribute welfare intergenerationally towards these generations. We demonstrate that the optimal Ramsey allocation is determined by a time-invariant aggregate saving rate s, defined as the share of aggregate labor income devoted to capital accumulation. This saving rate, characterized in closed form, can be implemented as competitive equilibrium with a proportional tax on capital that is also constant over time, strictly increasing in the extent of income risk and strictly decreasing in the welfare weights on future generations. Back-of-the-envelope calculations illustrate that for realistic degrees of income risk the associated optimal capital income tax is substantially positive at 16% even when maximizing steady state welfare.

To place our contribution in the literature, it is instructive to relate it to Davila et al. (2012), the paper most closely related to our work. They study constrained efficient allocations in the Bewley-Huggett-Aiyagari incomplete markets model where a planner chooses allocations, but cannot transfer resources between households with different shock realizations to provide direct insurance.[1] Most relevant for our work is their study of a simple two period model (in Section 2 of their paper) with ex-ante identical individuals that face uninsurable idiosyncratic income risk in the second period of their life. In the context of this model they show that, relative to the constrained efficient allocation there is overaccumulation of capital in the competitive equilibrium, and that the constrained-efficient allocation can be implemented as a competitive equilibrium by means of a tax on capital that reduces saving and thus increases the equilibrium interest rate and reduces the equilibrium wage. They then show that with ex ante household heterogeneity in wealth higher wages and lower interest rates (induced by subsidizing instead of taxing capital) have the benefits of redistributing from the wealth-rich to the wealth-poor.[2] The optimal tax on capital in their simple model is then determined by trading off these two forces, in the same way that in the OLG economy we study the optimal tax rate is determined by the tension between the same pecuniary externality (calling for positive taxes on capital) and the redistributional benefits towards future generations that higher wages and a higher capital stock induced by more saving entail. If one shuts down their redistributional concern by abstracting from wealth inequality and our intergenerational concerns (by setting the Pareto weights on future generations to zero and allocating all second period income to the current old), then the optimal capital tax rates in both papers coincide, and are unambiguously positive. Thus, while both papers share the same fundamental reason for taxing capital, the redistributive benefit pushing in the other direction is distinct from theirs in the OLG model: intragenerational distribution in their paper vis-a-vis inter-generational distribution in ours.

This trade-off in our model permits us to connect the results on optimal taxation of capital to the classical discussion of overaccumulation of capital in the canonical OLG model. Along this dimension we establish a surprising result. Consider the optimal Ramsey tax rate when the government places all weight in the social welfare function on generations

[1] The notion of constrained efficiency follows Diamond (1967), Stiglitz (1982) and Geanakoplos and Polemarchakis (1986), and refers to a planner problem with the constraint that the planner cannot directly overcome a friction implied by missing markets.

[2] In Davila et al. (2012) asset-income poor households benefit from an increase of the capital stock and thus wages. Park (2018) introduces endogenous human capital accumulation so that welfare of human-capital poor households might be improved by lower wages, which adds an additional distribution effect, with welfare implications of changing factor incomes opposite to those by Davila et al. (2012).

living in the steady state and the benefits to future generations of a higher capital stock are maximally potent. If this tax rate is positive (which is true if income risk is sufficiently high), then a government implementing this constant tax rate along the transition generates a Pareto-improving transition from the steady state competitive equilibrium. This holds even if the original equilibrium capital stock is below the golden rule capital stock maximizing aggregate consumption, and the tax on capital reduces aggregate consumption along the transition path. The steady state utility maximizing tax rate takes into account the welfare losses induced by the crowding out of capital. Since the capital stock monotonically decreases along the transition, welfare losses from this crowding-out effect monotonically increase along the transition. Instead, the utility gains from a tax-induced reduction of the saving rate (and thus higher first period consumption) are highest in the initial period, decrease monotonically along the transition but remain positive even in the long-run. Thus, setting the tax rate in all periods to the long-run welfare maximizing rate induces welfare gains for all generations.

In the last part of the paper we probe the robustness of our results to key modeling assumptions. Since the main market failure giving rise to positive taxes on capital is uninsurable labor income risk, and since a progressive labor income tax (a proportional tax rate of 100% and lump-sum rebate) can perfectly tackle this risk if labor supply is exogenous, we first show that endogenous labor supply and labor income taxes leaves the optimal saving rate chosen by the Ramsey planner unchanged, and gives rise to an associated optimal tax on capital with the same qualitative properties as in the benchmark model. Second, we explore whether the introduction of other tools for intergenerational redistribution such as social security, government debt or intergenerational altruism invalidates our results (we find that they do not), prior to discussing additional extensions, such as ex ante heterogeneity, idiosyncratic capital return risk and risk aversion different from unity.[3]

1.1 Relation to the Existing Literature

In addition to the work cited above, our paper builds on the general literature studying optimal Ramsey capital taxation in Bewley models, starting from Aiyagari (1995). The argument for taxing capital income in this literature is based on two steps. First, production efficiency in models with infinitely lived agents requires that a Ramsey government with sufficient fiscal instruments implements the modified golden rule in the long run steady

[3]We show that our results extend to Epstein-Zin-Weil utility (EZW utility, see Epstein and Zin (1989, 1991), Weil (1989)) as long as the inter-temporal elasticity of substitution (IES) is equal to one.

state (assuming convergence to a steady state), equating the return to capital to the subjective time discount rate.[4] Second, since the long-run competitive equilibrium interest rate absent taxes is below the time discount rate due to precisely the precautionary saving against idiosyncratic income risk emphasized in our paper, asset income has to be taxed to offset the precautionary saving and restore the modified golden rule. This is in contrast to the celebrated Chamley (1986), Judd (1985) result of zero optimal capital taxes in the representative agent neoclassical growth model.[5]

In our OLG model with log-utility we can characterize the entire time path of optimal Ramsey allocations analytically, and thus can demonstrate that the allocation indeed converges to a steady state, rather than having to assume it.[6] Furthermore we obtain a complete characterization of optimal capital tax rates and associated allocations along the transition.[7]

Complementary to our work, a recent theoretical literature studies optimal capital income taxes in models with idiosyncratic *investment* risk, see Evans (2015), Panousi (2015), and Panousi and Reis (2019). Their key focus is on the role of capital income taxes in providing insurance or redistribution; none of these papers focuses on the role of general equilibrium feedback from precautionary saving behavior on optimal capital income taxation. Finally, our work connects to the literature on optimal capital income taxation in

[4] Aiyagari (1995) shows this in a model with endogenous government spending, and Açikgöz et al. (2018) generalize it to exogenous government spending. Both papers assume that the Ramsey allocation converges to a steady state. Chen et al. (2019) argue that this convergence assumption to a steady state with idiosyncratic consumption risk and an interest rate below the modified golden rule is not innocuous, and Chen et al. (2021) construct a version of the Bewley model where the Ramsey government issues sufficiently large government debt that the Ramsey allocation converges to full insurance in the long run, making capital taxation in the steady state unnecessary to implement the modified golden rule. Finally note that in these papers the Ramsey government has fiscal policy tools to achieve desired redistribution, whereas the constrained planner in Davila et al. (2012) does not, and therefore the latter does in general not implement the modified golden rule.

[5] Chamley (2001) develops a partial equilibrium model to clarify that the Chamley-Judd zero optimal tax result, re-evaluated recently by Straub and Werning (2020), depends on the assumption of complete markets and breaks down if households face income risk and a borrowing constraint. In Chamley (2001)'s partial equilibrium analysis, the general equilibrium effects that are crucial to our results are missing by construction.

[6] Heathcote, Storesletten, and Violante (2017) develop an analytically tractable model with infinitely lived households facing idiosyncratic productivity risk, endogenous labor supply but without capital accumulation. Their focus is on characterizing the optimal progressivity of labor income taxation.

[7] Quantitative work in infinite horizon economies by Dyrda and Pedroni (2018) and Açikgöz et al. (2018) analyze optimal fiscal policy along the economy's transition from the status quo to the long-run steady state and find robustly positive capital income taxes. A similar finding is obtained by Gottardi et al. (2015) in a model with risky human capital originally proposed by Krebs (2003). These papers extend the work by Domeij and Heathcote (2004) analyzing the welfare consequences of abolishing capital income taxes in a Aiyagari-Bewley-Huggett economy taking into account the transition. Whereas idiosyncratic labor income risk plays a key role in these papers, none of them emphasizes how the general equilibrium price effects affect the optimal allocation chosen by the Ramsey planner as we do.

life-cycle economies, as in the two-period models of Pestieau (1974) and Atkinson and Sandmo (1980). Erosa and Gervais (2001, 2002), Conesa et al. (2009), Garriga (2017) and Peterman (2016) extend these studies to multiple periods and emphasize that capital income taxes are only zero under strong assumptions on preferences, or if labor income tax rates can depend on household age. The general equilibrium price effects of precautionary savings on prices in the Ramsey problem are not addressed in these papers.

The paper is organized as follows. Section 2 presents the model and Section 3 characterizes the competitive equilibrium. Section 4 lays out the Ramsey problem and presents its analytical solution. Section 5 discusses the efficiency properties of the Ramsey equilibrium and gives conditions under which implementing the long-run optimal policy induces a Pareto improving transition. Section 6 explores the sensitivity of our results to key modeling assumptions and Section 7 concludes. All proofs and detailed derivations are contained in the appendix.

2 Model

Time is discrete and extends from $t = 0$ to $t = \infty$. In each period a new generation is born that lives for two periods. Thus at any point in time there is a $y$oung and an $o$ld generation. We normalize household size to 1 for each age cohort. In addition there is an initial old generation that has one remaining year of life.

2.1 Household Endowments, Preferences and Budget Sets

Each household has one unit of time in both periods, supplied inelastically to the market. Labor productivity when young is equal to $(1 - \kappa)$, and, as in Harenberg and Ludwig (2015), in the second period labor productivity is given by $\kappa\eta_{t+1}$, where $\kappa \in [0, 1)$ is a parameter that captures relative labor income of the old, and η_{t+1} is an idiosyncratic labor productivity shock. We assume that the cdf of η_{t+1} is given by $\Psi(\eta_{t+1})$ in every period and denote the corresponding pdf by $\psi(\eta_{t+1})$. We assume that Ψ is both the population distribution of η_{t+1} as well as the cdf of the productivity shock for any given individual (that is, we assume a Law of Large Numbers, LLN). We make the following

Assumption 1. *The shock η_{t+1} takes positive values Ψ-almost surely and*

$$\int \eta_{t+1} d\Psi = 1.$$

Each member of the initial old generation is additionally endowed with assets equal to a_0. The asset endowment is independent of the household's realization of the shock η.

In order to obtain the sharpest analytical results we assume logarithmic period utility $u(c) = \ln(c)$. A household of generation $t \geq 0$ then ranks consumption allocations $c_t^y, c_{t+1}^o(\eta_{t+1})$ according to

$$V_t = \ln(c_t^y) + \beta \int \ln(c_{t+1}^o(\eta_{t+1}))d\Psi. \tag{1}$$

Lifetime utility of the initial old generation is determined as

$$V_{-1} = \int \ln(c_0^o(\eta_0))d\Psi.$$

However, we show in Section J that our results generalize to an Epstein-Zin-Weil (Epstein and Zin 1989; Epstein and Zin 1991; Weil 1989) utility function as long as the intertemporal elasticity of substitution is unity.

The budget constraints in both periods of a household born in period t read as

$$c_t^y + a_{t+1} = (1-\kappa)w_t \tag{2a}$$

$$c_{t+1}^o = a_{t+1}R_{t+1}(1-\tau_{t+1}) + \kappa\eta_{t+1}w_{t+1} + T_{t+1}, \tag{2b}$$

where w_t, w_{t+1} are the aggregate wages in period t and $t+1$, $R_{t+1} = 1 + r_{t+1}$ is the gross interest rate between period t and $t+1$, T_{t+1} are lump-sum transfers to the old generation, and η_{t+1} is the age-2 period-$t+1$ idiosyncratic shock to wages.[8]

2.2 Technology and Firm Profit Maximization

The representative firm operates a Cobb-Douglas technology with capital elasticity α:

$$F(K_t, L_t) = K_t^\alpha (L_t)^{1-\alpha}.$$

[8]Notice that instead of working with a tax on capital τ_t, one could equivalently work with standard capital income taxes τ_t^k. We discuss this equivalence in detail in Section 6.3.2 of the paper.

Furthermore we assume that capital fully depreciates between two (30 year) periods. The firm's first order conditions with respect to capital and labor read as

$$R_t = \alpha k_t^{\alpha-1} \tag{3a}$$

$$w_t = (1-\alpha) k_t^{\alpha} \tag{3b}$$

where

$$k_t = \frac{K_t}{L_t} = \frac{K_t}{1-\kappa+\kappa\int \eta_t d\Psi} = K_t$$

is the capital-labor ratio. Since $L_t = 1$, we do not need to distinguish between the aggregate capital stock K_t and the capital-labor ratio k_t.

2.3 Government

The government levies a time varying tax τ_t on capital, and rebates the proceeds in a lump-sum fashion to all members of the current old generation as a transfer T_t. Note that the restriction that transfers accrue exclusively to old households implies that the government has no direct tool for intergenerational redistribution, an assumption we relax below in Section 6.[9] We assume that the government has the following social welfare function

$$W = \sum_{t=-1}^{\infty} \omega_t V_t, \tag{4}$$

where $\{\omega_t\}_{t=-1}^{\infty}$ are the Pareto weights on different generations and satisfy $\omega_t \geq 0$. Since lifetime utilities of each generation will be bounded, so will be the social welfare function as long as $\sum_{t=-1}^{\infty} \omega_t < \infty$. We will also consider the case $\omega_t = 1$ for all t, in which case we will take the social welfare function to be defined as $W = \lim_{T\to\infty} \frac{\sum_{t=-1}^{T} V_t}{T}$, which is equivalent to maximizing steady state welfare.

[9] It also implies that, conditional on a beginning of the period capital stock given by past household decisions, the government cannot alter lifetime utility of the newborn generation in period t by changing the current tax τ_t. Since tax revenues from the current old are rebated back to this generation, remaining lifetime utility of the old is unaffected by τ_t. This in turn insures that the government has no incentive to deviate in period t from the period zero tax plan $\{\tau_t\}$. That is, given the restriction on the set of policies, Ramsey tax policies are time-consistent in our model with exogenous labor supply.

2.4 Competitive Equilibrium

We are now ready to define a competitive equilibrium.[10]

Definition 1. *Given the initial condition $a_0 = k_0$ and a sequence of tax policies $\boldsymbol{\tau} = \{\tau_t\}_{t=0}^{\infty}$, a competitive equilibrium is an allocation $\{c_t^y, c_t^o, L_t, a_{t+1}, k_{t+1}\}_{t=0}^{\infty}$, prices $\{R_t, w_t\}_{t=0}^{\infty}$ and transfers $\{T_t\}_{t=0}^{\infty}$ such that*

1. *given prices $\{R_t, w_t\}_{t=0}^{\infty}$ and policies $\{\tau_t, T_t\}_{t=0}^{\infty}$ for each $t \geq 0$, $(c_t^y, c_{t+1}^o(\eta_{t+1}), a_{t+1})$ maximizes (1) subject to (2a) and (2b) (for each realization of η_{t+1});*

2. *consumption $c_0^o(\eta_0)$ of the initial old satisfies (2b) (for each realization of η_0):*

$$c_0^o = a_0 R_0 (1 - \tau_0) + \kappa \eta_0 w_0 + T_0;$$

3. *prices satisfy equations (3a) and (3b);*

4. *the government budget constraint is satisfied in every period: for all $t \geq 0$*

$$T_t = \tau_t R_t k_t;$$

5. *markets clear*

$$\begin{aligned} L_t &= L = 1 \\ a_{t+1} &= k_{t+1} \\ c_t^y + \int c_t^o(\eta_t) d\Psi + k_{t+1} &= k_t^{\alpha}. \end{aligned}$$

Denote by $W(\boldsymbol{\tau})$ social welfare associated with an equilibrium for given tax policy $\boldsymbol{\tau}$. We will show below that for given tax policy $\boldsymbol{\tau}$ a competitive equilibrium exists and is unique and thus the function $W(\boldsymbol{\tau})$ is well-defined as long as $\tau_t \in (-\infty, 1)$ for all t.

Definition 2. *Given the initial condition $a_0 = k_0$, a Ramsey equilibrium is a sequence of tax policies $\hat{\boldsymbol{\tau}} = \{\hat{\tau}_t\}_{t=0}^{\infty}$ and equilibrium allocations, prices and transfers associated with*

[10] Since our results below focus on equilibria with positive returns on capital (a capital stock below the golden rule) we assume that the only asset households trade is physical capital, and rule out equilibria with bubbles or with fiat money issued by the government. Our equilibrium definition reflects this.

$\hat{\boldsymbol{\tau}}$ *(in the sense of the previous definition) such that*

$$\hat{\boldsymbol{\tau}} \in \arg\max_{\boldsymbol{\tau}} W(\boldsymbol{\tau}).$$

3 Analysis of Equilibrium for a Given Tax Policy

3.1 Partial Equilibrium

We first analyze the household problem for given prices and policies. The optimal asset choice of the household satisfies the standard intertemporal Euler equation

$$1 = \beta(1-\tau_{t+1})R_{t+1} \int \frac{[u'(a_{t+1}R_{t+1}(1-\tau_{t+1}) + \kappa\eta_{t+1}w_{t+1} + T_{t+1})]}{u'((1-\kappa)w_t - a_{t+1})} d\Psi(\eta_{t+1}).$$

Defining the saving rate as

$$s_t = \frac{a_{t+1}}{(1-\kappa)w_t} \tag{5}$$

the Euler equation with logarithmic utility can be written as

$$1 = \beta(1-\tau_{t+1}) \int \frac{1-s_t}{s_t(1-\tau_{t+1}) + \frac{\kappa w_{t+1}}{(1-\kappa)w_t R_{t+1}}\eta_{t+1} + \frac{T_{t+1}}{(1-\kappa)w_t R_{t+1}}} d\Psi(\eta_{t+1}). \tag{6}$$

Equation (6) defines the optimal partial equilibrium saving rate $s_t = s(w_t, w_{t+1}, R_{t+1}, \tau_{t+1}, T_{t+1}; \beta, \kappa, \Psi)$.

3.2 General Equilibrium

In equilibrium factor prices and transfers are given by

$$w_t = (1-\alpha)k_t^{\alpha} \tag{7a}$$

$$R_{t+1} = \alpha k_{t+1}^{\alpha-1} \tag{7b}$$

$$T_{t+1} = \tau_{t+1}R_{t+1}k_{t+1}. \tag{7c}$$

Using equation (7a) and the asset market clearing condition $a_{t+1} = k_{t+1}$ the saving rate and, consequently, the aggregate law of motion for the capital stock can be written as

$$s_t = \frac{k_{t+1}}{(1-\kappa)(1-\alpha)k_t^{\alpha}} \tag{8a}$$

$$k_{t+1} = s_t(1-\kappa)(1-\alpha)k_t^{\alpha}. \tag{8b}$$

For a given sequence of capital taxes $\{\tau_t\}_{t=0}^{\infty}$ and an initial condition k_0 the competitive equilibrium is a sequence of savings rates and associated capital stocks $\{s_t, k_{t+1}\}_{t=0}^{\infty}$ that solves the Euler equation (6) when factor prices have been substituted out. Using (7a)-(7c) the Euler equation becomes

$$1 = \alpha\beta(1-\tau_{t+1})\left(\frac{1-s_t}{s_t}\right)\Gamma. \tag{9}$$

where the constant

$$\Gamma = \int (\kappa\eta_{t+1}(1-\alpha)+\alpha)^{-1}\, d\Psi(\eta_{t+1}) = \Gamma(\alpha,\kappa;\Psi) \tag{10}$$

fully captures the impact of idiosyncratic income risk on the equilibrium saving rate.[11]

3.3 Characterization of the Saving Rate

Equation (9) has a closed form solution for the saving rate s_t in general equilibrium, and we can give a complete analytical characterization of its comparative statics properties.

Proposition 1. *Suppose assumption 1 is satisfied. Then for all $\tau_{t+1} \in (-\infty, 1)$ the unique saving rate $s_t = s(\tau_{t+1}; \Gamma)$ is given by*

$$s_t = \frac{1}{1+\left[(1-\tau_{t+1})\alpha\beta\Gamma(\alpha,\kappa;\Psi)\right]^{-1}}, \tag{11}$$

which is strictly increasing in Γ, strictly decreasing in τ_{t+1} and independent of the beginning of the period capital stock k_t.

The next corollary states that any saving rate $s_t \in (0,1]$ can be implemented as a competitive equilibrium by choice of the capital tax rate τ_{t+1}. This corollary permits to cast the Ramsey policy problem directly in terms of the government choosing saving rates.

Corollary 1. *For each saving rate $s_t \in (0,1]$ there exists a unique tax rate $\tau_{t+1} \in (-\infty, 1)$ that implements that saving rate s_t as part of a competitive equilibrium.*

[11] To interpret Γ, an old household's share in total output k_{t+1}^{α} is given by $\alpha + (1-\alpha)\kappa\eta_{t+1}$ since capital income (share α) accrues to the old, and a share κ of labor income (share $1-\alpha$) is subject to the idiosyncratic shock η_{t+1}. Thus Γ measures expected marginal utility of the share of output accruing to an old household. Since with log-utility marginal utility is convex, Γ is strictly increasing in income risk η_{t+1}.

From Proposition 1 s_t depends on income risk η only through the constant Γ. Furthermore, Γ is a strictly convex function of η, and Jensen's inequality implies the following:

Observation 1. *Assume that $\alpha \in (0,1)$ and $\kappa > 0$. Then*

1. *$\Gamma(\alpha, \kappa; \Psi)$ is strictly increasing in income risk, in the sense that if the distribution $\tilde{\Psi}$ over η is a mean-preserving spread of Ψ, then $\Gamma(\alpha, \kappa; \Psi) < \Gamma(\alpha, \kappa; \tilde{\Psi})$.*

2. *Define the degenerate distribution with $\eta \equiv 1$ as $\bar{\Psi}$, then for any nondegenerate Ψ*

$$1 < \bar{\Gamma} := \Gamma(\alpha, \kappa; \bar{\Psi}) < \Gamma(\alpha, \kappa; \Psi)$$

We can immediately deduce the following:

Corollary 2. *The equilibrium saving rate is strictly increasing in income risk.*

This result follows directly from the fact that $s_t = s(\tau_{t+1}; \Gamma)$ is strictly increasing in Γ and Γ is strictly increasing in income risk. With this characterization of the competitive equilibrium for given tax policy $\{\tau_{t+1}\}_{t=0}^{\infty}$ we now turn to the analysis of *optimal tax policy*.

4 The Ramsey Problem

The objective of the government is to maximize social welfare $W(k_0) = \sum_{t=-1}^{\infty} \omega_t V_t$ by choice of capital taxes $\{\tau_{t+1}\}_{t=0}^{\infty}$ where V_t is the lifetime utility of generation t in the competitive equilibrium associated with the sequence $\{\tau_{t+1}\}_{t=0}^{\infty}$. Making use of Corollary 1 we can substitute out taxes to write lifetime utility in terms of the saving rate s_t yielding

$$\begin{aligned} V(k_t, s_t) &= \ln((1-s_t)(1-\kappa)(1-\alpha)\, k_t^{\alpha}) \\ &+ \beta \int \ln\left(\kappa \eta_{t+1} w(s_t) + R(s_t) s_t (1-\kappa)(1-\alpha) k_t^{\alpha}\right) d\Psi(\eta_{t+1}), \end{aligned} \tag{12}$$

with[12]

$$w(s_t) = (1-\alpha)\,[k_{t+1}(s_t)]^{\alpha} \tag{13a}$$

$$R(s_t) = \alpha\,[k_{t+1}(s_t)]^{\alpha-1} \tag{13b}$$

$$k_{t+1}(s_t) = s_t(1-\kappa)(1-\alpha)k_t^{\alpha}. \tag{13c}$$

[12]We could now substitute factor prices in the lifetime utility function, but for the purpose of better interpretation of the results we refrain from doing so at this point.

Remaining lifetime utility of the initial old is given by (with factor prices substituted out)

$$V_{-1} = V(k_0, \tau_0) = \int \ln\left(\left[\alpha + \kappa\eta_0(1-\alpha)\right] k_0^{\alpha}\right) d\Psi(\eta_0) = V(k_0) \tag{14}$$

Note that τ_0 is irrelevant for welfare of the initial old generation (and all future generations) since k_0 is a fixed initial condition, τ_0 is nondistortionary, is lump-sum rebated and the government is assumed to balance the budget period-by-period. Expression (14) shows that with the set of policies considered lifetime utility of the initial old cannot be affected at all, and thus we do not need to include it in the social welfare function.[13]

Corollary 1 implies that the Ramsey government can implement any sequence of saving rates $\{s_t\}_{t=0}^{\infty}$ as a competitive equilibrium and thus can choose private saving rates directly. For $\sum_{t=0}^{\infty} \omega_t < \infty$ we can therefore restate the Ramsey problem as[14]

$$W(k_0) = \max_{\{s_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \omega_t V(k_t, s_t) \tag{15}$$

subject to (13a)–(13c).

4.1 Recursive Formulation and Characterization of Ramsey Problem

The Ramsey problem lends itself to a recursive formulation, under the following assumption on the social welfare weights[15]

Assumption 2. *The social welfare weights satisfy* $\omega_t > 0$ *and* $\frac{\omega_{t+1}}{\omega_t} = \theta \in (0,1), \forall t \geq 0.$

Under this assumption, the recursive formulation of the problem reads as

$$\begin{aligned} W(k) &= \max_{s\in[0,1)} \ln((1-s)(1-\kappa)\,(1-\alpha)\,k^{\alpha}) \\ &+ \beta \int \ln\left(\kappa\eta w(s) + R(s)s(1-\kappa)(1-\alpha)k^{\alpha}\right) d\Psi(\eta) + \theta W(k'(s)) \end{aligned} \tag{16}$$

[13]For a given capital stock k_t, the same argument applies to an arbitrary old generation at period t, in that remaining lifetime utility of this old generation cannot any longer be affected by τ_t. Since the same is true for lifetime utility of newborns in period t, the government has no incentives to ex post (after capital k_t is installed) deviate from its period zero Ramsey plan, in contrast to the typical time consistency problem often encountered in the optimal capital income tax literature, as discussed in footnote 9. This fact also implies that we can write the Ramsey problem recursively, as done in the next subsection.

[14]Recall that for $\omega_t = 1$ in all t we accordingly have $W(k_0) = \max_{\{s_t\}_{t=0}^{\infty}} \lim_{T\to\infty} \frac{\sum_{t=0}^{\infty} V(k_t,s_t)}{T}$.

[15]We can fully characterize the solution to the Ramsey problem for arbitrary social welfare weights $\{\omega_t\}_{t=0}^{\infty}$ using the sequential formulation of the problem, as Appendix B shows. The recursive formulation of the problem allows us to derive and interpret the solution in the most transparent manner.

subject to (13a)–(13c).

The recursive formulation of the problem highlights the three effects the Ramsey government considers when choosing the current saving rate s and thus the tax rate on capital τ that implements it as a competitive equilibrium. First, for fixed wages and interest rates, a higher s (lower tax on capital) directly reduces consumption when young and increases it when old. Private households consider exactly this trade-off, henceforth denoted as partial equilibrium effect $PE(s)$.

Second, a change in the saving rate s impacts this generation indirectly through general equilibrium effects of changed wages $w(s)$ and interest rates $R(s)$ when old, an effect we denote as general equilibrium effect, $GE(s)$. This effect encodes the pecuniary externality arising from precautionary saving increasing wages and thus the risky income component in the second period of life. This provides the key force for positive taxes on capital.[16]

Third, a change in the current saving rate increases the capital stock from tomorrow on, impacting lifetime utility of future generations. This effect, denoted as future generations effect $FG(s)$, encodes the motivation of the government to engage in intergenerational redistribution, by taxing capital less or subsidizing it in the market economy.[17]

Taking first order conditions for problem (16) yields:

$$\begin{aligned}
&(1-\kappa)(1-\alpha)k^{\alpha}\left[\frac{-1}{c^y}+\beta R(s)\int\frac{1}{c^o(\eta)}d\Psi(\eta)\right] \\
&\qquad+\beta\int\frac{[\kappa\eta w'(s)+(1-\kappa)(1-\alpha)k^{\alpha}R'(s)s]}{c^o(\eta)}d\Psi(\eta)+\theta W'(k'(s))\frac{dk'(s)}{ds}=0 \\
&PE(s)+GE(s)+FG(s)=0
\end{aligned}\tag{17}$$

Denote by s^{CE} the saving rate households choose in competitive equilibrium in the absence of capital taxes. We observe that $PE(s^{CE})=0$, and thus if $\theta=0$ (the Ramsey government does not value future generations) and general equilibrium effects are absent, the Ramsey government finds it optimal to implement the laissez-faire saving rate by setting the tax on capital to zero. This benchmark demonstrates that in the absence of the two forces motivating taxes on capital (the $GE(s)$ and $FG(s)$ effects) we obtain the expected result that the government should not tax or subsidize capital.

[16]The $GE(s)$ also includes a redistribution effect because higher wages redistribute income from this generation when old to the new young generation, an effect also present in the absence of income risk.

[17]The $FG(s)$ also encodes a pecuniary externality on future generations because the current generation's precautionary saving raises the future capital stock and thus the risky income component of future generations.

Appendix A demonstrates that for an arbitrary period utility function the general equilibrium effect $GE(s)$ is unambiguously negative, driving down the desired saving rate and shifting up the associated capital tax rate. Higher wages when old exacerbate uninsurable idiosyncratic income and thus consumption risk and therefore it is optimal for the Ramsey government to reduce labor income risk by reducing savings incentives, other things equal.

The advantage of logarithmic utility is that the Ramsey optimal tax problem has a complete analytical characterization, akin to the well-known result from macroeconomics that in the standard neoclassical growth model the recursive problem with log-utility has a unique closed-form solution, which can be obtained by the method of undetermined coefficients, see Appendix B. This closed-form solution has the following form for the three terms in the first order condition of the Ramsey government

$$PE(s) = \frac{-1}{(1-s)} + \frac{\alpha\beta}{s}\Gamma(\alpha,\kappa;\Psi) \tag{18a}$$

$$GE(s) = \frac{\alpha\beta}{s}\left[1-\Gamma(\alpha,\kappa;\Psi)\right] \tag{18b}$$

$$FG(s) = \frac{\theta\alpha(1+\alpha\beta)}{(1-\alpha\theta)s}, \tag{18c}$$

where we recall that the constant summarizing the impact of income risk satisfies

$$\Gamma(\alpha,\kappa;\Psi) \geq \frac{1}{\kappa(1-\alpha)+\alpha} \geq 1.$$

The first inequality is strict if Ψ is nondegenerate and $\kappa > 0$ and the second inequality is strict as long as $\kappa < 1$. Thus $[1-\Gamma(\alpha,\kappa;\Psi)] < 0$ if $\kappa < 1$. This implies

$$GE(s) < 0 \quad \text{and} \quad FG(s) > 0.$$

Recall that the saving rate s^{CE} in the competitive equilibrium with zero taxes satisfies $PE(s^{CE}) = 0$. Thus, the only reason to tax capital is the general equilibrium $GE(s)$ effect, which unambiguously lowers the desired saving rate and pushes the associated tax rate up above zero.

Against this works the potential desire of the government to intergenerationally redistribute, encoded in the $FG(s)$ and with size controlled by θ, which calls for a higher saving rate and thus a negative tax rate. Observe that income risk represented by Γ does not enter the expression for $FG(s)$, a finding specific to log-utility. In general equilibrium, consumption when old can be written as the product of a risk term and aggregate wages so that risk enters additively in the objective function with log-utility. Finally note that

$$PE(s) + GE(s) = \frac{-1}{(1-s)} + \frac{\alpha\beta}{s} \tag{19}$$

and thus the private partial equilibrium incentive to save more when income risk rises is *exactly* offset by the general equilibrium effect on factor prices. Given the explicit characterization of the $PE(s), GE(s), FG(s)$ terms the first order condition (17) and the implementation equation (11) immediately imply the following characterization of the solution of the optimal Ramsey tax problem.

Proposition 2. *Suppose assumptions 1 and 2 are satisfied. Then the solution of the Ramsey problem is characterized by a constant saving rate*[18]

$$s_t = s^* = \frac{\alpha(\beta+\theta)}{1+\alpha\beta} \tag{20}$$

and a sequence of capital stocks that satisfy

$$k_{t+1} = s^*(1-\kappa)(1-\alpha)k_t^{\alpha}$$

with initial condition k_0. The associated value function and its derivative are given by

$$\begin{aligned} W(k) &= \Theta_0 + \frac{\alpha(1+\alpha\beta)}{(1-\alpha\theta)}\ln(k) \\ W'(k) &= \frac{\alpha(1+\alpha\beta)}{(1-\alpha\theta)k}. \end{aligned}$$

Since the competitive equilibrium saving rate is given by $s_t = \frac{1}{1+[(1-\tau_{t+1})\alpha\beta\Gamma]^{-1}}$, the Ramsey saving rate is implemented with a constant capital tax $\tau^* = \tau(\beta,\theta,\kappa,\alpha;\Psi)$

$$1-\tau^* = \frac{s^*}{\alpha\beta\Gamma(1-s^*)} = \frac{(\theta+\beta)}{(1-\alpha\theta)\,\beta\Gamma(\alpha,\kappa;\Psi)}, \tag{21}$$

[18] Appendix B shows, using the sequential formulation of the problem, that for arbitrary welfare weights the optimal saving rate is still independent of the capital stock and given by

$$s_t = \frac{1}{1+\left(\alpha\beta+\alpha\left(1+\alpha\beta\right)\sum_{j=1}^{\infty}\frac{\omega_{t+j}}{\omega_t}\alpha^{j-1}\right)^{-1}}.$$

The saving rate in the proposition is a special case under the assumption $\frac{\omega_{t+1}}{\omega_t} = \theta$ for all t.

Corollary 3. *The optimal saving rate is independent of the extent of income risk and strictly increasing in the social discount factor θ and the individual discount factor β.*

Corollary 4. *The optimal capital tax rate is strictly increasing in income risk Γ, strictly decreasing in θ and in the labor income share κ of the old and strictly increasing in β.*

It is noteworthy that not only is the optimal saving rate constant and does not depend on the level of the capital stock, but it also is independent of the extent of income risk η. This is true despite the fact that for a *given* tax policy higher income risk induces a higher individually optimal saving rate, as shown in Section 3.3. The Ramsey government finds it optimal to implement a capital tax that is increasing in the amount of income risk, exactly offsetting the partial equilibrium incentive to save more as income risk increases.

4.2 Discussion of Optimal Tax Rates

We now use the closed-form characterization of the optimal Ramsey capital tax from equation (21) to discuss further how it depends on the extent of income risk Γ (and thus the pecuniary externality) and the desire for intergenerational redistribution θ. The following proposition, which follows immediately from inspection of (21), gives conditions under which the optimal Ramsey capital tax is positive, and complementary conditions under which capital is subsidized. Recall that for $\theta = 0$ only the utility of the first generation receives weight in the social welfare function, whereas $\theta = 1$ amounts to the Ramsey government maximizing steady state welfare.

Proposition 3. *For all $\theta \geq \bar{\theta}$ capital is subsidized whereas for all $\theta < \bar{\theta}$ it is taxed in every period,*[19] *where the threshold social discount factor $\bar{\theta}$ is given by*

$$\bar{\theta} = \frac{(\Gamma - 1)\,\beta}{1 + \alpha\beta\Gamma} > 0.$$

[19] If $\theta = 0$ then we can, by inserting the private Euler equation in the $PE(s)$ effect, directly derive the optimal tax rate on capital for an arbitrary strictly concave and differentiable utility function and a CRTS production function $f(k, 1)$ with strictly positive and strictly decreasing marginal products, as

$$\tau = \frac{-f_{lk}(k'(s))}{f_k(k'(s))} \times \frac{E[u'(c(\eta))[\kappa\eta - 1]]}{E[u'(c(\eta))]} > 0.$$

By defining $f(k, 1) = F(k, 1) + (1 - \delta)k$ we also observe that this result does not rely on full depreciation of capital. Note that although this result establishes that the optimal capital tax rate is positive in the two period model ($\theta = 0$), it does not give the optimal tax rate in closed form since consumption $c(\eta)$ is endogenous.

If $\bar{\theta} \geq 1$, capital is taxed even when the Ramsey government maximizes steady state welfare. If $\bar{\theta} < 1$ the Ramsey government maximizing steady state welfare should subsidize capital.

This proposition implies, as discussed in the introduction, that if $\theta = 0$ and the government has no inter-generational redistribution motive towards future generation, then taxes on capital are unambiguously positive due to the $GE(s)$ effect. In fact, in the limit as all labor income (in addition to capital income) also accrues to the old ($\kappa = 1$), only the pecuniary externality shapes the optimal tax on capital and it coincides with the one in Davila et al. (2012) in the two-period model they study in the first part of their paper.[20]

The results in Proposition 3 also apply when there is no income risk. In Appendix D we show that in this case a Ramsey government maximizing steady state welfare sets $\tau < 0$ if and only if the competitive equilibrium without taxes is has an interest rate $R > 1$, or equivalently, a capital stock below the golden rule capital stock k^{GR} that maximizes aggregate consumption. This suggests the possibility that without income risk in the competitive economy the Ramsey government optimally *subsidizes* capital in the steady state, but with sufficiently large income risk the Ramsey government finds it optimal to tax capital in the steady state. The next proposition, proved in Appendix D, shows that this is indeed the case.

Proposition 4. *Suppose the Ramsey government maximizes steady state welfare ($\theta = 1$) and denote by s^* the optimal saving rate; denote by s^{CE} the steady state equilibrium saving rate in the absence of government policy and by s^{GR} the golden rule saving rate maximizing steady state aggregate consumption. Finally assume that $\beta < \left[(1-\alpha)\bar{\Gamma} - 1\right]^{-1}$.*[21]

1. *Let income risk be large: $\Gamma > \frac{1}{\beta\left[(1-\alpha)-1/\bar{\Gamma}\right]}$. Then the steady state competitive equilibrium capital stock exceeds the golden rule, $s^{CE} > s^{GR}$ and $s^* < s^{CE}$ and the optimal capital tax rate is positive, $\tau > 0$.*

2. *Let income risk be intermediate:*

$$\Gamma \in \left[\frac{1+\beta}{(1-\alpha)\,\beta}, \frac{1}{\left[(1-\alpha) - 1/\bar{\Gamma}\right]\beta}\right].$$

[20] If $\kappa < 1$, then part of labor income accrues to the next generation which the government, at $\theta = 0$, does not value, and there is an additional incentive of the government to shift general equilibrium prices away from wages towards higher returns. Thus, the optimal capital tax in our model is larger than the one in Davila et al. (2012) even with $\theta = 0$ unless $\kappa = 1$.

[21] If this condition is violated, then the steady state competitive equilibrium is has a capital stock above the golden rule even without income risk and only case 1 of Proposition 4 is relevant, that is, the optimal capital tax rate is positive *for all degrees of income risk.* This condition also implies that $s^*(\bar{\Psi}) < s^{GR}$.

Then the steady state competitive equilibrium capital stock is below the golden rule, $s^* < s^{CE} < s^{GR}$, but optimal capital taxes are nevertheless positive, $\tau > 0$.

3. *Let income risk be small:*
$$\Gamma \in \left[\bar{\Gamma}, \frac{1+\beta}{(1-\alpha)\,\beta}\right).$$
Then the steady state competitive equilibrium capital stock is below the golden rule, $s^{CE} < s^{GR}$ and $s^{CE} < s^*$, and optimal capital taxes are negative.

The interesting result is case 2: in the presence of income risk the Ramsey government maximizing steady state welfare might want to tax capital even though this reduces aggregate consumption because of the GE effect: a lower capital stock shifts away income from risky labor income to non-risky capital income, and for moderate income risk this effect dominates the future generations effect as parameterized by θ.[22]

4.3 Quantitative Relevance: A Back of the Envelope Calculation

In the previous section we have characterized the optimal Ramsey saving rate and associated tax rate, and provided conditions under which capital should be taxed in the presence of idiosyncratic income risk, but subsidized in its absence (case 2 in the previous proposition). Although a full-fledged quantitative analysis is beyond the scope of this paper given that we have analyzed a stylized two period OLG model, we now provide a back-of-the-envelope calculation suggesting that for an empirically plausible extent of income risk the optimal tax on capital is indeed significantly positive when for the same parameterization it would be negative in the absence of uninsurable income risk.

Table 1: Parameter Values

Parameter		Value	Empirical Target
α	Capital Share in Production	0.2082	Labor Income Share in PSID
κ	Share of Labor Income of Old	0.3117	Rel. mean labor income, age 55-84 to 25-54
$\underline{\eta}$	Lower bound of Support for η	0.0242	η_{min} / η_{median} = 3.35%
σ_η	η-risk	0.8051	$Var[\log(\eta)]$ = 0.6480
τ^k	Capital Income Tax	0.28	Chari and Kehoe (2006)

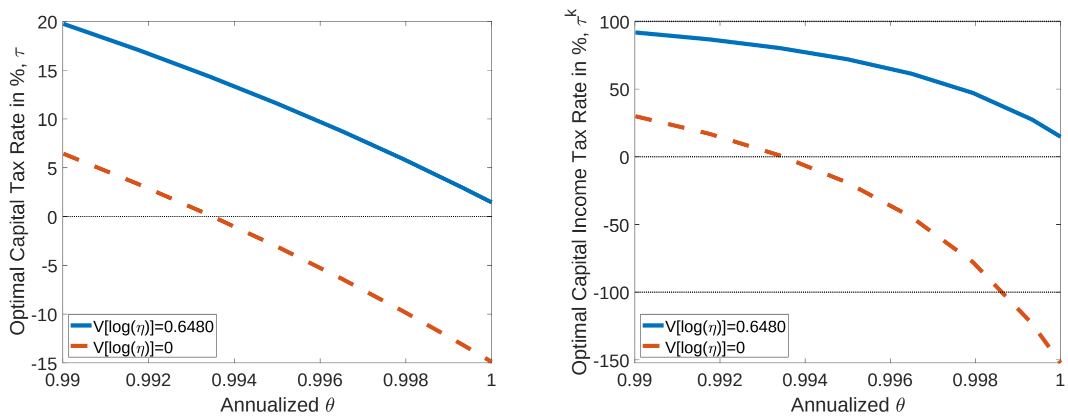
Notes:. This table summarizes the exogenously calibrated parameters used for the quantitative analysis.

[22] The bounds in the previous proposition can be directly defined in terms of the variance of the idiosyncratic income shock η, to a second order approximation of the integral defining Γ (see Appendix K.2).

To do so, we now parameterize the model. We set the initial capital income tax rate to $\tau^k = 28\%$ based on the estimate of Chari and Kehoe (2006). We next set the household time discount factor $\beta = 1$, which delivers an empirically plausible annual risk-free interest rate in the competitive equilibrium with $\tau^k = 28\%$ of 80 basis points in the economy without risk and 38bp in the economy with risk. We then exploit data from the US Panel Study of Income Dynamics (PSID) to calibrate the remaining parameters of the model (α, κ) as well as the distribution of idiosyncratic income risk Ψ. We assume that η is distributed according to a truncated (at the bottom) log-normal distribution with mean 1, a lower bound $\underline{\eta}$ and variance σ_η^2. Table 4.3 summarizes the income (risk) parameterization and Appendix C provides details how the empirical targets are constructed from the PSID.

Figure 1: Optimal Tax on Capital

(a) Optimal Capital Tax Rate τ^* (b) Optimal Capital Income Tax Rate τ^{k^*}



Notes:. This figure shows the optimal capital tax rate τ^* in Panel (a) and the optimal capital income tax rate τ^{k^*} in Panel (b) as a function of the annualized value of the Ramsey government's discount factor θ.

Figure 1 plots the Ramsey optimal tax on capital τ and, for easier comparison to empirically observed tax systems, the associated optimal capital income tax τ^k against the social discount factor θ, both in the model with empirically plausible income risk and the model without old-age income risk.[23] The figures show, as predicted by the theoretical results in the previous subsection, that the optimal capital (income) tax is increasing in income risk and decreasing in the desire of the government to redistribute towards future generations (higher θ). For a large range of social discount factors realistic old-age income risk turns the optimal tax on capital from negative without risk to positive (with income risk). This

[23]The two tax rates are related by $1 + (R - 1)(1 - \tau^k) = (1 - \tau)$, where R is the long-run gross interest rate associated with a given tax on capital τ. Appendix H discusses the monotonic relationship between the tax on capital, and the capital income tax further.

result emerges despite the fact that the economy with income risk has a capital stock below the golden rule and a positive net return on capital.

Table 2: Annualized Interest Rate [in %] and Optimal Tax Rate [in %]

Income Risk	Interest Rate [Initial CE]	Optimal Capital Tax Rate τ	Optimal Capital Income Tax Rate τ^k
	Annualized Discount Factor $\theta = 0.995$		
V[log(η)]=0.6480	0.38%	11.60%	72.05%
V[log(η)]=0.0	0.80%	-3.06%	-18.99%
	Annualized Discount Factor $\theta = 1.0$		
V[log(η)]=0.6480	0.38%	1.45%	14.79%
V[log(η)]=0.0	0.80%	-14.89%	-152.23%

Notes: This table shows the equilibrium interest rate at the initial competitive equilibrium, the optimal capital tax rate τ^*, and the optimal capital income tax rate τ^{k^*} for the stochastic and the deterministic economy.

To see this, Table 2 displays optimal policies for a government a government that discounts future generations at half a percent per year ($\theta = 0.995$) in the upper part of the table and for a government that maximizes steady state welfare ($\theta = 1$) in the lower part. First, we observe that given our parameterization the equilibrium risk free interest rate is small, but significantly positive (between 0.38% and 0.80% per annum), confirming that the pre-reform capital stock is below the golden rule. Second, our results in the table show that even though the optimal capital (income) tax rate depends quite strongly on the social discount factor, income risk turns an optimal capital subsidy without risk into a substantially positive optimal capital income tax in the presence of income risk (16% when maximizing steady state welfare, 72% when discounting future generations at half a percent per year) in the presence of income risk. Although these findings are derived in a simple two-period model where only the old face uninsurable idiosyncratic labor income risk, they suggest that both forces shaping optimal capital income taxes in the model are quantitatively potent.

5 Efficiency Properties of the Ramsey Equilibrium

The Ramsey allocation is the best allocation, given the social welfare weights, that a government needing to respect equilibrium behavior and restricted to proportional capital taxes can implement. We establish two efficiency results. First, defining constrained efficient allocations as those chosen by a social planner that cannot transfer consumption across households of different ages and with different idiosyncratic shocks, as in Davila et al.

(2012), we show that the Ramsey equilibrium is constrained efficient. Second, we prove that if the optimal Ramsey saving rate maximizing steady state welfare $s^*(\theta = 1)$ is smaller than the steady state saving rate in the competitive equilibrium without government s^{CE}, then implementing $s^*(\theta = 1)$ through positive capital taxes yields a Pareto-improving transition from the initial steady state towards the steady state associated with $s^*(\theta = 1)$. This is true even if the steady state equilibrium capital stock lies below the golden rule.

5.1 Constrained Efficiency of Ramsey Equilibria

The Ramsey government cannot implement fully Pareto efficient allocations, characterized in Appendix E.1. Can the government at least achieve constrained efficiency with the set of instruments it has? A constrained efficient allocation is defined as an allocation of capital and consumption that maximizes social welfare subject to the constraint that the allocation does not permit transfers across old households with different η realizations. Consequently, define the set of allocations that are feasible for the *constrained* planner as

$$c_t^y + \int c_t^o(\eta_t) d\Psi + k_{t+1} = k_t^\alpha \tag{22a}$$

$$c_t^o(\eta_t) = k_t MPK(k_t) + \kappa \eta_t MPL(k_t). \tag{22b}$$

The first constraint is simply the aggregate resource constraint. The second constraint restricts transfers across different η households: old age consumption is required to equal capital income plus an η household's share of labor income, where the returns to capital and labor are equal to the factors' relative productivities. The constrained planner might find it optimal, though, to change factor prices by choosing a sequence of capital stocks that differs from the competitive equilibrium. Note that these constraints also imply that

$$\int c_t^o(\eta_t) d\Psi = k_t MPK(k_t) + \kappa MPL(k_t) \tag{23a}$$

$$c_t^y = (1-\kappa) MPL(k_t) - k_{t+1} \tag{23b}$$

so that no *intergenerational* transfers are permitted either, relative to the competitive equilibrium. A constrained efficient allocation maximizes societal welfare $W = \sum_{t=-1}^{\infty} \omega_t V_t$ subject to $(22a)$ and $(22b)$. The question is whether the simple tax policy we consider is sufficient to offset the precautionary savings externality on factor prices, and implement the constrained efficient allocation. The answer is yes as we prove in Appendix E.2).[24]

Proposition 5. *The Ramsey allocation is constrained-efficient.*[25]

5.2 Pareto-Improving Tax Reforms

We now show that starting from the *steady state competitive equilibrium without taxes*, switching to the Ramsey optimal savings and tax policy that maximizes steady state welfare yields a Pareto improvement. This is true as long as the optimal Ramsey saving rate is smaller than the steady state competitive equilibrium saving rate, and is true even if the original competitive steady state equilibrium satisfies $k_0 < k^{GR}$ (and thus $R_0 > 1$), where k^{GR} is the golden rule capital stock maximizing steady state aggregate consumption and k_0 is the initial steady state equilibrium capital stock.

Proposition 6. *Let s^{CE} denote the saving rate in a steady state competitive equilibrium with zero taxes. Assume that $s^{CE} > s^*$. Then a government policy that sets $\tau_t = \tau^* > 0$ leads to a Pareto improving transition from the initial steady state with capital k_0 towards the new steady state associated with tax policy τ^*.*

The proof of this proposition in Appendix E.3 shows that *all generations* benefit from the government implementing a saving rate lower than the initial competitive equilibrium rate despite the fact that it lowers the capital stock, aggregate wages and consumption along the transition. Utility gains arise from higher consumption when young due to the lower saving rate. Since along the transition $c_t^y = (1 - s^*)(1 - \kappa)(1 - \alpha)k_t^\alpha$ and since the capital stock is decreasing along the transition, utility gains are highest in the first period

[24]With ex-ante heterogeneity in the robustness Section 6.3.4, the Ramsey planner can no longer implement the constrained efficient outcome, because the constrained planner can achieve some redistribution through mandating saving rates that differ across productivity type whereas the Ramsey planner cannot. A similar result applies in infinite horizon Aiyagari-style models, as emphasized by Davila et al. (2012).

[25]This result relates our analysis to optimal Mirrleesian capital income taxation, see e.g. Golosov et al. (2003) and Farhi and Werning (2012). Consider a Mirrleesian planner who chooses optimal allocations under the constraint that η-shocks are private information of households. Also assume that the Mirrleesian planner is not permitted to use *intergenerational* transfers, i.e. impose constraints (23). The planner wants to implement transfers across $\hat{\eta}$-types (where $\hat{\eta}$ denotes the reports of households) to provide insurance against low $\hat{\eta}$ realizations. The resource constraint and absence of intergenerational transfers implies that these transfers net out to zero in every period. Under such a transfer scheme all high-η households have an incentive to report low $\hat{\eta}$ and any transfer scheme is not incentive compatible. Furthermore the planner has no other means to incentivize truthful reporting (e.g. by making future consumption or labor supply contingent on the $\hat{\eta}$ reports). Thus, transfers across $\hat{\eta}$-households are infeasible and constraint (22b) emerges as consequence of incentive compatibility in the Mirrleesian problem. The planner therefore implements the constrained efficient allocation of Section 5.1, which coincides with the Ramsey optimum, as shown in the proposition.

of the transition and monotonically decreasing along the transition. In contrast, utility losses emerge from lower consumption when old, which in general equilibrium is $c^o_{t+1} = [\alpha + \kappa\eta_{t+1}(1-\alpha)]\, k^{\alpha}_{t+1}$, and is monotonically decreasing along the transition. Thus, the net welfare gains are highest for the initial generations and monotonically decline along the transition. But by the choice of s^* the government insures that generations in the new steady state benefit from the reform, and the monotonicity of the net welfare gains along the transition insures that all generations living through the transition are also better off from implementing s^* through positive taxes on capital, $\tau^* > 0$.

The result in the previous proposition is of course not surprising if s^{CE} is larger than the golden rule implementing saving rate s^{GR}. However, for intermediate risk, Proposition 4 shows that $s^* < s^{CE} < s^{GR}$, and thus the steady state equilibrium capital stock is below the golden rule. Proposition 6 establishes that setting $\tau^* > 0$ implements a Pareto-improving transition even in this case. Note that Proposition 6 discusses a massive permanent policy reform from $\tau = 0$ to $\tau = \tau^*$. A reform decreasing the saving rate s^{CE} marginally but permanently by implementing a marginal tax hike $\tau = \varepsilon > 0$ also leads to a Pareto improvement, under the same conditions as in the previous proposition.[26]

The sources of the net utility gains along the transition (higher, and monotonically decreasing, consumption when young and lower, and monotonically decreasing, consumption when old) do not rest on the presence or extent of income risk. Whether the initial laissez-faire equilibrium satisfies the inequalities $s^* < s^{CE} < s^{GR}$ of course depends on risk.

Finally, note that the converse of Proposition 6 is not true: even if $s^{CE} < s^*$, implementing the Ramsey optimal steady state savings subsidy $\tau^* < 0$ and associated higher saving rate s^* does *not* lead to a Pareto improvement. Appendix E.4 shows that the generation born into the first period of a policy-induced transition loses from the policy reform. In fact, not only is implementing $\tau^* < 0$ not Pareto improving if $s^{CE} < s^*$, any marginal policy reform that induces a period 1 saving rate above the competitive saving rate s^{CE} does not result in a Pareto improvement since it makes the first generation worse off.

[26]Our results on Pareto improving transitions by implementing the optimal long-run Ramsey saving rate $s^* < s^{CE}$ or by a marginal reduction of the saving rate from s^{CE} hold for arbitrary additively separable strictly increasing lifetime utility function. All we require is an initial laissez-faire equilibrium allocation featuring $s^* < s^{CE} < s^{GR}$; the exact conditions for this inequality to be satisfied of course depends on the preference structure. Note that, in general, the tax rate on capital required to implement the Pareto-improving time-constant saving rate will be time-varying, rather than constant, as in the logarithmic case.

6 Discussion and Robustness

As discussed above, the optimal capital tax results in this paper emerge as the government trades off the correction of the pecuniary externality with the improvement of the well-being of future generations through a higher future capital stock. One could argue that taxes on capital are neither the most direct way to tackle uninsurable labor income risk (progressive labor income taxes would be) nor are they the obvious policy to redistribute resources across generations (social security or government debt would be). We therefore now investigate how robust our main results are to the introduction of these additional policy tools (labor income taxes in Section 6.1 and social security/government debt in Section 6.2). Finally, Section 6.3 briefly discusses robustness of the results to other extensions.

6.1 Endogenous Labor Supply and Labor Income Taxation

If the Ramsey government can complete markets through progressive taxation or income-contingent transfers, then it would, with exogenous labor supply, trivially provide full insurance against η-risk by taxing all labor income in the second period at 100% and rebating it in a lump-sum fashion among all households. Consequently, the pecuniary externality induced by private precautionary savings disappears. With endogenous labor supply, however, taxing labor income at a confiscatory rate is no longer optimal, which raises the question how the government should tax capital and labor income in our model. We now show that our previous results with log-utility qualitatively go through unchanged, and quantitatively the tax on capital remains significantly positive even if the government places large weights on future generations.

Assume households have lifetime utility defined over consumption when young, and stochastic consumption and labor allocations when old $c_t^y, c_{t+1}^o(\eta), l_{t+1}^o(\eta)$ given by

$$V_t = \ln(c_t^y) + \beta \int \left[\ln(c_{t+1}^o(\eta)) + \gamma \ln(1 - l_{t+1}^o(\eta)) \right] d\Psi(\eta). \tag{24}$$

where $\gamma > 0$ is a parameter.[27] The budget constraints of a typical generation now read as

$$\begin{aligned} c_t^y + a_{t+1} &= (1-\kappa) w_t && \text{(25a)} \\ c_{t+1}^o(\eta) &= (1-\tau_{t+1}) R_{t+1} a_{t+1} + \kappa \eta (1-\tau_{t+1}^l) w_{t+1} l_{t+1}^o(\eta) + T_{t+1}, && \text{(25b)} \end{aligned}$$

[27]The utility function of the initial old is similarly defined.

where τ^l_{t+1} is the proportional labor income tax rate. As before, tax revenues from capital and labor income taxes on the old are rebated to them in a lump-sum fashion:

$$T_{t+1} = \tau_{t+1}R_{t+1}a_{t+1} + \tau^l_{t+1}\kappa w_{t+1}\int \eta l^o_{t+1}(\eta)d\Psi(\eta). \tag{26}$$

The production technology transforming capital and labor (K_t, L_t) into output and the associated firm problem remains unchanged, and equilibrium in the labor market obtains if

$$L_t = 1 - \kappa + \kappa\int \eta l^o_t(\eta)d\Psi(\eta). \tag{27}$$

We provide a complete definition of equilibrium for given tax policy in Section F of the appendix. The Ramsey government maximizes social welfare, as in the benchmark model, cf. equation (4), by choice of proportional taxes on labor income and capital $\{(\tau^l_t, \tau_t)\}^\infty_{t=0}$.

6.1.1 Theoretical Results

We now argue that the optimal saving rate the Ramsey government chooses is identical to the one in Proposition 2, and is implemented with a tax on capital that is strictly increasing in idiosyncratic labor productivity risk η and strictly decreasing in the social welfare weights on future generations. The analysis proceeds in two steps, with details relegated to Appendix F.2. There we first show that a given sequence of private aggregate saving rates $s_t = \frac{K_{t+1}}{(1-\kappa)w_t} \in (0, 1)$ and aggregate labor $L_t \geq 0$ can be implemented by choice of a sequence of capital and labor income tax rates (τ_{t+1}, τ^l_t). This implies that the Ramsey government can directly choose sequences $\{s_t, L_t\}$ and then implement them as competitive equilibrium with proportional capital and labor income taxes. Second, we show that the optimal Ramsey saving rate is identical to the one with exogenous labor supply, and the capital tax rate implementing it has the same form as before.

Competitive Equilibrium for Given Tax Policy Starting with the first step, exploiting the individual optimality conditions for optimal labor supply, at an interior allocation[28] and substituting out factor prices we can write aggregate labor supply of the old L^o_{t+1}, and thus

[28]Given the Inada conditions implied by the utility function, the only potentially binding corner is $l^o_{t+1}(\eta) \geq 0$. Since $l^o_{t+1}(\eta)$ is strictly increasing in η, this constraint is not binding for η large enough. Assumption 4 in the appendix gives an assumption on the support of η to insure that $l^o_{t+1}(\eta)$ is positive almost surely. This lower bound can be made arbitrarily small for sufficiently small leisure utility weight γ.

aggregate labor supply of all households L_{t+1} as

$$L^o_{t+1}(\tau^l_{t+1}) = \int \eta l^o_{t+1}(\eta) d\Psi(\eta) = \frac{1-\tau^l_{t+1} - \frac{\gamma\alpha(1-\kappa)}{\kappa(1-\alpha)}}{1-\tau^l_{t+1} + \frac{\gamma}{1-\alpha}} \tag{28a}$$

$$L_{t+1}(\tau^l_{t+1}) = 1-\kappa+\kappa L^o_{t+1}(\tau^l_{t+1}) = 1-\kappa+\kappa\frac{1-\tau^l_{t+1} - \frac{\gamma\alpha(1-\kappa)}{\kappa(1-\alpha)}}{1-\tau^l_{t+1} + \frac{\gamma}{1-\alpha}}. \tag{28b}$$

Crucially, aggregate equilibrium labor supply is independent of the saving rate and the tax rate on capital shaping the dynamics of the economy, and exclusively depends on the labor income tax rate when old.[29] Define $\bar{\tau}^l_{t+1} = \frac{\kappa(1-\alpha)-\gamma\alpha(1-\kappa)}{\kappa(1-\alpha)} < 1$. We have the following

Proposition 7. *For any tax rate* $\tau^l_{t+1} \in \left(-\infty, \bar{\tau}^l_{t+1}\right)$ *aggregate labor supply is given by equation* $(28b)$. *Aggregate labor supply is strictly decreasing in the labor income tax rate* τ^l_{t+1}. *Thus, for any aggregate labor supply* $L^o_{t+1} \in (0,1)$ *there exists a unique labor income tax rate* τ^l_{t+1} *that implements this* L^o_{t+1} *as part of a competitive equilibrium.*

Note that the labor income tax rate implementing any positive labor supply of the old is strictly less than 1. Optimal individual labor supply $l^o_{t+1}(\eta;\tau^l_{t+1})$ (see equation (66) of Appendix F.2.1) is only a function of the labor income tax rate and the idiosyncratic shock. With this result in hand we can now proceed as in the benchmark model with exogenous labor supply. As in equation (11), the competitive equilibrium saving rate is determined from the household intertemporal Euler equation, with equilibrium factor prices and transfers substituted out, as

$$s_t = \frac{1}{1+\left[\alpha\beta(1-\tau_{t+1})\Gamma(\alpha,\kappa;\tau^l_{t+1},\Psi)\right]^{-1}} = s(\alpha,\kappa;\tau_{t+1},\tau^l_{t+1},\Psi), \tag{29}$$

where

$$\Gamma(\alpha,\kappa;\tau^l_{t+1},\Psi) = \int\left(\frac{1}{\alpha+\frac{(1-\alpha)\kappa}{L_{t+1}(\tau^l_{t+1})}\left[(1-\tau^l_{t+1})\eta l^o_{t+1}(\eta;\tau^l_{t+1})+\tau^l_{t+1}L^o_{t+1}(\tau^l_{t+1})\right]}\right)d\Psi(\eta) \tag{30}$$

completely summarizes the impact of idiosyncratic productivity and thus income risk on the optimal saving decision. Note that the labor income tax rate affects the risk term Γ

[29] We assume that the utility weight on leisure γ is sufficiently small and/or the productivity κ of the old sufficiently large that the Ramsey government finds it optimal to implement positive aggregate labor supply of the old. Otherwise old households do not work, and idiosyncratic risk trivially becomes irrelevant. A tax rate $\tau^l_{t+1} < \bar{\tau}^l_{t+1}$ insures this.

through its impact on individual and aggregate labor supply. However, since for every labor income tax rate τ_{t+1}^{l} satisfying the restriction in proposition 7 the term Γ is a positive constant, we immediately have the following

Proposition 8. *For any labor income tax rate* $\tau_{t+1}^{l} \in \left(-\infty, \frac{\kappa(1-\alpha)-\gamma\alpha(1-\kappa)}{\kappa(1-\alpha)}\right)$ *and any tax rate on capital* $\tau_{t+1} \in (-\infty, 1)$ *the aggregate equilibrium saving rate* $s_t \in (0,1)$ *is given in equation* (29)*. Consequently, for any saving rate* $s_t \in (0,1)$ *and given a labor income tax rate and associated labor allocation there exists a unique capital tax rate* $\tau_{t+1} \in (-\infty, 1)$ *that implements this saving rate as part of a competitive equilibrium.*

The previous two propositions demonstrate the sequential nature of solving for the competitive equilibrium, given tax policy. In each period $t \geq 0$, given a labor income tax rate τ_t^l, we can solve for equilibrium labor supply $(l_t^o(\eta), L_t^o, L_t)$. Then, given this labor allocation, which in turn determines $\Gamma(\alpha, \kappa; \tau_{t+1}^l, \Psi)$, and given a tax on capital τ_{t+1}, one solves for the equilibrium saving rate s_t. Finally, the capital stock K_t and the saving rate s_t today determine the aggregate capital stock in period $t+1$. Thus, given an initial condition K_0, any aggregate allocation of labor and savings $\{L_t, s_t\}$ and associated allocation of individual labor $\{l_t^o(\eta)\}$ and capital $\{K_{t+1}\}$ in equation (64) can be implemented as a competitive equilibrium through a suitable choice of labor income and capital tax rates $\{\tau_t^l, \tau_{t+1}\}$.

The Ramsey Optimal Policy The previous results imply that, as in the benchmark model, we can solve the Ramsey problem directly in terms of allocations $\{(L_t, s_t)\}$. Our benchmark characterization of the optimal saving rate and associated tax rate on capital goes through unchanged.

Proposition 9. *The optimal Ramsey saving rate with endogenous labor supply is given by*

$$s_t = \frac{1}{1 + \left(\alpha\beta + \alpha\left(1+\alpha\beta\right)\sum_{j=1}^{\infty}\frac{\omega_{t+j}}{\omega_t}\alpha^{j-1}\right)^{-1}} \in (0,1). \tag{31}$$

Furthermore, if we assume that relative social welfare weights are constant, $\frac{\omega_{t+1}}{\omega_t} = \theta$ *for all t, then the optimal Ramsey saving rates in* (31) *are constant over time and given by*

$$s_t = s = \frac{\alpha(\beta+\theta)}{1+\alpha\beta} \in (0,1).$$

The optimal tax on capital implementing this saving rate as competitive equilibrium is *given by equation* (29), *or explicitly, as*

$$1-\tau_{t+1}=\frac{s_t}{(1-s_t)\,\alpha\beta\Gamma(\alpha,\kappa;\tau^l_{t+1},\Psi)}, \tag{32}$$

where $\Gamma(\alpha,\kappa;\tau^l_{t+1},\Psi)$ *was defined in equation* (30).

This result demonstrates that the optimal saving rate s_t chosen by the Ramsey planner is independent of the optimal labor allocation and the labor income tax rates that implement them, and identical to the one with exogenous labor supply. It again is independent of the extent of idiosyncratic labor income risk. The optimal tax on capital τ_{t+1} is strictly increasing in income risk Γ, and depends on the optimal labor allocation determining this risk, and thus on the optimal labor income tax rate that governs it.[30] Crucially, since this optimal rate is less than one, idiosyncratic labor income risk continues to be present, and the precautionary savings channel and associated pecuniary externality remains operational.

6.1.2 Quantification

To get a sense of the extent to which endogenous labor supply and labor income taxation affects the optimal tax on capital quantitatively, we extend the calibration from Section 4.3 to endogenous labor supply. Apart from the idiosyncratic productivity process we keep all parameters the same, and we recalibrate the distribution of η, together with the new leisure utility parameter γ in such a way that minimum, mean and log-variance of labor income is the same as in the benchmark economy, and average hours worked are 1/3 of total time. Table 3 contains the resulting parameter values, Figure 2 plots the optimal capital and labor income tax rates against the (annualized) social discount factor and Table 4 summarizes optimal policies for $\theta = 0.995$ and $\theta = 1$.

Panel (a) of Figure 2 contrasts the optimal tax rate on capital in the benchmark economy (exogenous labor) with that in the endogenous labor economy, with and without idiosyncratic income risk. The figure shows that endogenous labor supply and labor income taxes shift the optimal capital tax rate down by $8\%p$ (blue, solid line). Notably, this rate remains positive for all social discount factors below an annualized social discount factor of $\theta = 0.997$. As in the benchmark economy, the capital tax rate is strictly decreasing, and the labor income tax rate is strictly increasing in θ. Turning to Panel (b) we see that

[30] The optimal allocation of labor L_{t+1} is determined from a static first order condition of the Ramsey problem which has no closed-form solution but is straightforward to solve numerically. Proposition 7 then gives the associated optimal labor income tax rate implementing this labor allocation.

Table 3: Parameter Values: Endogenous Labor Supply

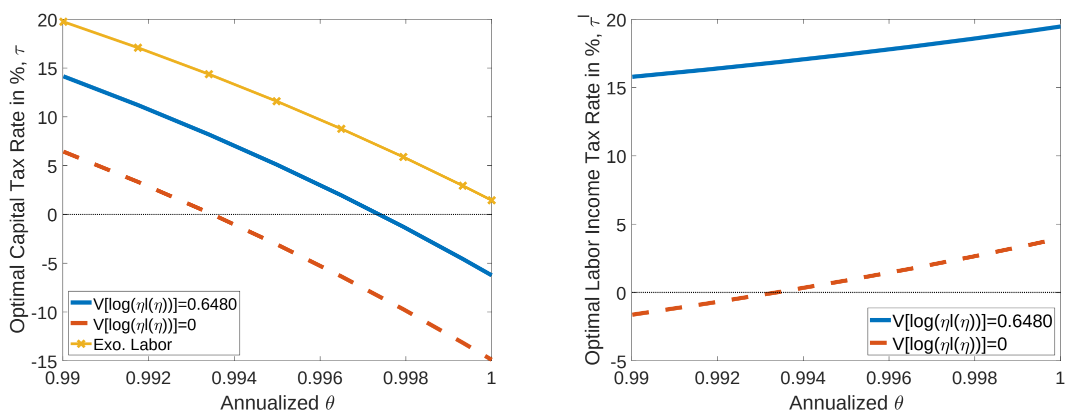
Parameter		Exo. Labor	Target	End. Labor
γ	Labor utility weight	0	$E[l(\eta)] = 0.3333$	0.73
$\underline{\eta}$	η support lower bound	0.05	$\underline{\eta}l(\underline{\eta})/\eta_{med}l(\eta_{med})$ = 3.35%	1.0
σ_η	η-risk	0.81	$Var[\log(\eta l(\eta))]$ = 0.648	0.45
$E[\eta]$	Mean idiosyn. shock	1	$E[\eta l(\eta)] = 1$	2.68

Notes:. This table summarizes the endogenously calibrated parameters used for the quantitative analysis of the model with endogenous labor supply. Parameters α, κ and the initial tax rate τ^k used in calibration are given in Table 4.3.

the optimal labor income tax turns strongly positive, in the ballpark of $16 - 20\%$, in the presence of income risk (relative to a world without income risk), indicating its important role in providing insurance against idiosyncratic labor income risk. Crucially, however, since labor supply responds endogenously to labor income taxes, the optimal rate is far below the 100% required to provide full insurance. Therefore, significant income risk remains, and our argument for the taxation of capital based on precautionary savings against idiosyncratic risk and the associated pecuniary externality retains its validity quantitatively, although it becomes less potent, as Panel (a) of Figure 2 and Table 4 show. In the absence of this risk (red-dashed line in Figure 2), in contrast, no strong case for taxing capital can be made in our model.

Figure 2: Optimal Taxes [in %] with Endogenous labor Supply

(a) Optimal Capital Tax Rate τ^* (b) Optimal Labor Tax Rate τ^{l^*}



Notes: This figure shows the optimal capital tax rate τ^* in Panel (a) and the optimal labor income tax rate τ^{l^*} in Panel (b) as a function of the annualized value of the Ramsey government's discount factor θ in the model with endogenous labor supply.

Table 4: Annualized Interest Rate [in %] and Optimal Tax Rates [in %]

Parameter Configuration	Interest Rate (Initial CE)	Optimal Capital Tax Rate τ^*	Opt. Cap. Income Tax Rate τ^{k^*}
	Annualized Discount Factor $\theta = 0.995$		
V[log($\eta l(\eta)$)]=0.6480	0.57%	5.11%	31.73%
V[log($\eta l(\eta)$)]=0	0.80%	-3.06%	-19.00%
	Annualized Discount Factor $\theta = 1.0$		
V[log($\eta l(\eta)$)]=0.6480	0.58%	-6.24%	-63.80%
V[log($\eta l(\eta)$)]=0	0.80%	-14.90%	-152.27%

Notes: This table shows the equilibrium interest rate at the initial competitive equilibrium, the optimal capital tax rate τ^*, and the optimal capital income tax rate τ^{k^*} for the stochastic and the deterministic economy with endogenous labor supply. Corresponding results for the model with exogenous labor supply were shown in Table 2.

6.2 Other Forms of Intergenerational Redistribution

We now discuss extensions to more direct intergenerational redistribution policies such as social security and government debt, as well as private intergenerational linkages through altruism and bequests. We ask whether the argument for capital taxes remains operative in the presence of these more direct mechanisms for intergenerational transfers.

6.2.1 Fiscal Instruments for Intergenerational Redistribution: Social Security and Government Debt

Consider an economy with a Pay-As-You-Go (PAYG) pension system in which labor income $(1-\kappa)w_t$ of young households in period t is subject to a payroll tax τ_t^p. These payroll taxes finance pension income $b_t = \tau_t^p(1-\kappa)w_t$ of period t old households. The budget constraints of a household born in t are now given by

$$\begin{aligned} c_t^y + a_{t+1} &= w_t(1-\kappa)(1-\tau_t^p) \\ c_{t+1}^o(\eta) &= a_{t+1}R_{t+1}(1-\tau_{t+1}) + \kappa\eta w_{t+1} + b_{t+1} + T_{t+1}. \end{aligned}$$

Appendix G.1 derives closed form expressions for general Ramsey government discount functions $\boldsymbol{\omega}$ showing that the saving rate out of net income $s_t = \frac{a_{t+1}}{(1-\tau_t^p)w_t}$ takes the same form as we derived for the analysis with capital income taxes, see equation (20) and footnote 18. While a closed form solution for the optimal contribution rate to the pension system does in general not exist, we show in the appendix that *both* the contribution rate as well as the capital (income) tax rate are increasing in risk.

We simplify the remainder of our analysis by focusing on a government maximizing steady state utility, $\theta = 1$. First, consider the degenerate case with zero income risk. With exogenous labor supply the pension system is nondistortionary, and the Ramsey government uses the PAYG pension system to implement the golden rule capital stack and per capita consumption at its maximally feasible level, with optimal capital taxes equal to zero. If the competitive equilibrium of the deterministic economy has a steady state capital stock below the golden rule, the welfare maximizing pension contribution rate is negative, otherwise it is positive.

Next, consider an increase of second period income risk. This gives rise to precautionary saving, which in general equilibrium increases wages and thus the risky income component. Now the Ramsey government strikes a balance between providing inter-generational redistribution, partially insuring households against their income risk through a risk-free pension income, and manipulating relative prices to reduce the risk exposure of households. Since the Ramsey government only has two instruments to achieve these three objectives and since, by construction, it cannot perfectly insure idiosyncratic income risk, it finds it optimal to increase both its instruments when risk rises. Contribution rates to the pension system implement intergenerational redistribution, and increase with risk to reduce the direct income risk exposure, but cannot do so perfectly. Capital taxes rise with income risk in order to address the *remaining* negative pecuniary externality arising from the increase of precautionary savings.

Moreover, if we impose a constraint of non-negative pension contribution rates and if the no-income risk economy has a capital stock below the golden rule, then there is an income-risk threshold such that for risk below that threshold the optimal pension contribution rate is exactly zero and all of our results on optimal capital income taxes from Proposition 4 go through completely unchanged.

Finally, note that intergenerational redistribution does not need to take the form of a PAYG pension system. We show in Appendix G.2, again for a general government discount function $\boldsymbol{\omega}$, that a suitably chosen sequence of explicit government debt achieves the same allocation as the PAYG pension system.

6.2.2 Private Intergenerational Linkages: Bequests and Altruism

To investigate the robustness of our main findings with respect to private rather than public intergenerational transfers we first consider intergenerational transfers motivated by warm-glow bequest motives, with details provided in appendix G.3. Assume that households survive to old age at rate $\varsigma \in (0,1]$; our benchmark results are nested for $\varsigma = 1$. Utility of a period t cohort from second period consumption in case of survival is now weighted by $\beta\varsigma$, and flow utility from bequeathed wealth to the period $t+1$ young households in case of death receives utility weight $\beta(1-\varsigma)\varphi$. The utility function of cohort t then is

$$\ln(c_t^y) + \beta \mathbb{E}_t \left[\varsigma \ln(c_{t+1}^o) + (1-\varsigma)\varphi \ln\left(a_{t+1}^o R_{t+1}(1-\tau_{t+1})\right)\right].$$

Maximization is subject to the per period budget constraints

$$\begin{aligned} c_t^y + a_{t+1}^o &= (1-\kappa)w_t + a_t^y + T_t^y \\ c_{t+1}^o &= a_{t+1}^o R_{t+1}(1-\tau_{t+1}) + \frac{\kappa}{\varsigma}\eta w_{t+1} + T_{t+1}^o \end{aligned}$$

where $a_t^y = (1-\varsigma)a_t^o R_t(1-\tau_t)$ is the inherited wealth of the young households and where in each period the capital income tax revenue $T_t = a_{t+1}^o R_{t+1}\tau_t$ is used to finance transfers T_t^y to the young and transfers T_t^o to the old households. The scaling of second period productivity by $\frac{1}{\varsigma}$ achieves that aggregate labor in the economy still integrates to one. We focus on characterizing the optimal long-run policy ($\theta = 1$) and can still establish closed form solutions for the competitive equilibrium saving rate and show that it is independent of the capital stock, increasing in risk and decreasing in capital taxes. The optimal Ramsey saving rate remains independent of risk, constant in time and can be implemented with a capital tax that is strictly increasing in risk. Thus, our results fully extend to a model with survival risk and an operative warm-glow bequest motive.

Finally, Appendix G.4 provides an extension of the model to intergenerational transfers induced by one-sided altruism. In this economy, in each period $t = 0, 1, \ldots$ a new dynastic household is born that lives for two periods and values the utility of future generations through altruism parameter $\delta \geq 0$ with preferences given by

$$\tilde{V}_t = u(c_t^y) + \beta E_t \left[u(c_{t+1}^o) + \delta\left(u(c_{t+1}^y) + \beta u(c_{t+2}^o)\right) + \delta^2\left(u(c_{t+2}^y) + \beta u(c_{t+3}^o)\right) + \ldots\right].$$

Our benchmark model is obtained when $\delta = 0$. We consider inter-vivos transfers $b_{t+1} \geq 0$ from the period $t+1$ old to the period $t+1$ young households, and also augment the model by a standard borrowing constraint. Since intergenerational linkages now induce an endogenous wealth distribution, this borrowing constraint is potentially binding. The budget constraints are now given by

$$
\begin{aligned}
c_t^y + a_{t+1} &= (1-\kappa)w_t + b_t \\
c_{t+1}^o + b_{t+1} &= a_{t+1}R_{t+1}(1-\tau_{t+1}) + \kappa\eta_{t+1}w_{t+1} + T_{t+1} \\
a_{t+1} &\geq 0, \text{ and } b_{t+1} \geq 0.
\end{aligned}
$$

In the appendix we show that this structure gives rise to an Aiyagari (1994)-style economy with a strategic interaction through inter-vivos transfers between altruistic households, in which closed form solutions for the competitive equilibrium saving rates and the optimal Ramsey saving rate are not available any longer. Precautionary saving behavior of the dynastic household now emerges due to prudence and due to potentially binding borrowing constraints $a_{t+1} \geq 0, b_{t+1} \geq 0$. We show that a Ramsey government that only weighs explicitly the utility of the dynastic period 0 young households exactly shares the objective of the dynastic period 0 young households but internalizes the pecuniary externality arising from the private saving behavior. Since it now also has a motive to redistribute intratemporally between the wealth-rich and poor, the model features precisely the same mechanisms emphasized in Davila et al. (2012) as discussed in the introduction. If the government places additional welfare weights on future generations through a social discount factor $\theta > \delta$, then an inter-generational distribution motive similar to the one in our model is also operative. Thus, a dynastic framework gives rise to potentially interesting dynamics beyond a pure Aiyagari (1994)-style model, but one that is no longer analytically tractable (as the main body of the Davila et al. (2012) paper already shows).

6.3 Other Extensions and Robustness Analyses

This subsection briefly discusses other extensions and alternative assumptions of the model that, although not central for the existence of the main trade-off associated with capital taxation between insurance against idiosyncratic risk and intergenerational redistribution, could affect its quantitative significance.

6.3.1 General Production Function and Capital Depreciation

We thus far have assumed that the production function has a Cobb-Douglas form and capital depreciates at a full 100% within one period. The first assumption is common in macroeconomic analyses of long-run growth and business cycles. The latter assumption is common in two-period OLG models in which a period should be interpreted as lasting 30 years, and with an annual depreciation rate of 5% the 30-year depreciation rate is 79%, with a 10% annual rate it is 96%. As we argue in footnote 19, our finding that uninsurable idiosyncratic income risk drives the optimal tax rate on capital above zero in the absence of intergenerational redistribution concerns is a general feature of the model, and obtained for any depreciation rate δ and any neoclassical constant returns to scale production function with strictly positive and strictly decreasing marginal products.[31]

6.3.2 Taxation of Capital and Capital Income Taxation

Our discussion of optimal taxation thus far has focused on the taxation of the stock of capital (or wealth). We could have equivalently phrased our results in terms of capital income taxes τ^k. The two tax rates are related by $1 + (R - 1)(1 - \tau^k) = (1 - \tau)$, where R is the gross interest rate associated with a given tax on capital τ. Although not necessary for our results, Appendix H gives a sufficient condition for the Ramsey equilibrium to have uniformly positive net returns ($R_t > 1$ for all t), in which case capital income tax rates have the same sign as the corresponding taxes on the stock of capital.

6.3.3 Idiosyncratic Return Risk

Now suppose that households also face idiosyncratic return shocks, denoted by ϱ, as modeled by Benhabib et al. (2011) and empirically investigated by Fagereng et al. (2020). After-tax gross returns in the second period of life are now risky and given by $R_{t+1}\varrho_{t+1}(1-\tau_{t+1})$, and return risk and labor income risk η_{t+1} may be correlated. We further assume that transfers are contingent on the return realization, $T_{t+1}(\varrho) = a_{t+1}R_{t+1}\varrho_{t+1}\tau_{t+1}$.[32]

With these assumptions we show in Appendix I.1 that our results go through unchanged, with the impact of idiosyncratic risk now expressed in terms of the random variable $\delta_{t+1} = \frac{\eta_{t+1}}{\varrho_{t+1}}$ and its distribution Π. The constant $\Gamma = \Gamma(\alpha, \kappa; \Pi)$ reflecting risk in (11) is now determined by the distribution Π. An increase in labor income risk still increases the competitive

[31]To be sure, as in the neoclassical growth model obtaining closed-form solutions to the dynamic consumption-saving problem in general equilibrium does require Cobb-Douglas utility and, for $\kappa > 0$, full depreciation.

[32]If the capital tax returns were redistributed as lump-sum transfers, the capital tax can be used to insure idiosyncratic return risk, and the Ramsey government would want to use it for this purpose also, in addition to implementing a desirable aggregate saving rate. Our implementation result would not hold anymore, and we could not solve for the saving rate directly in the Ramsey problem any longer. Note, however, that such an insurance effect would give an additional rationale for using a positive tax on capital, in the same way the labor income tax is being used to provide insurance against idiosyncratic labor income risk in Section 6.1.

equilibrium saving rate, and an increase in returns risk decreases it as saving becomes less attractive. The optimal Ramsey saving rate continues to be given by equation (20), implemented with a tax rate according to (21), with Γ now derived from the distribution Π.

6.3.4 Ex-Ante Heterogeneity in Labor Productivity

Our results go through unchanged if households differ ex ante in permanent labor productivity ν. Labor productivity of type ν is given by $\nu(1-\kappa)$ when young and $\kappa\nu\eta$ when old. Further assume that the distribution of second period shocks η is independent of permanent productivity type ν and that the cross-sectional distribution of ν has mean 1. The government continues to tax capital at a uniform rate and rebates revenues lump-sum within each ν-type according to the groups' tax payments. In Appendix I.2 we show that, not surprisingly given homotheticity of preferences, the general equilibrium saving rate is identical across all ν-type households, and still given by equation (11) from the benchmark model. The optimal saving rate s implemented by the utilitarian (across ν) Ramsey government remains unchanged from equation (20), and so is the implementing optimal tax rate (21).

6.3.5 Time Varying Productivity and Population Growth

Our results fully extend to a model with deterministic technological progress. Assume that production is given by $Y_t = K_t^{\alpha}\left(A_t L_t\right)^{1-\alpha}$ where productivity $A_t = (1+g_t)A_{t-1}$ grows at a time varying rate g_t. This time-varying growth rate cancels in the household optimization problem and only adds maximization-irrelevant additive terms to the Ramsey problem, leaving the optimal saving rate and capital tax rate implementing it unchanged.

In contrast, positive population growth at a constant rate n reduces the equilibrium capital-labor ratio, lowers wages and increasing asset returns faced by cohort t in period $t+1$ thereby increasing the competitive equilibrium saving rate for given taxes. The optimal Ramsey saving rate, however, is not affected by population growth, but the optimal tax on capital implementing that rate is.[33] Finally, in the steady state of the model, positive population growth shrinks but does not eliminate the size of the intermediate risk interval characterized in Proposition 4 for which the economy has capital below the golden rule yet capital is taxed at a positive rate, and the reform towards that tax rate is Pareto improving.

[33] If the population growth rate is time varying, then so is the general equilibrium saving rate. With a Utilitarian objective we can then no longer characterize the solution to the Ramsey problem in closed form. If the planner instead maximizes discounted per capita utilities, then the closed form results go through unchanged and the optimal Ramsey saving rate continues to be a constant.

6.3.6 General Intertemporal Elasticity of Substitution ρ and Risk Aversion σ

Our results thus far were derived under the assumption that households maximized time-separable expected utility, with logarithmic period utility. We can extend our results to a more general utility function with intertemporal elasticity of substitution ρ and risk aversion σ, as in Epstein and Zin (1989, 1991) and Weil (1989), of the form

$$V_t = u(c_t^y) + \beta u(v(c_{t+1}^o)) \tag{33}$$

where the period utility function is, for $\rho \neq 1$, $u(x) = \frac{x^{1-\frac{1}{\rho}}}{1-\frac{1}{\rho}}$ for $x \in \{c_t^y, v(c_{t+1}^o)\}$, and $\rho = 1$ corresponds to log-utility analyzed thus far. The certainty equivalent over old-age consumption is given by

$$v(c_{t+1}^o) = \begin{cases} \left(\int c_{t+1}^o(\eta)^{1-\sigma} d\Psi(\eta)\right)^{\frac{1}{1-\sigma}} & \text{for } \sigma \neq 1 \\ \exp\left(\int \ln\left(c_{t+1}^o(\eta)\right) d\Psi(\eta)\right) & \text{for } \sigma = 1. \end{cases}$$

This preference specification was first introduced into the literature by Selden (1978, 1979). The parameter ρ measures the IES and the parameter σ governs risk aversion.[34] If $\sigma = 1/\rho$ then the utility function takes the standard CRRA form.

Appendix J shows that the closed form solution of the Ramsey problem goes through unchanged for a unit IES, $\rho = 1$. The optimal Ramsey saving rate is still given by

$$s^* = \frac{\alpha(\beta + \theta)}{1 + \alpha\beta}.$$

The tax rate τ implementing this saving rate as a competitive equilibrium is given by

$$(1 - \tau) = \frac{s}{(1 - s)\alpha\beta\tilde{\Gamma}}.$$

It is increasing in term $\tilde{\Gamma}$, which captures the effects of income risk on households' savings

[34] This specification of Epstein-Zin-Weil preferences is also used by other papers in the literature, e.g., in Bommier et al. (2020). Note that V_t represents the same ordinal ranking over current consumption c_t^y and the certainty equivalent over future risky consumption $c_{t+1}^o(\eta_{t+1})$ as the more commonly used specification

$$\tilde{V}_t = \left\{(1 - \tilde{\beta})(c_t^y)^{1-\frac{1}{\rho}} + \tilde{\beta}\left[v(c_{t+1}^o)\right]^{1-\frac{1}{\rho}}\right\}^{\frac{1}{1-\frac{1}{\rho}}}$$

since one is a monotone transformation of the other, $V_t = \frac{\tilde{V}_t^{1-\frac{1}{\rho}}}{(1-\tilde{\beta})\left(1-\frac{1}{\rho}\right)} - \frac{(1+\beta)}{1-\frac{1}{\rho}}$, where $\beta = \frac{\tilde{\beta}}{1-\tilde{\beta}}$.

behavior and itself depends on risk aversion σ.

For a general IES satisfying $\frac{1}{\sigma} \neq \rho \neq 1$ the optimal saving rate and tax rate are neither time invariant nor independent of income risk. We show that an increase in income risk increases the optimal steady state Ramsey saving rate s^* if $\rho < 1$ and decreases it if $\rho > 1$. Thus the direction of the change in s with respect to income risk is exclusively determined by the IES ρ, with log-utility as the watershed. We can also determine the impact of income risk on optimal steady state capital taxes. If $\rho \leq 1$, then an increase in income risk increases the optimal tax rate on capital. Similarly, if $\rho > 1$ and $\sigma \leq 1/\rho$, then an increase in income risk increases the optimal tax rate on capital. Finally, if $\rho > 1$ and $\sigma > 1/\rho$, an increase in income risk might lead to a strict reduction in the capital tax rate τ, but only if the competitive equilibrium saving rate for given τ is strictly decreasing in income risk, i.e., if households do not engage in precautionary saving.

7 Conclusion

We have analyzed optimal capital taxes in a canonical OLG model with idiosyncratic labor income risk. We obtain a full analytical characterization of the Ramsey allocation and tax policy along the transition path when the IES equals one. The optimal aggregate saving rate is independent of idiosyncratic income risk, and is implemented by a tax rate that is increasing in income risk, and positive if and only if income risk is sufficiently large.

By showing that the Ramsey government can implement constrained efficient allocations through a proportional capital tax we argue that capital income taxation, in the context of our model, is the appropriate fiscal tool to deal with the externality on equilibrium factor prices induced by private precautionary savings behavior against idiosyncratic income risk. However, we also demonstrate that capital should not necessarily be taxed, and should be subsidized when the government cares strongly about future generations. Judiciously chosen assumptions permit us to make these points in a fully analytically tractable and transparent manner. The next step would be to investigate numerically, whether in richer life cycle models with idiosyncratic income risk and thus heterogeneity in income and wealth *within generations* the optimal Ramsey tax policy is well approximated by the simple linear, time-invariant tax on capital that we have shown theoretically to be optimal.

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Online Appendix: Not for Publication

A Derivation of the Current Generations $GE(s)$ Effects

From equations (13a) and (13b) we find that

$$
\begin{aligned}
w'(s) &= (1-\alpha)\alpha\left[k'(s)\right]^{\alpha-1}\frac{dk'(s)}{ds} = (1-\alpha)\alpha\left[(1-\kappa)(1-\alpha)k^{\alpha}\right]^{\alpha}[s]^{\alpha-1} \\
R'(s) &= \alpha(\alpha-1)\left[k'(s)\right]^{\alpha-2}\frac{dk'(s)}{ds} = \alpha(\alpha-1)\left[(1-\kappa)(1-\alpha)k^{\alpha}\right]^{\alpha-1}[s]^{\alpha-2}
\end{aligned}
$$

and thus

$$
\kappa\eta w'(s) + (1-\kappa)(1-\alpha)k^{\alpha}R'(s)s = (1-\alpha)\alpha\left[(1-\kappa)(1-\alpha)k^{\alpha}\right]^{\alpha}[s]^{\alpha-1}\left[\kappa\eta-1\right].
$$

For a general period utility function $u(.)$ the general equilibrium effect reads as

$$
GE(s) = (1-\alpha)\alpha\left[(1-\kappa)(1-\alpha)k^{\alpha}\right]^{\alpha}[s]^{\alpha-1}\beta\int u'\left(c^{o}(\eta)\right)\left[\kappa\eta-1\right]d\Psi(\eta). \tag{34}
$$

If the utility function is logarithmic, equation (34) specializes, after substitution for $c^{o}(\eta)$ from the budget constraint, to equation $(18b)$ in the main text. Note that

$$
\begin{aligned}
\int u'\left(c^{o}(\eta)\right)\left[\kappa\eta-1\right]d\Psi(\eta) &= (\kappa-1)\int u'(c^{o}(\eta))d\Psi(\eta) + Cov\left[u'(c^{o}(\eta)),(\kappa\eta-1)\right] \\
&< (\kappa-1)\int u'(c^{o}(\eta))d\Psi(\eta) < 0.
\end{aligned}
$$

Thus, the general equilibrium effect is unambiguously negative as asserted in the main text.

B Derivation of Optimal Saving Rate for Log-Utility

B.1 Sequential Formulation

In this section we provide a full solution to the Ramsey optimal taxation problem for the case of logarithmic utility in its sequential formulation, for an arbitrary set of social welfare weights. We first recognize from the aggregate law of motion that

$$
\begin{aligned}
\ln(k_{t+1}) &= \ln(1-\alpha)+\ln(1-\kappa)+\alpha\ln(k_t)+\ln(s_t)\\
&= \varkappa+\sum_{i=0}^{t}\alpha^i\ln(s_{t-i})+\alpha^{t+1}\ln(k_0)\\
&= \varkappa_{t+1}+\sum_{i=0}^{t}\alpha^i\ln(s_{t-i}),
\end{aligned}
$$

where $\varkappa_{t+1}=\ln(1-\alpha)+\ln(1-\kappa)+\alpha^{t+1}\ln(k_0)$. Therefore the objective of the Ramsey government is given by (suppressing maximization-irrelevant constants)

$$
\begin{aligned}
\sum_{t=0}^{\infty}\omega_t V(k_t,s_t) &= \sum_{t=0}^{\infty}\omega_t\left[\ln(1-s_t)+\alpha\beta\ln(s_t)+\alpha\left(1+\alpha\beta\right)\ln(k_t)\right]\\
&= \chi+\sum_{t=0}^{\infty}\omega_t\left[\ln(1-s_t)+\alpha\beta\ln(s_t)+\alpha\left(1+\alpha\beta\right)\sum_{i=1}^{\infty}\alpha^{i-1}\ln(s_{t-i})\right]\\
&= \chi+\sum_{t=0}^{\infty}\left[\omega_t\ln(1-s_t)+\ln(s_t)\left(\alpha\beta\omega_t+\alpha\left(1+\alpha\beta\right)\sum_{i=t+1}^{\infty}\omega_i\alpha^{i-(t+1)}\right)\right]
\end{aligned}
$$

and thus the social welfare function can be expressed purely in terms of saving rates as

$$
W(\{s_t\}_{t=0}^{\infty})=\chi+\sum_{t=0}^{\infty}\omega_t\left[\ln(1-s_t)+\ln(s_t)\left(\alpha\beta+\alpha\left(1+\alpha\beta\right)\sum_{j=1}^{\infty}\frac{\omega_{t+j}}{\omega_t}\alpha^{j-1}\right)\right],
$$

where χ is a constant that depends positively on the initial capital stock k_0, but is again irrelevant for maximization. Taking first order conditions with respect to s_t and setting it to zero delivers the optimal saving rate in the main text:

$$
s_t=\frac{1}{1+\left(\alpha\beta+\alpha\left(1+\alpha\beta\right)\sum_{j=1}^{\infty}\frac{\omega_{t+j}}{\omega_t}\alpha^{j-1}\right)^{-1}}.
$$

B.2 Recursive Formulation

To obtain the closed form solution of the recursive version of the problem for $\frac{\omega_{t+1}}{\omega_t}=\theta$ by the method of undetermined coefficients guess that the value function takes the following log-linear form:

$$
W(k)=\Theta_0+\Theta_1\ln(k).
$$

Using this guess and equations $(13a)$-$(13c)$ (and writing $k_{t+1}(s_t)$ recursively as $k'(s)$) rewrite the Bellman equation (16) as:

$$
\begin{aligned}
W(k) &= \Theta_0 + \Theta_1 \ln(k) \qquad (35) \\
&= \max_{s\in[0,1]} \{\ln((1-s)(1-\kappa)(1-\alpha)\,k^{\alpha}) \\
&\quad +\beta \int \ln(\kappa\eta w(s) + R(s)k'(s))\, d\Psi(\eta) + \theta W(k')\Big\} \\
&= \max_{s\in[0,1]} \{\ln((1-s)(1-\kappa)(1-\alpha)\,k^{\alpha}) \\
&\quad +\beta \int \ln([\kappa(1-\alpha)\eta + \alpha][s(1-\kappa)(1-\alpha)k^{\alpha}]^{\alpha})\, d\Psi(\eta) + \theta W(k')\Big\} \\
&= (1+\alpha\beta)\ln((1-\kappa)(1-\alpha)) + \beta \int \ln(\kappa\eta(1-\alpha)+\alpha)\, d\Psi(\eta) \\
&\quad +\theta\Theta_0 + \theta\Theta_1 \ln[(1-\kappa)(1-\alpha)] + \left[\alpha + \alpha^2\beta + \alpha\theta\Theta_1\right]\ln(k) \\
&\quad + \max_{s\in[0,1]} \{\ln(1-s) + (\alpha\beta + \theta\Theta_1)\ln(s)\}\,.
\end{aligned}
$$

For the Bellman equation to hold, the coefficient Θ_1 has to satisfy

$$\Theta_1 = \frac{\alpha(1+\alpha\beta)}{(1-\alpha\theta)}.$$

We also immediately recognize that the optimal saving rate chosen by the Ramsey planner is independent of the capital stock k and determined by the first order condition

$$\frac{1}{1-s} = \frac{\alpha\beta + \theta\Theta_1}{s}$$

and thus

$$s^* = \frac{\alpha\beta + \theta\Theta_1}{1+\alpha\beta+\theta\Theta_1} = \frac{\alpha(\beta+\theta)}{1+\alpha\beta} \qquad (36)$$

as given by equation (20) in the main text. Plugging in s^* and Θ_1 into the Bellman equation (35) yields a linear equation in the constant Θ_0 whose solution completes the full analytical characterization of the Ramsey optimal taxation problem.

C Details of the Empirical Analysis

C.1 Sample Selection

We use data from the Panel Study of Income Dynamics (PSID), which interviews households in the United States annually from 1968 to 1997 and every other year since then.[35] The representative core sample consists of about 2,000 households in each wave, and we use data from 1977–2012.[36] Household pre-government income is defined as labor income before taxes, which we calculate as the sum of head and spouse annual labor income. We impute taxes using Taxsim, and add 50% of the estimated payroll taxes to the sum of head and spouse labor incomes to obtain pre-government income. We deflate all nominal values with the annual CPI, and select households if the household head is between 25 and 84 years of age. The minimum of household pre-government income needs to be above a constant threshold, which is defined as the income from working 520 hours at half the minimum wage.

Labor Income Share. We take our pre-government income measure to compute the ratio of labor income to total income (defined as the sum of labor income and capital income) for each household in the sample and take the average. This gives 0.792, suggesting that $\alpha \approx 0.208$.

Estimate of κ. In our model, young workers have average productivity $1-\kappa$, and old workers have average productivity κ. Thus the ratio of average earnings of old to young workers is $d = \frac{\kappa}{1-\kappa}$ and thus $\kappa = \frac{d}{1+d}$. We define young workers as workers in the age range 25 to 54 and old workers as workers of age 55 to 84. As ratio of their earnings we obtain $d = 0.453$ and thus $\kappa \approx 0.312$.

Lower bound support of η. Based on our income measure we compute the ratio of the lowest income in our sample of old workers of age 55 to 84 to the median income in that group giving 3.35%.

Residual Income Variance We run a panel regression, with log income as dependent variable and time dummies, a cubic in age, a control for the number of adult household

[35] We thank Chris Busch for helping us with the data.

[36] We do not use earlier waves because of poor coverage of income transfers before the 1977 wave.

members and an additional cubic in years of education for college workers as independent variables giving a variance of 0.648.

D Overaccumulation of Capital in the Competitive Equilibrium and Positive Capital Taxation

In this section we provide the details of the relation between the solution to the Ramsey problem in the steady state and the overaccumulation of capital (a capital stock above the golden rule capital stock) in the steady state equilibrium absent government policy, including the proof of Proposition 4 in the main text.

D.1 Definitions

First, and as usual, define the golden rule capital stock as the capital stock that maximizes aggregate (per capita) steady state consumption $C = k^{\alpha} - k$. Thus, the golden rule capital stock, saving rate and associated gross real interest rate are given by:

$$\begin{aligned} k^{GR} &= \alpha^{\frac{1}{1-\alpha}} \\ s^{GR} &= \frac{\alpha}{(1-\kappa)(1-\alpha)} \\ R^{GR} &= 1. \end{aligned}$$

A capital stock and associated saving rate is inefficiently high if it is larger than the golden rule level, and thus the associated gross real interest rate is less than 1. In this case aggregate consumption can be increased by lowering the capital stock in this case.

Now let us turn to the steady state of a competitive equilibrium. In any such steady state, the gross real interest rate is related to the steady state capital stock k through

$$R = \alpha k^{\alpha-1}.$$

From the law of motion of capital (equation (8b)) in a steady state

$$k = s(1-\kappa)(1-\alpha)k^{\alpha}$$

the steady state equilibrium interest rate R is related to the saving rate s by

$$R = \frac{\alpha}{s(1-\kappa)(1-\alpha)}.$$

The steady state equilibrium saving rate s itself is given by (see equation (11))

$$s = \frac{1}{1 + [(1-\tau)\alpha\beta\Gamma]^{-1}} = \frac{(1-\tau)\alpha\beta\Gamma}{1 + (1-\tau)\alpha\beta\Gamma}$$

which leads to a steady state relation between the real interest rate and the tax rate:

$$R = \frac{\frac{1}{(1-\tau)\beta\Gamma} + \alpha}{(1-\kappa)(1-\alpha)} = R(\tau;\Gamma).$$

A higher tax rate τ reduces the saving rate, the capital stock and increases the real interest rate. Holding τ constant the steady state interest rate is decreasing in the amount of income risk measured by Γ. The steady state interest rate in the absence of government policy $(\tau = 0)$ is given by

$$R(\tau = 0;\Gamma) = \frac{\frac{1}{\beta\Gamma} + \alpha}{(1-\kappa)(1-\alpha)}.$$

D.2 Overaccumulation of Capital in the Competitive Equilibrium

Recall that $\bar{\Gamma} = \frac{1}{\kappa(1-\alpha)+\alpha}$. The steady state competitive equilibrium in the absence of taxes has overaccumulated capital (a capital stock above the golden rule and $R(\tau = 0;\Gamma) < 1$) if and only if

$$\begin{aligned} \frac{\frac{1}{\beta\Gamma} + \alpha}{(1-\kappa)(1-\alpha)} &< 1 \\ \Gamma^{\text{eff}} := \frac{1}{[1-\alpha-1/\bar{\Gamma}]\beta} &< \Gamma \end{aligned} \tag{37}$$

The constant Γ^{eff} gives the first bound used in Proposition 4.

The optimal Ramsey steady state (i.e., $\theta = 1$) tax rate (see equation (21)) is given by

$$1 - \tau = \frac{1+\beta}{(1-\alpha)\,\beta\Gamma}$$

and thus the optimal Ramsey tax rate is positive, $\tau > 0$, if and only if

$$\begin{aligned} \frac{1+\beta}{(1-\alpha)\,\beta\Gamma} &< 1 \\ \Gamma^{\tau=0} : \frac{1+\beta}{(1-\alpha)\,\beta} &< \Gamma. \end{aligned} \tag{38}$$

In the proposition we made the assumption that $\beta < \frac{1}{(1-\alpha)\bar{\Gamma}-1} = \frac{\kappa(1-\alpha)+\alpha}{(1-\kappa)(1-\alpha)-\alpha}$ to insure that all cases of the proposition can occur. Under this assumption $\Gamma^{\tau=0} < \Gamma^{\text{eff}}$ and the interval in the second part of the proposition is nonempty (the equilibrium capital stock can be below the golden rule yet capital is taxed at a positive rate) since

$$\begin{aligned} \Gamma^{\tau=0} := \frac{1+\beta}{(1-\alpha)\,\beta} &< \frac{1}{[1-\alpha-1/\bar{\Gamma}]\beta} := \Gamma^{\text{eff}} \\ \frac{1+\beta}{1-\alpha} &< \frac{1}{1-\alpha-1/\bar{\Gamma}} \\ 1+\beta &< \frac{1-\alpha}{(1-\kappa)(1-\alpha)-\alpha} \\ \beta &< \frac{\kappa(1-\alpha)+\alpha}{(1-\kappa)(1-\alpha)-\alpha}, \end{aligned}$$

which holds on account of the assumption made in the proposition. Thus if the steady state competitive equilibrium capital stock is above the golden rule the optimal tax on capital is positive, but the opposite is not necessarily true. If there is no risk, however, then $\Gamma = \bar{\Gamma}$ and conditions (37) and (38) coincide:

$$\begin{aligned} \frac{1}{[1-\alpha-1/\bar{\Gamma}]\beta} &< \bar{\Gamma} \\ \frac{1}{\beta}+1 &< (1-\alpha)\bar{\Gamma} \\ \frac{1+\beta}{(1-\alpha)\,\beta} &< \bar{\Gamma} \end{aligned}$$

This results in the following proposition, referenced in the main text:

Proposition 10. *Let $\theta = 1$ and the assumption in Proposition 4 be satisfied. If the steady state competitive equilibrium capital stock is larger than the golden rule, the optimal Ramsey tax rate τ is positive. If η is degenerate at $\eta = 1$, then the reverse is true as well: $\tau > 0$ only if the steady state competitive equilibrium capital stock is larger than the golden rule.*

It remains to show the ranking of the savings rates in the different parts of Proposition 4. Recall that the savings rates are defined as

$$s^{CE} = \frac{1}{1+[\alpha\beta\Gamma]^{-1}} \tag{39a}$$

$$s^{*} = \frac{\alpha(1+\beta)}{1+\alpha\beta} \tag{39b}$$

$$s^{GR} = \frac{\alpha}{(1-\kappa)(1-\alpha)}. \tag{39c}$$

It then follows directly from the definition of Γ^{eff} that $s^{CE} < s^{GR}$ if and only if $\Gamma < \Gamma^{\text{eff}}$, and it follows directly from the definition of $\Gamma^{\tau=0}$ that $s^{CE} < s^{*}$ if and only if $\Gamma < \Gamma^{\tau=0}$. Finally, the condition on β in the proposition implies that $s^{*} < s^{GR}$.

E Characterization of Efficient Allocations

E.1 Characterization of Pareto Efficient Allocations

In this section we derive the solution to the unconstrained social planner problem and study whether the Ramsey government implements Pareto efficient allocations. The obvious answer is no, since an unconstrained social planner would provide full insurance against idiosyncratic η shocks, which, given the market structure, is ruled out in any competitive equilibrium. More interesting is the question how the saving rate chosen by the unconstrained planner compares to that selected by a constrained planner and the Ramsey government. The planner maximizes social welfare

$$\omega_{-1}\beta\int \ln(c_0^o(\eta_0))d\Psi(\eta_0) + \sum_{t=0}^{\infty}\omega_t\left[\ln(c_t^y) + \beta\int \ln(c_{t+1}^o(\eta_{t+1}))d\Psi(\eta_{t+1})\right]$$

subject just to the sequence of resource constraints

$$c_t^y + \int c_t^o(\eta_t)d\Psi(\eta_t) + k_{t+1} = k_t^{\alpha}.$$

We again restrict attention to geometrically declining welfare weights: $\omega_{t+1}/\omega_t = \theta \leq 1$. Trivially, the social planner provides full insurance against idiosyncratic income risk so that $c_t^o(\eta) = c_t^o$ for all η and all t. Thus the problem simplifies to

$$\max_{\{c_t^y, c_t^o, k_{t+1}\}} \omega_{-1} \beta \ln(c_0^o) + \sum_{t=0}^{\infty} \omega_t \left[\ln(c_t^y) + \beta \ln(c_{t+1}^o) \right] \qquad s.t.$$
$$c_t^y + c_t^o + k_{t+1} = k_t^\alpha$$

with $k_0 > 0$ given. The first order conditions are given by

$$\begin{aligned} \frac{\omega_t}{c_t^y} &= \lambda_t \\ \frac{\beta \omega_{t-1}}{c_t^o} &= \lambda_t \\ \lambda_t &= \lambda_{t+1} \alpha k_{t+1}^{\alpha-1} \\ c_t^y + c_t^o + k_{t+1} &= k_t^\alpha. \end{aligned}$$

The optimal allocation of consumption across two generations at a given time t is then given by

$$\frac{c_t^o}{c_t^y} = \frac{\beta \omega_{t-1}}{\omega_t}$$

and over time for a given generation it is characterized by

$$\frac{c_{t+1}^o}{c_t^y} = \beta \alpha k_{t+1}^{\alpha-1}.$$

In contrast to the Ramsey problem, consumption of the old in the first period is no longer irrelevant for maximization because the social planner can redistribute resources intergenerationally whereas the Ramsey planner, given the assumed restriction on instruments cannot. Thus, we characterize optimal allocations in period 0 and in an arbitrary period $t > 0$ separately.

Periods $t > 0$. Since we have assumed that $\frac{\omega_{t+1}}{\omega_t} = \theta$ we obtain

$$\frac{c_t^o}{c_t^y} = \frac{\beta \omega_{t-1}}{\omega_t} = \frac{\beta}{\theta}$$

and thus from the resource constraint we get

$$\begin{aligned} c_t^y &= \frac{\theta}{\theta+\beta}\left(k_t^\alpha - k_{t+1}\right) \\ c_t^o &= \frac{\beta}{\theta+\beta}\left(k_t^\alpha - k_{t+1}\right). \end{aligned}$$

Define, similarly to the Ramsey problem, the saving rate of the social planner as

$$s_t = \frac{k_{t+1}}{(1-\kappa)(1-\alpha)k_t^\alpha}.$$

Then from the first order conditions we obtain

$$\begin{aligned} \frac{1}{c_t^y} &= \frac{\beta}{c_{t+1}^o}\alpha k_{t+1}^{\alpha-1} \\ \frac{k_{t+1}}{\left(k_t^\alpha - k_{t+1}\right)} &= \frac{\alpha\theta k_{t+1}^\alpha}{\left(k_{t+1}^\alpha - k_{t+2}\right)} \\ \left(1-(1-\kappa)(1-\alpha)s_{t+1}\right) &= \alpha\theta\left(\frac{1}{(1-\kappa)(1-\alpha)s_t} - 1\right). \end{aligned}$$

As in the neoclassical growth model we can show that the only solution to this first order difference equation that does not eventually violate the non-negativity constraint of consumption and does not violate the transversality condition of the social planner is a constant saving rate s solving

$$\left(1-(1-\kappa)(1-\alpha)s\right) = \alpha\theta\left(\frac{1}{(1-\kappa)(1-\alpha)s} - 1\right).$$

Define $\tilde{s} = (1-\kappa)(1-\alpha)s$ then we have

$$1-\tilde{s} = \alpha\theta\left(\frac{1}{\tilde{s}} - 1\right)$$

with solutions $\tilde{s} = 1$ (and thus $s > 1$) and $\tilde{s} = \alpha\theta$. Therefore the constant saving rate that solves the social planner problem from period $t = 1$ onward is given by:

$$s^{SP} = \frac{\alpha\theta}{(1-\kappa)(1-\alpha)}.$$

The optimal sequence of capital stocks, starting from k_0, is therefore given by

$$
\begin{aligned}
k_{t+1} &= (1-\kappa)(1-\alpha)s_t k_t^{\alpha} \\
&= \alpha\theta k_t^{\alpha}.
\end{aligned}
$$

Period 0. Let us next characterize the allocation in period $t = 0$. We get

$$
\frac{c_0^o}{c_0^y} = \frac{\beta\omega_{-1}}{\omega_0}
$$

and thus only the ratio of the first two welfare weights matters. Therefore we can, without loss of generality, normalize $\omega_0 = 1$ so that

$$
\frac{c_0^o}{c_0^y} = \beta\omega_{-1}.
$$

Using this in the resource constraint one obtains

$$
\begin{aligned}
c_0^y &= \frac{1}{1+\beta\omega_{-1}}\left(k_0^{\alpha} - k_1\right) \\
c_0^o &= \frac{\beta\omega_{-1}}{1+\beta\omega_{-1}}\left(k_0^{\alpha} - k_1\right)
\end{aligned}
$$

$$
k_1 = s_0(1-\kappa)(1-\alpha)k_0^{\alpha}.
$$

Then from the first order conditions we get

$$
\begin{aligned}
\frac{1}{c_0^y} &= \frac{\beta}{c_1^o}\alpha k_1^{\alpha-1} \\
\frac{k_1\left(1+\beta\omega_{-1}\right)}{\left(k_0^{\alpha}-k_1\right)} &= \frac{\alpha\left(\theta+\beta\right)k_1^{\alpha}}{\left(k_1^{\alpha}-k_2\right)} \\
\frac{s_0(1-\kappa)(1-\alpha)\left(1+\beta\omega_{-1}\right)}{\left(1-s_0(1-\kappa)(1-\alpha)\right)} &= \frac{\alpha\left(\theta+\beta\right)}{\left(1-\alpha\theta\right)}
\end{aligned}
$$

and thus

$$
\begin{aligned}
s_0^{SP} &= \frac{\alpha\left(\theta+\beta\right)}{(1-\kappa)(1-\alpha)\left[\left(1+\beta\omega_{-1}\right)\left(1-\alpha\theta\right)+\alpha\left(\theta+\beta\right)\right]} \\
&= \frac{\alpha\theta}{(1-\kappa)(1-\alpha)\left[\frac{\left(\theta+\beta\theta\omega_{-1}\right)}{\theta+\beta}\left(1-\alpha\theta\right)+\alpha\theta\right]}.
\end{aligned}
\tag{40}
$$

Now, suppose that $\omega_{-1} = 1/\theta$. Then (40) simplifies to

$$s_0^{SP} = s^{SP} = \frac{\alpha\theta}{(1-\kappa)(1-\alpha)}. \tag{41}$$

We summarize these results in the following

Proposition 11. *The solution to the social planner problem, for any $k_0 > 0$, is given by*

$$s_0^{SP} = \frac{\alpha\theta}{(1-\kappa)(1-\alpha)\left[\frac{(\theta+\beta\theta\omega_{-1})}{\theta+\beta}(1-\alpha\theta)+\alpha\theta\right]}$$

and associated capital stock in period 1

$$k_1 = s_0^{SP}(1-\kappa)(1-\alpha)k_0^{\alpha}.$$

and consumption allocations in period 0

$$c_0^y = \frac{1}{1+\beta\omega_{-1}}\left(1 - s_0^{SP}(1-\kappa)(1-\alpha)\right)k_0^{\alpha}$$
$$c_0^o = \frac{\beta\omega_{-1}}{1+\beta\omega_{-1}}\left(1 - s_0^{SP}(1-\kappa)(1-\alpha)\right)k_0^{\alpha}$$

and in all periods $t > 0$ by a constant saving rate

$$s^{SP} = \frac{k_{t+1}}{(1-\kappa)(1-\alpha)k_t^{\alpha}} = \frac{\alpha\theta}{(1-\kappa)(1-\alpha)}$$

and associated sequence of capital stocks

$$k_{t+1} = \alpha\theta k_t^{\alpha} \tag{42}$$

and consumption levels

$$c_t^y = \frac{\theta(1-\alpha\theta)k_t^{\alpha}}{\theta+\beta} \tag{43a}$$
$$c_t^o = \frac{\beta(1-\alpha\theta)k_t^{\alpha}}{\theta+\beta}. \tag{43b}$$

If, in addition $\omega_{-1} = \frac{1}{\theta}$ then $s_0^{SP} = s^{SP}$ and equations (42) and (43) apply for all periods $t \geq 0$.

Also notice that for $\theta = \omega_{-1} = 1$, i.e., for a planner that maximizes steady state utility and also weighs the initial generation equally, then the optimal saving rates in all $t \geq 0$ are

$$s_0^{SP} = s^{SP} = \frac{\alpha}{(1-\kappa)(1-\alpha)} = s^{GR}.$$

We summarize these insights in the next

Corollary 5. *If $\theta = 1$ (associated with maximizing steady state utility), then the social planner chooses the golden rule saving rate*

$$s^{SP} = s^{GR} = \frac{\alpha}{(1-\kappa)(1-\alpha)}$$

in all $t > 0$ and the capital stock converges, in the long run, to its golden rule level

$$k^{GR} = \alpha^{\frac{1}{1-\alpha}}$$

which satisfies

$$\alpha \left[k^{GR}\right]^{\alpha-1} = 1$$

and associated consumption levels

$$\begin{aligned} c^y &= \frac{(1-\alpha)}{1+\beta}\alpha^{\frac{\alpha}{1-\alpha}} \\ c_t^o &= \frac{\beta(1-\alpha)}{1+\beta}\alpha^{\frac{\alpha}{1-\alpha}} \end{aligned}$$

Therefore, the social planner chooses the golden rule capital stock k^{GR} maximizing net output $y^{GR} = \left(k^{GR}\right)^{\alpha} - k^{GR}$ and splits it efficiently between c^y and c^o according to the rule $c^o = \beta c^y$. If, in addition, $\omega_{-1} = 1$ then also

$$s_0^{SP} = s^{SP} = \frac{\alpha}{(1-\kappa)(1-\alpha)}.$$

Obviously, the Ramsey equilibrium is not Pareto efficient because it does not provide full consumption insurance against idiosyncratic income risk. What is more remarkable is that even though the optimal Ramsey saving rate is independent of income risk (and the same as in a model where income risk is absent), it is in general different from the saving rate optimally chosen by the social planner (who fully insures the idiosyncratic income risk). This result is summarized in the next

Corollary 6. *For a fixed social discount factor $\theta \in [0,1]$, the optimal Ramsey saving rate equals the saving rate chosen by the social planner if and only if the following knife edge condition is satisfied:*

$$(1-\kappa) = \frac{\theta(1+\alpha\beta)}{(1-\alpha)(\beta+\theta)}$$

Note that the Ramsey government *can* implement the saving rate desired by the social planner through an appropriate choice of taxes, but unless the condition above is satisfied, it is suboptimal to do so. The reason is that the Ramsey government has no instruments to transfer resources across generations and thus forcing the planner saving rate onto households (by appropriate choice of the capital tax rate) results in an equilibrium allocation of consumption across the young and the old that is typically suboptimal.[37]

E.2 Proof of Constrained Efficiency of Ramsey Allocation

Proof. Define the saving rate of the constrained planner as

$$s_t = \frac{k_{t+1}}{(1-\kappa)MPL(k_t)} = \frac{k_{t+1}}{(1-\alpha)(1-\kappa)k_t^{\alpha}}.$$

Thus, the law of motion for the effective capital stock for the constrained planner is

$$k_{t+1} = s_t(1-\alpha)(1-\kappa)k_t^{\alpha}$$

as in the Ramsey problem. Furthermore, from the constraints on the constrained planner

$$\begin{aligned}
c_t^y &= (1-\kappa)MPL(k_t) - k_{t+1} = (1-s_t)(1-\kappa)(1-\alpha)k_t^{\alpha} \\
c_{t+1}^o(\eta_{t+1}) &= k_{t+1}MPK(k_{t+1}) + \kappa\eta_{t+1}MPL(k_{t+1}) \\
&= \alpha k_{t+1}^{\alpha} + \kappa\eta_{t+1}(1-\alpha)k_{t+1}^{\alpha} \\
&= \left[\alpha + \kappa\eta_{t+1}(1-\alpha)\right]k_{t+1}^{\alpha}.
\end{aligned}$$

Thus the consumption allocation is the same as in the Ramsey equilibrium and the solution, in terms of saving rates, of the constrained planner problem is the same as the Ramsey equilibrium. □

[37] Finally note that if one were to treat the social discount factor θ as a *free parameter*, then one concludes that the Ramsey optimal saving rate is efficient, in that it is identical to the choice of the social planner with a *different social discount rate* $\theta^{SP} = \frac{(\beta+\theta)(1-\kappa)(1-\alpha)}{1+\alpha\beta}$.

E.3 Proof of Pareto-Improving Tax-Induced Transition

E.3.1 Log Utility

Proof of Proposition 6. The capital stock evolves according to

$$k_{t+1} = s_t(1-\kappa)(1-\alpha)k_t^\alpha.$$

Therefore if the Ramsey government implements s^* through positive capital taxes in the first period of the transition this will lead to a falling capital stock along the transition. Recall from (1) that utility of a generation born in period t is given by

$$V_t = \ln(c_t^y) + \beta \int \ln(c_{t+1}^o(\eta_{t+1}))d\Psi.$$

Now, suppose that the policy is implemented (as a surprise) in period 1 where $k_1 = k_0$. The initial old are unaffected by this policy and thus indifferent to the tax reform. Now we need to characterize the utility consequences for all generations born along the transition. Denoting by $s_0 = s^{CE}$ the equilibrium saving rate in the initial steady state, we have

$$\Delta V_t = V_t(s^*) - V_t(s_0) = \ln(c_t^y(s^*)) - \ln(c_t^y(s_0)) + \beta \int \left(\ln(c_{t+1}^o(s^*)) - \ln(c_{t+1}^o(s_0))\right) d\Psi.$$

where the consumption allocations are

$$\begin{aligned} c_t^y(s_t) &= (1-s_t)(1-\kappa)(1-\alpha)k_t^\alpha \\ c_{t+1}^o(\eta_{t+1}; s_t) &= s_t(1-\kappa)(1-\alpha)k_t^\alpha \alpha k_{t+1}^{\alpha-1} + \kappa\eta_{t+1}(1-\alpha)k_{t+1}^\alpha \\ &= \left[\alpha + \kappa\eta_{t+1}(1-\alpha)\right] k_{t+1}^\alpha. \end{aligned}$$

Thus

$$\Delta V_t = \underbrace{\ln\left[((1-s^*)k_t^\alpha)\right] - \ln\left[((1-s_0)k_0^\alpha)\right]}_{=\Delta V_t^+} + \alpha\beta\Gamma_2 \left(\underbrace{\ln\left[k_{t+1}\right] - \ln\left[k_0\right]}_{=\Delta V_t^-}\right)$$

Since the capital stock is monotonically decreasing along the transition, $\Delta V_t^- < 0$ for all $t > 0$ and $\Delta V_s^- < \Delta V_t^- < 0$ for all $s > t > 0$, and we call ΔV_t^- the "loss" term. From the monotonically decreasing capital stock it also follows that ΔV_t^+ is monotonically decreasing along the transition. Since in the limit we have $\lim_{t\to\infty} \Delta V_t > 0$ (because s^*

maximizes steady state utility), it follows that $\Delta V_t^+ > 0$ for all $t > 0$ and we therefore refer to ΔV_t^+ as the "gains" term. Finally, since gains are monotonically decreasing and losses—the absolute value $\left|V_t^-\right|$—are monotonically increasing we achieve the smallest gains and largest losses for $t \to \infty$ and since $\lim_{t\to\infty} \Delta V_t > 0$, it follows that $\Delta V_t > 0$ in all $t > 0$. $\square$

E.3.2 Generalization

The previous results generalize to additively separable life-time utility functions of the form

$$V_t = u(c_t^y) + g(c_{t+1}^o, \Psi) \tag{44}$$

with $u' > 0, u'' < 0$ for all $c_t^y > 0$ and $g' > 0, g'' < 0$ for all $c_{t+1}^o > 0$. Aggregating second period consumption with function $g(\cdot)$ nests standard (discounted) expected utility formulations as well as non-expected utility preferences such as Epstein-Zin-Weil preferences, analyzed in Section 6.3.6. As before, write consumption allocations in terms of the saving rate s as $(c_t^y(s), c_{t+1}^o(\eta, s))$. As shorthand, below we denote as $u_s = u'(c_t^y(s)) \times c_t^y(s)'$, with g_s defined correspondingly. Given this notation the first-order condition of the Ramsey problem for $\theta = 1$ is

$$\frac{\partial V_\infty}{\partial s} = u_s + g_s = 0 \qquad \Leftrightarrow \qquad -u_s = g_s. \tag{45}$$

We make the following additional

Assumption 3.

$$\lim_{s\to 1} -u_s > \lim_{s\to 1} g_s \tag{46}$$

and, for all $s \in (\alpha, 1)$,

$$\varepsilon_{u',c} = -\frac{u''(c_t^y(s))}{u'(c_t^y(s))} < \frac{c_t^y(s)''}{c_t^y(s)'} = \varepsilon_{c_s,s}, \tag{47}$$

where $\varepsilon_{u',c}$ is the semi-elasticity of marginal utility[38] *with respect to consumption c^y and $\varepsilon_{c_s,s}$ is the semi-elasticity of consumption c^y with respect to the saving rate s.*

[38] In in a static stochastic environment this would be equal to the measure of absolute risk aversion. We prefer the term semi-elasticity of marginal utility because first period consumption is not stochastic.

The next proposition generalizes Proposition 6 to additively separable utility functions with the above properties. It also provides conditions for existence and uniqueness of a solution to (45):

Proposition 12. *Let the utility function be given by (44). Under assumption 3 the solution to (45) gives a unique $s^* \in (\alpha, 1)$. Further assume that $s^{CE} > s^*$. Then implementing s^* in period $t = 0$ for all $t \geq 0$ leads to a Pareto improving transition.*

Before proving the above proposition, note that condition (46) is required for existence, and condition (47) for uniqueness of $s^* \in (\alpha, 1)$. We further show that condition (47) implies that $\frac{\partial V_\infty}{\partial s} < 0$ for $s^{CE} > s^*$ so that the generation born in the limit of the transition when the economy approaches the new steady state benefits from implementing $s^* < s^{CE}$. We later establish for Epstein-Zin-Weil preferences, which nest CRRA preferences as a special case, that all these conditions are satisfied. Thus, we show analytically that the conditions apply quite generally. For the general class of HARA utility functions

$$u(c) = \frac{1-\gamma}{\gamma}\left(\frac{\iota \cdot c}{1-\gamma} + \xi\right)^{\gamma}$$

with parameters $\iota > 0, \xi, \gamma$, and the restriction $\frac{\iota \cdot c}{1-\gamma} + \xi > 0$ and $\gamma \neq 1$ (ruling out linear utility) condition (46) may fail to hold so that there is no solution to the Ramsey problem. For instance, with exponential utility condition (46) may fail to hold since there is no Inada condition as consumption approaches zero, so that $\lim_{s \to 1} -u_s < \infty$.[39]

As for the assumption that $s^{CE} > s^*$ notice that we earlier established that s^{CE} is increasing with risk if there is precautionary savings. Thus, with sufficient risk we have $s^{CE} > s^*$. Also, as for the second part of the proposition on the Pareto improving transition, the proof follows exactly the same logic as the proof of Proposition 6.

This proposition does not address whether the equilibrium has overaccumulated capital. As before, the interesting case is where $s^* < s^{CE} < s^{GR}$, where $s^{GR} = \frac{\alpha}{(1-\alpha)(1-\kappa)}$ is the golden rule saving rate. Finally, notice that the proposition is silent about implementation. We address implementation under the assumption of existence of a unique s^* in the subsequent Proposition 13 for expected utility and later in Proposition 29 for EZW utility.

[39] Consider nested exponential utility, i.e., $\gamma = -\infty$, and $\xi = 1$. Further parameterize $\iota = 1$, $\alpha = 0.33$, $\kappa = 0.7$ and $\eta = 1$, i.e., a degenerate deterministic case. Also assume an expected utility formulation with $\beta = 1$

$$g(c^o; \Psi) = \beta \int u(c^o(\eta)) d\Psi(\eta).$$

Then condition (46) fails to hold, an interior s^* does not exist and the optimal saving rate is $s^* = 1$.

Proof of Proposition 12. First, we establish that s^* is unique and that with uniqueness we get for $s^{CE} > s^*$ that $\frac{\partial V_\infty}{\partial s} < 0$. To show this, we analyze the first-order condition of the Ramsey government (45). The next steps will establish that (i) $g_s > 0$ is a continuous and downward sloping function in s, (ii) $-u_s > 0$ for $s > \alpha$, and (iii) that condition (47) is required for a single crossing of g_s and $-u_s$. Findings (i)-(ii) together with (46) establish existence, the additional item (iii) then insures uniqueness of s^*.

Now start from the allocation in the long-run steady state. Recall from Section E.3 above that consumption when young and old is

$$c_t^y = (1 - s_t)(1 - \kappa)(1 - \alpha)k_t^\alpha,$$
$$c_{t+1}^o = (\alpha + \kappa(1 - \alpha)\eta)\, k_{t+1}^\alpha,$$

where

$$k_{t+1} = s_t(1 - \kappa)(1 - \alpha)k_t^\alpha. \tag{48}$$

In steady state we thus have

$$k = (s(1 - \kappa)(1 - \alpha))^{\frac{1}{1-\alpha}}$$

and therefore steady state consumption allocations are

$$c^y = (1 - s)s^{\frac{\alpha}{1-\alpha}} ((1 - \alpha)(1 - \kappa))^{\frac{1}{1-\alpha}} \tag{49a}$$
$$c^o = (\alpha + \kappa(1 - \alpha)\eta) ((1 - \alpha)(1 - \kappa))^{\frac{\alpha}{1-\alpha}} s^{\frac{\alpha}{1-\alpha}}. \tag{49b}$$

Use this in the social welfare function with $\theta = 1$ to obtain

$$\begin{aligned} V_\infty &= u(c^y) + g(c^o; \Psi) \\ &= u\left((1 - s)s^{\frac{\alpha}{1-\alpha}} ((1 - \alpha)(1 - \kappa))^{\frac{1}{1-\alpha}}\right) + g((\alpha + \kappa(1 - \alpha)\eta) ((1 - \alpha)(1 - \kappa))^{\frac{\alpha}{1-\alpha}} s^{\frac{\alpha}{1-\alpha}}; \Psi). \end{aligned}$$

From the above we readily observe that $g_s > 0$ as well as $g_{ss} < 0$ because of decreasing marginal utility.[40] To establish existence of s^* observe that

$$u_s = u'(c^y(s)) \times c^y(s)' = u'\left((1-\alpha)(1-\kappa)\right)^{\frac{1}{1-\alpha}}\left(-1+\frac{\alpha}{1-\alpha}(1-s)s^{-1}\right)s^{\frac{\alpha}{1-\alpha}}$$
$$< 0 \qquad \Leftrightarrow \qquad c^y(s)' < 0 \qquad \Leftrightarrow \qquad s > \alpha.$$

because $u'(c^y(s)) > 0$ and thus $u_s < 0$ for $s > \alpha$. If, in addition, condition (46) holds, then there exists at least one solution $s^* \in (\alpha, 1)$. Also notice that condition (46) holds if u satisfies the Inada condition, because then $\lim_{s\to 1} -u_s = \infty$ and $\lim_{s\to 1} g_s < \infty$.

To establish uniqueness we further require that $u'' < 0$ for all $s \in (\alpha, 1)$ so that $-u_s$ is upward sloping and continuous. Observe that

$$u_{ss} = u''(c^y)c^y(s)' + u'(c^y)c^y(s)'' < 0 \qquad \Leftrightarrow \qquad \varepsilon_{u',c} = -\frac{u''(c^y)}{u'(c^y)} < \frac{c^y(s)''}{c^y(s)'} = \varepsilon_{c_s,s}$$

which limits the (positive) semi-elasticity of marginal utility $\varepsilon_{u',c}$ from above. For the semi-elasticity of consumption $\varepsilon_{c_s,s}$ notice that we have already established that $c^y(s)' < 0$ for $s \in (\alpha, 1)$. We next show that for $s \in (\alpha, 1)$ also $c^y(s)'' < 0$ so that $\varepsilon_{c_s,s} > 0$. To see this, write

$$c^y(s)'' = \left((1-\alpha)(1-\kappa)\right)^{\frac{1}{1-\alpha}}\frac{\alpha}{1-\alpha}s^{\frac{\alpha}{1-\alpha}-1}\left[-2+(1-s)\frac{2\alpha-1}{1-\alpha}s^{-1}\right]$$

and thus $c^y(s)'' < 0$ if

$$-2+(1-s)\frac{2\alpha-1}{1-\alpha}s^{-1} < 0 \qquad \Leftrightarrow \qquad s > 2\alpha - 1$$

Before, we have shown that for $s > \alpha$ we have $c^y(s)' < 0$ and since $\alpha > 2\alpha - 1 \Leftrightarrow \alpha < 1$ we know that $s > \alpha$ implies that $c^y(s)'' < 0$ and thus for $s \in (\alpha, 1)$ we get $\frac{c^y(s)''}{c^y(s)'} > 0$. Also, since by property (47) the function $-u_s$ is continuous and upward sloping and since g_s is downward sloping we have that if $s^* \in (\alpha, 1)$ exists, then $s^{CE} > s^*$ implies that $V'_\infty(s) < 0$.

Along the transition, recall that the consumption allocations for generation t is

$$c^y_t(s_t) = (1-s_t)(1-\kappa)(1-\alpha)k^\alpha_t$$
$$c^o_{t+1}(\eta_{t+1}; s_t) = \left[\alpha + \kappa\eta_{t+1}(1-\alpha)\right]k^\alpha_{t+1}.$$

[40] Specifically, we have assumed that $g_c > 0$, $g_{cc} < 0$. Observe from (49b) that $c^o(s)' > 0$ so that $g_s > 0$ and $g_{ss} < 0$.

Thus, assuming a unique $s^* < s^{CE}$ we obtain

$$\Delta V_t = \underbrace{u\left((1-s^*)(1-\kappa)(1-\alpha)k_t^\alpha\right) - u\left((1-s^{CE})(1-\kappa)(1-\alpha)k_0^\alpha\right)}_{=\Delta V_t^+} +$$
$$\underbrace{g\left(\left[\alpha+\kappa\eta_{t+1}(1-\alpha)\right]k_{t+1}^\alpha;\Psi\right) - g\left(\left[\alpha+\kappa\eta_{t+1}(1-\alpha)\right]k_0^\alpha;\Psi\right)}_{=\Delta V_t^-}$$

and since $\frac{\partial c_t^y}{\partial k_t} > 0$ as well as $\frac{\partial c_{t+1}^o}{\partial k_{t+1}} > 0$ the same arguments on the behavior of V_t^+ and V_t^- along the transition as in the proof of Proposition 6 apply. □

E.3.3 Implementation

Observe that the proof above does not say anything about implementation of the saving rates though taxation of capital. The next proposition contains a fairly general implementation result for expected utility. Proposition 29 extends this result to EZW utility.

Proposition 13. *If the utility function in both periods is of the HARA form,*

$$u(c) = \frac{1-\gamma}{\gamma}\left(\frac{\iota c}{1-\gamma}+\xi\right)^\gamma, \tag{50}$$

with parameters $\iota > 0, \xi, \gamma$, $\gamma \neq 1$ such that $\frac{\iota c}{1-\gamma}+\xi > 0$, then in general equilibrium the saving rate s is strictly decreasing in the tax rate τ and any $s^* \in (\alpha, 1]$ can be implemented by a unique (but typically time-dependent) tax rate τ_{t+1}^*.

Proof. Start from the Euler equation for a given period t aggregate wage $w_t = (1-\alpha)k_t^\alpha$

$$u'\left[(1-\kappa)w_t(1-s(\tau_{t+1}))\right] =$$
$$\alpha\beta(1-\tau_{t+1})\left((1-\kappa)w_t\right)^{\alpha-1}s(\tau_{t+1})^{\alpha-1}\int u'\left[(\alpha+(1-\alpha)\kappa\eta)\left[(1-\kappa)w_t\right]^\alpha s(\tau_{t+1})^\alpha\right]d\Psi(\eta). \tag{51}$$

Totally differentiate (51) to get

$$
\begin{aligned}
&-(1-\kappa)w_t u''\left[(1-\kappa)w_t(1-s(\tau_{t+1}))\right]\frac{ds(\tau_{t+1})}{d\tau_{t+1}} = \alpha\beta\left((1-\kappa)w_t\right)^{\alpha-1} \\
&\left[-s(\tau_{t+1})^{\alpha-1}+(1-\tau_{t+1})(\alpha-1)s(\tau_{t+1})^{\alpha-2}\frac{ds(\tau_{t+1})}{d\tau_{t+1}}\right]\int u'\left[(\alpha+(1-\alpha)\kappa\eta)\left[(1-\kappa)w_t\right]^{\alpha}s(\tau_{t+1})^{\alpha}\right]d\Psi(\eta) \\
&\qquad + \alpha^2\beta(1-\tau_{t+1})\left((1-\kappa)w_t\right)^{2\alpha-1}s(\tau_{t+1})^{2(\alpha-1)}\frac{ds(\tau_{t+1})}{d\tau_{t+1}}\cdot \\
&\qquad \cdot\int u''\left[(\alpha+(1-\alpha)\kappa\eta)\left[(1-\kappa)w_t\right]^{\alpha}s(\tau_{t+1})^{\alpha}\right](\alpha+(1-\alpha)\kappa\eta)d\Psi(\eta).
\end{aligned}
$$

Now use the notation

$$
\begin{aligned}
c^y(s(\tau_{t+1})) &= (1-\kappa)w_t(1-s(\tau_{t+1})) \\
c^o(s(\tau_{t+1}),\eta) &= (\alpha+(1-\alpha)\kappa\eta)\left[(1-\kappa)w_t\right]^{\alpha}s(\tau_{t+1})^{\alpha}
\end{aligned}
$$

and divide by $(1-\kappa)w_t$ to rewrite this further as

$$
\begin{aligned}
&-u''\left[c^y(s(\tau_{t+1}))\right]\frac{ds(\tau_{t+1})}{d\tau_{t+1}} = -\alpha\beta\left((1-\kappa)w_t\right)^{\alpha-2} \\
&\quad\left[s(\tau_{t+1})^{\alpha-1}+(1-\tau_{t+1})(1-\alpha)s(\tau_{t+1})^{\alpha-2}\frac{ds(\tau_{t+1})}{d\tau_{t+1}}\right]E\left[u'(c^o(s(\tau_{t+1}),\eta))\right] \\
&\qquad + \alpha^2\beta(1-\tau_{t+1})\left((1-\kappa)w_t\right)^{2(\alpha-1)}s(\tau_{t+1})^{2(\alpha-1)}\frac{ds(\tau_{t+1})}{d\tau_{t+1}}\cdot \\
&\qquad \cdot\int u''\left[(\alpha+(1-\alpha)\kappa\eta)\left[(1-\kappa)w_t\right]^{\alpha}s(\tau_{t+1})^{\alpha}\right](\alpha+(1-\alpha)\kappa\eta)d\Psi(\eta).
\end{aligned}
$$

Since $u'>0$ and $u''<0$ ambiguity of implementation may come from the expression

$$
\int u''\left[(\alpha+(1-\alpha)\kappa\eta)\left[(1-\kappa)w_t\right]^{\alpha}s(\tau_{t+1})^{\alpha}\right](\alpha+(1-\alpha)\kappa\eta)d\Psi(\eta). \tag{52}
$$

Before proceeding observe that without risk implementation is unambiguous since then

$$
u''\left[(\alpha+(1-\alpha)\kappa)\left[(1-\kappa)w_0\right]^{\alpha}s(\tau)^{\alpha}\right](\alpha+(1-\alpha)\kappa)<0.
$$

With income risk, observe that with HARA utility (50) we have

$$u' = \iota\left(\frac{\iota c}{1-\gamma}+\xi\right)^{\gamma-1}, \qquad u'' = -\iota^2\left(\frac{\iota c}{1-\gamma}+\xi\right)^{\gamma-2}$$

and thus (52) becomes

$$\begin{aligned}
&-\int \iota^2\left[\frac{\iota}{1-\gamma}(\alpha+(1-\alpha)\kappa\eta)\left[(1-\kappa)w_t\right]^{\alpha}s(\tau_{t+1})^{\alpha}+\xi\right]^{\gamma-2}(\alpha+(1-\alpha)\kappa\eta)d\Psi(\eta)\\
&=-\iota^2\int\left[\left(\frac{\iota}{1-\gamma}(\alpha+(1-\alpha)\kappa\eta)\left[(1-\kappa)w_t\right]^{\alpha}s(\tau_{t+1})^{\alpha}+\xi\right)(\alpha+(1-\alpha)\kappa\eta)^{\frac{1}{\gamma-2}}\right]^{\gamma-2}d\Psi(\eta)\\
&=-\iota^2\int\left[\left(\frac{\iota}{1-\gamma}(\alpha+(1-\alpha)\kappa\eta)^{\frac{\gamma-1}{\gamma-2}}\left[(1-\kappa)w_t\right]^{\alpha}s(\tau_{t+1})^{\alpha}+\xi(\alpha+(1-\alpha)\kappa\eta)^{\frac{1}{\gamma-2}}\right)\right]^{\gamma-2}d\Psi(\eta)\\
&=\Lambda(s(\tau_{t+1});\iota,\xi,\alpha,\kappa,\gamma,\eta)<0
\end{aligned}$$

and thus for HARA preferences defined in the proposition s_t and τ_{t+1} are strictly negatively related, implying that for any saving rate there exists a unique tax rate implementing this saving rate as a competitive equilibrium. □

E.3.4 Marginal Reforms

The next corollary studies marginal tax reforms rather than implementing the full Ramsey equilibrium.

Corollary 7. *Let Assumption 3 hold and assume that $s^{CE} > s^*$. Implementing a saving rate $s^{CE} - \epsilon \geq s^*$ for small $\epsilon > 0$ in all periods $t \geq 0$ through a time-varying tax rate τ_{t+1} yields a Pareto improvement.*

Proof. Replace in the proof of Proposition 12 s^* by $s^{CE} - \epsilon \geq s^*$ to note that the same arguments on monotone transitions of the gains and loss terms apply. □

E.4 Savings Subsidy Does Not Induce Pareto Improvement

In this section we show, in contrast to the previous section, that even if $s^{CE} < s^*$, implementing the Ramsey (for $\theta = 1$) saving rate s^* through a savings subsidy $\tau^* < 0$ does *not* lead to a Pareto improving transition. We exploit the fact that in the first period of the transition the capital stock $k_1 = k_0$ is predetermined, and the capital stock in $t = 2$ satisfies

$$k_2 = s(1-\alpha)(1-\kappa)k_0^\alpha$$

for any saving rate implemented by a given tax policy. Then we can calculate lifetime utility of the first transition generation, as a function of an implemented saving rate s, as

$$\begin{aligned}
V_1(s) &= \ln\left((1-s)(1-\kappa)(1-\alpha)k_0^\alpha\right) + \beta \int \ln\left(\alpha + \kappa\eta_2(1-\alpha)\right)\left(s(1-\alpha)(1-\kappa)k_0^\alpha\right)^\alpha d\Psi(\eta) \\
&= \ln(1-s) + \beta\alpha\ln(s) + \ln\left((1-\kappa)(1-\alpha)k_0^\alpha\right) \\
&+ \beta \int \ln\left(\alpha + \kappa\eta_2(1-\alpha)\right)\left((1-\alpha)(1-\kappa)k_0^\alpha\right)^\alpha d\Psi(\eta)
\end{aligned}$$

and thus

$$\begin{aligned}
V_1'(s) &= -\frac{1}{1-s} + \frac{\alpha\beta}{s} \\
V_1''(s) &= -\frac{1}{(1-s)^2} - \frac{\alpha\beta}{s^2} < 0.
\end{aligned}$$

Therefore $V_1(s)$ is strictly concave in s. Therefore, if $V_1'(s = s^{CE}) \leq 0$, then $V(s = s^{CE}) > V(s)$ for all $s > s^{CE}$. We have

$$\begin{aligned}
V_1'(s = s^{CE}) = -\frac{1}{1-s^{CE}} + \alpha\beta\frac{1}{s^{CE}} \leq 0 \\
\Leftrightarrow \qquad s^{CE} \geq \frac{\alpha\beta}{1+\alpha\beta}
\end{aligned}$$

which is satisfied, exploiting expression (11) for the optimal competitive equilibrium saving rate (with zero taxes). Thus not only is implementing $\tau^* < 0$ not Pareto improving if $s^{CE} < s^*$, but in fact any policy reform that induces a saving rate in period 1 above the competitive saving rate with zero taxes, s^{CE}, will not result in a Pareto improvement, since it makes the first generation strictly worse off.

F Endogenous Labor Supply and Labor Income Taxation

In this section we provide the details of the analysis of the model with endogenous labor supply. We first define competitive equilibrium for given fiscal policy in this version of the model, prior to characterizing first the competitive equilibrium for given policy, and then the solution to the optimal Ramsey problem.

F.1 Definition of Competitive Equilibrium

Definition 3. *For a given initial capital stock K_0 and a given sequence of tax rates $\{(\tau_t^l, \tau_t)\}$ a competitive equilibrium is a sequence of allocations for households, $\{(c_t^y, a_{t+1}, c_{t+1}^o(\eta), l_{t+1}^o(\eta))\}$, a sequence of allocations for firms, $\{(K_t, L_t)\}$, a sequence of factor prices $\{(R_t, w_t)\}$ and a sequence of transfers $\{T_t\}$ such that*

1. *for all $t \geq 0$, given $(\tau_{t+1}^l, \tau_t, T_t)$ and (w_t, w_{t+1}, R_{t+1}) the allocation $(c_t^y, a_{t+1}, c_{t+1}^o(\eta), l_{t+1}^o(\eta))$ maximizes (1) subject to $(25a)$ and $(25b)$;*

2. *given (τ_0^l, τ_0, T_0) and (w_0, R_0) the allocation $(c_0^o(\eta), l_0^o(\eta))$ maximizes*

$$V_{-1} = \int [\ln(c_0^o(\eta)) + \gamma(1 - l_0^o(\eta))] d\Psi(\eta) \tag{53}$$

   subject to

$$c_0^o(\eta) = (1 - \tau_0) R_0 a_0 + \kappa\eta(1 - \tau_0^l) w_0 l_0^o(\eta) + T_0; \tag{54}$$

3. *For all $t \geq 0$, factor prices satisfy*

$$R_t = \alpha K_t^{\alpha-1} L_t^{1-\alpha} \tag{55a}$$

$$w_t = (1 - \alpha) K_t^{\alpha} L_t^{-\alpha}; \tag{55b}$$

4. *for all $t \geq 0$, the government budget constraint is satisfied:*

$$T_t = \tau_t R_t K_t + \tau_t^l \kappa w_t \int l_t^o(\eta) \eta d\Psi(\eta); \tag{56}$$

5. *the markets for labor, capital and final goods clear in every period: for all $t \geq 0$:*

$$L_t = 1 - \kappa + \kappa \int l_t^o(\eta) \eta d\Psi(\eta) \tag{57}$$

$$K_{t+1} = a_{t+1} \tag{58}$$

$$F(K_t, L_t) = c_t^y + \int c_t^o(\eta) d\Psi(\eta) + K_{t+1}. \tag{59}$$

F.2 Analysis

F.2.1 Analysis of Equilibrium for Given Tax Policy

First, we characterize the competitive equilibrium for a given sequence of capital and labor income tax rates. We first state the household optimality conditions, and then show how to aggregate them, exploiting the market clearing conditions and the government budget constraint in general equilibrium.

Optimal Household Decisions For given factor prices and tax policies, the household makes a labor-leisure choice and a consumption savings choice. The next lemma characterizes this choice.

Lemma 1. *Let assumption 1 hold, and assume that the allocations are interior.*[41] *Then the optimal choice of the saving rate* $s_t = \frac{a_{t+1}}{(1-\kappa)w_t}$ *and stochastic old-age labor supply and consumption* $(l^o_{t+1}(\eta), c^o_{t+1}(\eta))$ *are given by*

$$l^o_{t+1}(\eta) = 1 - \frac{\gamma c^o_{t+1}(\eta)}{\kappa\eta w_{t+1}(1-\tau^l_{t+1})} \tag{60}$$

$$1 = \beta(1-\tau_{t+1}) \int \left(\frac{1-s_t}{s_t(1-\tau_{t+1}) + \frac{\kappa w_{t+1}\eta l^o_{t+1}(\eta)(1-\tau^l_{t+1})}{(1-\kappa)w_t R_{t+1}} + \frac{T_{t+1}}{(1-\kappa)w_t R_{t+1}}} \right) d\Psi(\eta) \tag{61}$$

$$c^o_{t+1}(\eta) = (1-\tau_{t+1})R_{t+1}s_t(1-\kappa)w_t + \kappa\eta(1-\tau^l_{t+1})w_{t+1}l^o_{t+1}(\eta) + T_{t+1}. \tag{62}$$

General Equilibrium Now we aggregate the individual decisions and express aggregate labor supply and the aggregate private saving rate as a function of the policy instruments. Aggregate labor supply of the old, L^o_t, and thus total aggregate labor supply L_t are given by

$$L^o_t = \int l^o_t(\eta)\eta d\Psi(\eta)$$
$$L_t = 1-\kappa+\kappa L^o_t$$

[41] Consumption and leisure are strictly positive almost surely by the Inada conditions implied by log-utility. However, labor supply might optimally be equal to zero for sufficiently low η. We will below state a sufficient condition on the support of η such that labor supply is indeed interior η-almost surely.

and factor prices are determined as

$$w_t = (1-\alpha)K_t^{\alpha}L_t^{-\alpha}$$
$$R_t = \alpha K_t^{\alpha-1}L_t^{1-\alpha}.$$

Aggregate transfers are given by

$$T_{t+1} = \tau_{t+1}R_{t+1}K_{t+1} + \tau_{t+1}^{l}\kappa w_{t+1}L_{t+1}^{o}. \tag{63}$$

Finally, the aggregate capital stock, the endogenous state variable in this model, evolves as a function of the saving rate $s_t = \frac{K_{t+1}}{(1-\kappa)w_t} = \frac{a_{t+1}}{(1-\kappa)w_t}$

$$K_{t+1} = a_{t+1} = s_t(1-\kappa)w_t = s_t(1-\kappa)(1-\alpha)K_t^{\alpha}L_t^{-\alpha}. \tag{64}$$

Individual and Aggregate Labor Supply In order to obtain a tractable expression for aggregate labor supply, first insert the budget constraint when old (62) into the optimality condition for individual labor supply, equation (60). This delivers, after rearranging,

$$(1+\gamma)(1-\tau_{t+1}^{l})\eta l_{t+1}^{o}(\eta) = \eta(1-\tau_{t+1}^{l}) - \frac{\gamma\left[(1-\tau_{t+1})R_{t+1}s_t(1-\kappa)w_t + T_{t+1}\right]}{\kappa w_{t+1}}. \tag{65}$$

Now we can aggregate both sides of this equation by integrating with respect to idiosyncratic productivity shocks, to obtain

$$(1+\gamma)L_{t+1}^{o} = 1 - \frac{\gamma\left[(1-\tau_{t+1})R_{t+1}s_t(1-\kappa)w_t + T_{t+1}\right]}{(1-\tau_{t+1}^{l})\kappa w_{t+1}}$$

and exploiting the expression for aggregate wages, interest rates and transfers we obtain

$$(1+\gamma)L_{t+1}^{o} = 1 - \frac{\gamma\alpha\left(1-\kappa+\kappa L_{t+1}^{o}\right)}{(1-\tau_{t+1}^{l})\kappa(1-\alpha)} - \frac{\gamma\tau_{t+1}^{l}L_{t+1}^{o}}{(1-\tau_{t+1}^{l})}$$

which yields aggregate equilibrium labor L_{t+1}^{o} and thus L_{t+1} solely as a function of the labor income tax rate τ_{t+1}^{l}, as stated in equations (28a) and (28b) in the main text. This then immediately leads to Proposition 7 of the main text. The fact that aggregate labor supply is strictly decreasing in the labor income tax rate follows from the fact that

$$L_t^o = \frac{1-\tau_t^l - \frac{\gamma\alpha(1-\kappa)}{\kappa(1-\alpha)}}{1-\tau_t^l + \frac{\gamma}{1-\alpha}}$$

and thus, taking the derivative with respect to the labor income tax rate τ_t^l yields:

$$\begin{aligned}\frac{\partial L_t^o}{\partial \tau_t^l} =& \frac{-(1-\tau_t^l + \frac{\gamma}{1-\alpha}) + 1 - \tau_t^l - \frac{\gamma\alpha(1-\kappa)}{\kappa(1-\alpha)}}{(1-\tau_t^l + \frac{\gamma}{1-\alpha})^2} \\ =& -\frac{\frac{\gamma}{1-\alpha} + \frac{\gamma\alpha(1-\kappa)}{\kappa(1-\alpha)}}{(1-\tau_t^l + \frac{\gamma}{1-\alpha})^2} < 0.\end{aligned}$$

We also note that we can express individual labor supply exclusively in terms of individual labor productivity and labor income taxes, independent of the saving rate and independent of the tax rate on capital. Rewriting (65) yields

$$\begin{aligned} l_{t+1}^o(\eta) &= \frac{1}{(1-\tau_{t+1}^l)(1+\gamma)}\left(1-\tau_{t+1}^l - \frac{\gamma\alpha L_{t+1}}{\eta\kappa(1-\alpha)} - \frac{\gamma\tau_{t+1}^l L_{t+1}^o}{\eta}\right) \\ &= \frac{1}{(1-\tau_{t+1}^l)(1+\gamma)}\left(1-\tau_{t+1}^l - \frac{\gamma\alpha(1-\kappa) + [\alpha + (1-\alpha)\tau_{t+1}^l]\gamma\kappa L_{t+1}^o}{\eta\kappa(1-\alpha)}\right) \\ &= l_{t+1}^o(\eta;\tau_{t+1}^l), \end{aligned} \tag{66}$$

where we note that $L_{t+1}^o = L_{t+1}^o(\tau_{t+1}^l)$. We can now also state a condition to insure that individuals find it optimal to supply positive labor even at the lowest productivity level $\underline{\eta}$. For this we need

$$\underline{\eta} > \gamma \frac{\alpha\,(1-\kappa) + \left[\alpha + (1-\alpha)\tau_{t+1}^l\right]\kappa\left(\frac{1-\tau_{t+1}^l - \frac{\gamma\alpha(1-\kappa)}{\kappa(1-\alpha)}}{1-\tau_{t+1}^l + \frac{\gamma}{1-\alpha}}\right)}{(1-\tau_{t+1}^l)\kappa(1-\alpha)} = \Xi(\gamma, \tau_{t+1}^l). \tag{67}$$

We note that since $\Xi(\gamma = 0, \tau_{t+1}^l) = 0$, by continuity in γ for every $\tau_{t+1}^l \in (-\infty, 1)$ there exists a small enough γ such that this condition is satisfied and labor supply is positive for every possible productivity level. We therefore make

Assumption 4. *The lower bound of the productivity shock* $\underline{\eta}$ *satisfies equation* (67) *for all* $\tau_{t+1}^l \leq \bar{\tau}_{t+1}^l$.

The Aggregate Saving Rate We can now express the saving rate in (61) as a function of the allocation of labor, which we have shown in the previous subsection just to depend on the labor income tax rate τ^l_{t+1}. Using (61) and the expressions for wages, interest rates and transfers in general equilibrium yields

$$1 = \alpha\beta(1-\tau_{t+1})\left(\frac{1-s_t}{s_t}\right)\int\left(\frac{1}{\alpha + \frac{(1-\alpha)\kappa}{L_{t+1}(\tau^l_{t+1})}\left[(1-\tau^l_{t+1})\eta l^o_{t+1}(\eta;\tau^l_{t+1}) + \tau^l_{t+1}L^o_{t+1}(\tau^l_{t+1})\right]}\right)d\Psi(\eta)$$

which gives the equilibrium saving rate in equation (29) in the main text. Proposition 8 of the main text then immediately follows.

F.2.2 Optimal Ramsey Allocations and Tax Policy

The objective of the government is to maximize social welfare, cf. equation (4), by choice of capital taxes $\{\tau_t\}_{t=0}^{\infty}$ and labor taxes $\{\tau^l_t\}_{t=0}^{\infty}$ and where V_t is lifetime utility of generation t in the competitive equilibrium associated with the sequence $\{\tau_t, \tau^l_t\}_{t=0}^{\infty}$. From the previous implementation result we know that the Ramsey government can, for any $t \geq 0$, implement any desired aggregate labor supply allocation L^o_t, L_t with an appropriate choice of labor income taxes τ^l_t. Given these choices it can then implement any aggregate saving rate s_t with an appropriate choice of τ_{t+1}. Note that since the initial old already made their savings decisions and the revenue from the capital tax is lump-sum distributed to them, the tax rate τ_0 is irrelevant for welfare. We now express expected lifetime utility of a given generation directly in terms of aggregate allocations; the Ramsey government chooses these allocations to maximize social welfare and implements these allocations as a competitive equilibrium with taxes, as discussed above. Lifetime utility of generation t can be expressed purely as a function of the beginning of the period capital stock, and the aggregate saving rate and aggregate labor supply when young and when old:

$$V_t = V(K_t, s_t, L^o_t, L^o_{t+1}) = u((1-s_t)(1-\kappa)(1-\alpha)K^\alpha_t L_t(L^o_t)^{-\alpha}) + \beta\int u\Big(\kappa w(s_t, L^o_t, L^o_{t+1})\cdot$$
$$\Big[\eta l^o_{t+1}(\eta, L^o_{t+1})(1-\tau^l_{t+1}(L^o_{t+1})) + \tau^l_{t+1}(L^o_{t+1})L^o_{t+1}\Big] + R(s_t, L^o_t, L^o_{t+1})K_{t+1}(s_t, L^o_t), l^o_{t+1}(\eta, L^o_{t+1})\Big)d\Psi(\eta)$$

where the aggregate components are themselves given by

$$\begin{aligned}
L_t(L_t^o) &= 1-\kappa+\kappa L_t^o && (68a)\\
L_{t+1}(L_{t+1}^o) &= 1-\kappa+\kappa L_{t+1}^o && (68b)\\
\tau_{t+1}^l(L_{t+1}^o) &= \frac{\kappa(1-\alpha-L_{t+1}^o(1+\gamma-\alpha))-\gamma\alpha(1-\kappa)}{\kappa(1-\alpha)(1-L_{t+1}^o)} && (68c)\\
l_{t+1}^o(\eta, L_{t+1}^o) &= \frac{\left(1-\tau_{t+1}^l(L_{t+1}^o)-\frac{\gamma\alpha L_{t+1}(L_{t+1}^o)}{\eta\kappa(1-\alpha)}-\frac{\gamma\tau_{t+1}^l L_{t+1}^o}{\eta}\right)}{(1-\tau_{t+1}^l(L_{t+1}^o))(1+\gamma)} && (68d)\\
K_{t+1}(s_t, L_t^o) &= s_t(1-\kappa)(1-\alpha)K_t^\alpha L_t(L_t^o)^{-\alpha} && (68e)\\
w(s_t, L_t^o, L_{t+1}^o) &= (1-\alpha)\left[K_{t+1}(s_t, L_t^o)\right]^\alpha L_{t+1}(L_{t+1}^o)^{-\alpha} && (68f)\\
R(s_t, L_t^o, L_{t+1}^o) &= \alpha\left[K_{t+1}(s_t, L_t^o)\right]^{\alpha-1} L_{t+1}(L_{t+1}^o)^{1-\alpha} && (68g)
\end{aligned}$$

Similarly, remaining lifetime utility of the initial old (and already substituting out factor prices) is given by

$$V_{-1} = V(K_0, L_0^o) =$$
$$\beta\int u\Big(\kappa(1-\alpha)K_0^\alpha L_0(L_0^o)^{-\alpha}\left[\eta l_0^o(\eta, L_0^o)(1-\tau_0^l(L_0^o))+\tau_0^l(L_0^o)L_0^o\right]+\alpha K_0^\alpha L_0(L_0^o)^{1-\alpha}, l_0^o(\eta, L_0^o)\Big)d\Psi(\eta).$$

Exploiting the assumption of logarithmic utility in consumption and leisure, the objective of the Ramsey government (including the initial generation) can be written as

$$\begin{aligned}
W(K_0) = &\sum_{t=-1}^{\infty}\omega_t V_t\\
&= \omega_{-1}\beta\int\left[\alpha\log(K_0)-\alpha\log(L_0(L_0^o))+\log(\kappa(1-\alpha)[\eta l_0^o(\eta, L_0^o)(1-\tau_0^l(L_0^o))+\tau_0^l L_0^o]+\alpha L_0(L_0^o))\right.\\
&\left.+\gamma\log(1-l_0^o(\eta))\right]d\Psi(\eta)\\
&+\sum_{t=0}^{\infty}\omega_t\Bigg[\log(1-s_t)+\log(1-\kappa)+\log(1-\alpha)+\alpha\log(K_t)-\alpha\log(L_t(L_t^o))\\
&+\beta\int\Big(\alpha\log(s_t)+\alpha\log(1-\kappa)+\alpha\log(1-\alpha)+\alpha^2\log(K_t)-\alpha^2\log(L_t(L_t^o))-\alpha\log(L_{t+1}((L_{t+1}^o)))\\
&+\log[\kappa(1-\alpha)(\eta l_{t+1}^o(\eta,(L_{t+1}^o))(1-\tau_{t+1}^l((L_{t+1}^o)))+\tau_{t+1}^l(L_{t+1}^o)L_{t+1}^o)+\alpha L_{t+1}(L_{t+1}^o)]\\
&+\gamma\log(1-l_{t+1}^o(\eta), L_{t+1}^o)\Big)d\Psi(\eta)\Bigg],
\end{aligned}$$

where the log-capital stock $\log(K_t)$ can be expressed as:

$$\begin{aligned}
\log(K_t) &= (\log(1-\alpha)+\log(1-\kappa))\left(\frac{1-\alpha^t}{1-\alpha}\right)+\alpha^t\log(K_0) \\
&+ \sum_{\tau=1}^{t}\alpha^{\tau-1}\log(s_{t-\tau})-\sum_{\tau=1}^{t}\alpha^{\tau-1}\log(L_{t-\tau}) \\
&= \log K_t(K_0,\{s_\tau,L_\tau\}_{\tau=0}^{t-1}).
\end{aligned}$$

Thus we note that the objective function can be written purely in terms of the aggregate allocations $\{s_t, L_t^o\}_{t=0}^{\infty}$ and that it is additively separable in time between the savings rate s_t on one hand and aggregate labor supply L_t^o on the other hand. This in turn will greatly facilitate the characterization of the optimal Ramsey allocations.

Optimal Saving Rate Ignoring constants that are irrelevant for maximization *with respect to the savings rate* s_t, this part $W^s(K_0)$ of the social welfare function can be expressed as:

$$\begin{aligned}
W^s(K_0) &= \sum_{t=0}^{\infty}\omega_t\left[\log(1-s_t)+\alpha\beta\log(s_t)+\alpha(1+\alpha\beta)\log(K_t)\right] \\
&= \sum_{t=0}^{\infty}\omega_t\left[\log(1-s_t)+\log(s_t)\left(\alpha\beta+\alpha\,(1+\alpha\beta)\sum_{j=1}^{\infty}\frac{\omega_{t+j}}{\omega_t}\alpha^{j-1}\right)\right]
\end{aligned}$$

Taking first order conditions with respect to s_t and setting it to zero then immediately results in Proposition 9, with the implementing capital tax rate directly implied by equation (29) in the main text.

G Intergenerational Redistribution

G.1 Pension System

G.1.1 Setup

The budget constraints of households under a time varying capital tax τ_t, a time varying contribution rate to the pension system τ_t^p and a flat pension payment b_{t+1} in the two periods of life are

$$
\begin{aligned}
c_t^y + a_{t+1} &= w_t(1-\kappa)(1-\tau_t^p) \\
c_{t+1}^o(\eta) &= a_{t+1}R_{t+1}(1-\tau_{t+1}) + \kappa\eta w_{t+1} + b_{t+1} + T_{t+1}.
\end{aligned}
$$

We assume a PAYG pension system (balanced budget) so that

$$\tau_t^p(1-\kappa)w_t = b_t.$$

Furthermore, while our formal analysis also encompasses the case of an unrestricted pay-as-you-go pension system, we are mainly interested in a scenario where pension payments are restricted to be positive (i.e., there is no reverse pension system). In this scenario the constraint $\tau_t^p \geq 0$ applies.

Finally, as in the main text, the budget constraint of the capital tax system is

$$\tau_t R_t a_t = T_t.$$

G.1.2 Analysis

Define the net saving rate by

$$s_t = \frac{a_{t+1}}{(1-\kappa)w_t(1-\tau_t^p)},$$

and note that with this definition of the saving rate we obtain the law of motion for capital in general equilibrium as

$$k_{t+1} = a_{t+1} = s_t(1-\alpha)k_t^\alpha(1-\kappa)(1-\tau_t^p)$$

and can thus express consumption in the two periods in general equilibrium as

$$
\begin{aligned}
c_t^y &= (1-s_t)(1-\tau_t^p)(1-\alpha)(1-\kappa)k_t^\alpha = \frac{1-s_t}{s_t}k_{t+1} \\
c_{t+1}^o &= k_{t+1}\alpha k_{t+1}^{\alpha-1} + k_{t+1}^\alpha(1-\alpha)\left(\kappa\eta + \tau_{t+1}^p(1-\kappa)\right) \\
&= \left(\alpha + (1-\alpha)\left(\kappa\eta + \tau_{t+1}^p\left(1-\kappa\right)\right)\right)k_{t+1}^\alpha.
\end{aligned}
$$

Using this in the private household Euler equation in competitive equilibrium with log utility

$$1 = \alpha\beta k_{t+1}^{\alpha-1}(1-\tau_{t+1})\int \frac{c_t^y}{c_{t+1}^o(\eta)} d\Psi(\eta)$$

yields

$$\begin{aligned}1 &= \alpha\beta k_{t+1}^{\alpha-1}(1-\tau_{t+1})\int \frac{\frac{1-s_t}{s_t}k_{t+1}}{\left(\alpha+(1-\alpha)\left(\kappa\eta+\tau_{t+1}^p\left(1-\kappa\right)\right)\right)k_{t+1}^{\alpha}} d\Psi(\eta)\\ &= \alpha\beta(1-\tau_{t+1})\Gamma(\alpha,\kappa,\Psi;\tau_{t+1}^p)\frac{1-s_t}{s_t},\end{aligned}$$

where the constant summarizing the impact of income risk is now given by

$$\Gamma(\alpha,\kappa,\Psi;\tau_{t+1}^p) = \int \left[\alpha+(1-\alpha)\left(\kappa\eta+\tau_{t+1}^p\left(1-\kappa\right)\right)\right]^{-1} d\Psi(\eta) \tag{69}$$

and thus the private saving rate is only a function of the two tax rates and exogenous parameters:

$$s_t(\tau_{t+1},\tau_{t+1}^p) = \frac{1}{1+\left[\alpha\beta(1-\tau_{t+1})\Gamma(\alpha,\kappa,\Psi;\tau_{t+1}^p)\right]^{-1}}. \tag{70}$$

From equation (70) the following observation immediately follows:

Observation 2. *The private saving rate in general equilibrium with a PAYG pension system has the following properties:*

1. $\frac{\partial\Gamma(\cdot;\tau_{t+1}^p)}{\partial\tau_{t+1}^p} > 0$ *and thus* $\frac{\partial s(\tau_{t+1},\tau_{t+1}^p)}{\partial\tau_{t+1}^p} < 0$.
2. $\frac{\partial s(\tau_{t+1},\tau_{t+1}^p)}{\partial\tau_{t+1}} < 0$
3. *A mean-preserving spread in* η *increases* $\Gamma(\cdot;\tau_{t+1}^p)$ *and thus* $s(\tau_{t+1},\tau_{t+1}^p)$ *by less the larger is* τ_{t+1}^p.

The key implication of this result is that for given τ_{t+1}^p we can implement any desired saving rate s_t by choice of τ_{t+1}. The saving rate increases in income risk, but less so with a larger pension system since the latter provides partial consumption insurance in old age, and thus reduces the precautionary saving incentives of private households.

Thus the implementation results from the main paper extend unchanged to the case with a PAYG pension system. If, in addition, the constraint $\tau_{t+1}^p \geq 0$ is imposed and is binding, then the implementation result from the main text applies unchanged (since the PAYG system is not operative in that case).

G.1.3 The Ramsey Tax Problem

From the implementation result we observe that any saving rate $s_t \in (0,1)$ can be implemented for a given contribution rate τ_t^p with some capital tax rate $\tau_t \in (-\infty, 1)$. In light of this, we define the Ramsey problem as one of directly choosing the saving rate s_t and the contribution rate to the pension system τ_t^p, which constitutes a hybrid between a primal and an indirect utility approach to optimal taxation.

The government's social welfare function is

$$W = \omega_{-1}\beta \int \ln(c_0^o(\eta_0)) d\Psi(\eta_0) + \sum_{t=0}^{\infty} \omega_t \left[\ln(c_t^y) + \beta \int \ln(c_{t+1}^o(\eta_{t+1})) d\Psi(\eta_{t+1}) \right] \quad (71)$$

which using the expressions

$$\begin{aligned} c_t^y &= (1-s_t)(1-\tau_t^p)(1-\alpha)(1-\kappa)k_t^\alpha \\ c_{t+1}^o &= \left(\alpha + (1-\alpha)\left(\kappa\eta + \tau_{t+1}^p(1-\kappa)\right)\right) k_{t+1}^\alpha \\ k_{t+1} &= s_t(1-\tau_t^p)(1-\kappa)(1-\alpha)k_t^\alpha \end{aligned}$$

can be rewritten as

$$\begin{aligned} W = \Xi + \beta\omega_{-1} & \int \ln\left[(\alpha + (1-\alpha)(\kappa\eta_0 + \tau_0^p(1-\kappa)))\right] d\Psi(\eta_0) \\ & + \sum_{t=0}^{\infty} \omega_t \left[\ln(1-s_t) + \ln(1-\tau_t^p) + \alpha\ln(k_t)\right] \\ & + \beta \sum_{t=0}^{\infty} \omega_t \left[\int \ln\left[\left(\alpha + (1-\alpha)\left(\kappa\eta_{t+1} + \tau_{t+1}^p(1-\kappa)\right)\right)\right] d\Psi(\eta_{t+1}) \right. \\ & \quad \left. + \alpha\ln(1-\tau_t^p) + \alpha\ln(s_t) + \alpha^2\ln(k_t) \right]. \end{aligned}$$

Now follow the analogous steps to those in Appendix B to write the dynamics of the capital stock as

$$\ln(k_t) = \varkappa_t + \sum_{j=0}^{t-1} \alpha^j \left(\ln\left(s_{t-j}\right) + \ln\left(1 - \tau_{t-j}^p\right)\right)$$

and collect terms to get

$$W = \tilde{\Xi} + \beta \sum_{t=0}^{\infty} \omega_{t-1} \int \ln\left[\left(\alpha + (1-\alpha)\left(\kappa\eta_0 + \tau_t^p\left(1-\kappa\right)\right)\right)\right] d\Psi(\eta_t)$$
$$+ \sum_{t=0}^{\infty} \omega_t \left[\ln(1-s_t) + (\ln(s_t) + \ln(1-\tau_t^p))\left(\alpha\beta + \alpha(1+\alpha\beta)\sum_{j=1}^{\infty} \frac{\omega_{t+j}}{\omega_t}\alpha^{j-1}\right)\right].$$

We directly observe from the above that the first-order condition with respect to s_t is the same as derived in Appendix B and therefore

$$s_t^* = \frac{1}{1 + \left(\alpha\beta + \alpha\left(1+\alpha\beta\right)\sum_{j=1}^{\infty} \frac{\omega_{t+j}}{\omega_t}\alpha^{j-1}\right)^{-1}}. \tag{72}$$

For τ_t^p we obtain from the respective first-order condition in all $t = 0, \ldots$ the function

$$f(\tau_t^p) = \beta\Gamma(\tau_t^{p^*})(1-\alpha)(1-\kappa) - \frac{1}{1-\tau_t^p}\frac{\omega_t}{\omega_{t-1}}\left(\alpha\beta + \alpha\left(1+\alpha\beta\right)\sum_{j=1}^{\infty}\frac{\omega_{t+j}}{\omega_t}\alpha^{j-1}\right) \tag{73}$$

which since $\Gamma(\tau_t^p) > 0$ is increasing in τ_t^p and since $\frac{1}{1-\tau_t^p}$ is convex and upward sloping in τ_t^p with $\lim_{\tau_t^p \to 1} \frac{1}{1-\tau_t^p} = \infty$ and $\lim_{\tau_t^p \to -\infty} \frac{1}{1-\tau_t^p} = 0$ implicitly defines the optimal $\tau_t^{p^*} \in (-\infty, 1)$. Furthermore, since $\Gamma(\tau_t^p)$ is increasing in risk for a given τ_t^p we find that τ^{p^*} is increasing in risk (but need not be positive).

Furthermore, we know from (70), evaluated at the optimal saving rate s_t^* that the optimal capital tax rate required to implement s_t^* is given by

$$\tau_{t+1}(s_t^*, \tau_{t+1}^{p^*}) = 1 - \frac{s_t^*}{\alpha\beta(1-s_t^*)\Gamma(\tau_{t+1}^{p^*})}. \tag{74}$$

Holding the contribution rate to the pension system constant, the optimal τ_{t+1}^* is increasing in risk. However, to characterize the complete response of the capital tax rate to income risk we have to take into account that the contribution rate to the pension system also rises, reducing overall second period income risk and thus precautionary saving (and therefore the need to tax capital income). Introducing the notation that an increase of risk is measured by an increase of the variance σ_η^2 of η we therefore must evaluate the total derivative:[42]

$$\frac{\partial \tau_{t+1}}{\partial \sigma_\eta^2} = \frac{s_t^*}{\alpha\beta(1-s_t^*)\Gamma(\tau_{t+1}^{p^*})^2}\left(\underbrace{\frac{\partial \Gamma(\tau_{t+1}^p, \Psi)}{\partial \sigma_\eta^2}}_{>0} + \underbrace{\frac{\partial \Gamma(\tau_{t+1}^p, \Psi)}{\partial \tau_{t+1}^p}}_{<0}\underbrace{\frac{\partial \tau_{t+1}^p(\Psi)}{\partial \sigma_\eta^2}}_{>0}\right). \tag{75}$$

We next show that $\frac{\partial \tau_{t+1}}{\partial \sigma_\eta^2} > 0$. From the first-order condition (73) we note that by the implicit function theorem $\frac{d\tau_{t+1}^p}{d\sigma_\eta^2} = -\frac{\partial f(\cdot)/\partial \sigma_\eta^2}{\partial f(\cdot)/\partial \tau_{t+1}^p}$ with the partial derivatives

$$\frac{\partial f(\cdot)}{\partial \sigma_\eta^2} = \beta(1-\alpha)(1-\kappa)\frac{\partial \Gamma(\tau_{t+1}^p)}{\partial \sigma^2} > 0$$
$$\frac{\partial f(\cdot)}{\partial \tau_{t+1}^p} = -\left(\frac{\omega_t}{\omega_{t-1}}\left(\alpha\beta + \alpha\left(1+\alpha\beta\right)\sum_{j=1}^{\infty}\frac{\omega_{t+j}}{\omega_t}\alpha^{j-1}\right)\frac{1}{(1-\tau^p)^2} - \beta(1-\alpha)(1-\kappa)\frac{\partial \Gamma(\tau_{t+1}^p)}{\partial \tau_{t+1}^p}\right) < 0.$$

We thus find (as we had already argued informally above) that the optimal social security contribution rate is strictly increasing in income risk:

$$\frac{d\tau_{t+1}^p}{d\sigma_\eta^2} = \frac{\partial \Gamma/\partial \sigma_\eta^2}{\frac{\omega_t}{\omega_{t-1}}\frac{\alpha\beta+\alpha(1+\alpha\beta)\sum_{j=1}^{\infty}\frac{\omega_{t+j}}{\omega_t}\alpha^{j-1}}{\beta(1-\kappa)(1-\alpha)}\frac{1}{(1-\tau^p)^2} - \partial \Gamma(\tau_{t+1}^p)/\partial \tau_{t+1}^p} > 0$$

We now use this result to sign the overall effect of income risk on the optimal capital tax rate. For this, note that

$$\frac{\partial \Gamma(\tau_{t+1}^p, \Psi)}{\partial \tau_{t+1}^p}\frac{\partial \tau_{t+1}^p(\Psi)}{\partial \sigma_\eta^2} = \frac{\partial \Gamma/\partial \sigma_\eta^2}{\frac{\omega_t}{\omega_{t-1}}\frac{\alpha\beta+\alpha(1+\alpha\beta)\sum_{j=1}^{\infty}\frac{\omega_{t+j}}{\omega_t}\alpha^{j-1}}{\beta(1-\kappa)(1-\alpha)}\frac{1}{(1-\tau^p)^2}\frac{1}{\partial \Gamma(\tau_{t+1}^p)/\partial \tau_{t+1}^p} - 1}$$

[42]In Appendix K.2 we show that the notion of an increase in income risk (mean-preserving spread in the distribution of η) is equivalent to an increase in the variance of η to a second order approximation of the integral Γ. Here expressing income risk in terms of the variance is simply a matter of notation, and stands in for a mean-preserving spread in η.

and we can rewrite (75) as

$$\frac{\partial \tau}{\partial \sigma_\eta^2} = \frac{s_t^*}{\alpha\beta(1-s_t^*)\Gamma(\tau_{t+1}^{p^*})^2} \frac{\partial \Gamma(\tau_{t+1}^p, \Psi)}{\partial \sigma_\eta^2} \cdot \left(1 - \frac{1}{1 - \frac{\omega_t}{\omega_{t-1}} \frac{\left(\alpha\beta + \alpha(1+\alpha\beta)\sum_{j=1}^{\infty} \frac{\omega_{t+j}}{\omega_t}\alpha^{j-1}\right)}{\beta(1-\kappa)(1-\alpha)} \frac{1}{(1-\tau^p)^2} \frac{1}{\partial\Gamma(\tau_{t+1}^p)/\partial\tau_{t+1}^p}}\right)$$

and since $\frac{\partial \Gamma}{\partial \tau^p} < 0$ we obtain

$$\begin{aligned}
& 1 - \frac{\omega_t}{\omega_{t-1}} \frac{\left(\alpha\beta + \alpha\left(1+\alpha\beta\right)\sum_{j=1}^{\infty} \frac{\omega_{t+j}}{\omega_t}\alpha^{j-1}\right)}{\beta(1-\kappa)(1-\alpha)} \frac{1}{(1-\tau^p)^2} \frac{1}{\partial\Gamma(\tau_{t+1}^p)/\partial\tau_{t+1}^p} > 1 \\
\Leftrightarrow \quad & 1 - \frac{1}{1 - \frac{\omega_t}{\omega_{t-1}} \frac{\left(\alpha\beta + \alpha(1+\alpha\beta)\sum_{j=1}^{\infty} \frac{\omega_{t+j}}{\omega_t}\alpha^{j-1}\right)}{\beta(1-\kappa)(1-\alpha)} \frac{1}{(1-\tau^p)^2} \frac{1}{\partial\Gamma(\tau_{t+1}^p)/\partial\tau_{t+1}^p}} > 0 \\
\Leftrightarrow \quad & \frac{\partial \tau}{\partial \sigma_\eta^2} > 0.
\end{aligned}$$

Therefore, the direct effect of a marginal increase of income risk on households savings in competitive equilibrium dominates the indirect effect from a reduction of consumption risk due to an increase of the optimal social security contribution and benefit system. The mitigation of the additional income risk through a marginal increase of the social security contribution rate τ_{t+1}^p is not strong enough to offset the effect of the marginal increase of risk. Intuitively, the Ramsey government, when optimally determining the social security contribution rate, has two motives. First, it aims at inter-generational redistribution. Second, it aims at reducing the direct effect of income risk. Since it has these two motives, it will not be optimal for the Ramsey government to completely offset a marginal increase of income risk so that $\Gamma(\tau_{t+1}^p; \Psi)$ increases even after the optimal adjustment of τ_{t+1}^p. Since the household saving rate in competitive equilibrium therefore increases due to the precautionary saving motive and since the Ramsey government aims at implementing a constant saving rate in order to offset the negative pecuniary externality from that increase of risk—just as in our model from the main text—the capital tax rate has to increase with income risk in order to implement that constant saving rate.

Finally, notice that with geometric discounting of the government $\omega_t = \theta^t$ we get

$$\sum_{j=1}^{\infty} \frac{\omega_{t+j}}{\omega_t} \alpha^{j-1} = \frac{\theta}{1-\alpha\theta}$$

and thus $\tau_t^{p^*} = \tau^{p^*}$, $s_t^* = s^*$ and $\tau_{t+1}^* = \tau^*$ for all t. We have the following:

Proposition 14. *The optimal saving rate s_t^* is independent of risk and the optimal pension contribution rate $\tau_{t+1}^{p^*}$ and the optimal capital tax rate τ_{t+1}^* are strictly increasing in idiosyncratic income risk. If, in addition, $\omega_t = \theta^t$ then $s_t^* = \frac{\alpha(\beta+\theta)}{1+\alpha\beta}$, $\tau_t^{p^*} = \tau^{p^*}$ and $\tau_{t+1}^* = \tau^*$ for all $t = 0, \ldots, \infty$.*

G.1.4 Ramsey Tax Problem in Steady State

We next aim to relate results in the pension system to those stated in Proposition 4 and thus from now on focus on maximizing steady state utility where $\omega_t = \theta = 1$. First, consider the deterministic economy. In this case the optimal pension contribution rate solves

$$\frac{1+\alpha\beta}{1-\alpha} \frac{1}{1-\tau^p} = \frac{\beta(1-\alpha)(1-\kappa)}{\alpha + (1-\alpha)\left(\kappa + \tau^p(1-\kappa)\right)}. \tag{76}$$

Solving this equation for the optimal contribution rate under certainty (denoted by $\Psi = \bar{\Psi}$) delivers

$$\tau^{p^*}(\bar{\Psi}) = \frac{\frac{\beta}{1+\beta} - \frac{\alpha}{1-\alpha}}{1-\kappa} - \frac{\kappa}{1-\kappa}. \tag{77}$$

The optimal capital tax rate is determined from our implementation result, equation (70):

$$\frac{1}{\alpha\beta\Gamma(\tau^{p^*}(\bar{\Psi}), \bar{\Psi})} \frac{s^*}{1-s^*} = 1 - \tau^*(\bar{\Psi}). \tag{78}$$

We have

$$\frac{1}{\Gamma(\tau^{p^*}(\bar{\Psi}), \bar{\Psi})} = \alpha + (1-\alpha)\left(\kappa + (1-\kappa)\tau^{p^*}(\bar{\Psi})\right) = \frac{(1-\alpha)\beta}{1+\beta}.$$

Using this result and the expression for s^* in (78) we obtain for the optimal tax rate on capital $\tau^*(\bar{\Psi}) = 0$. The associated steady state capital stock is

$$k^*(\bar{\Psi}) = \left(s^*(1 - \tau^{p^*}(\bar{\Psi}))(1-\alpha)(1-\kappa)\right)^{\frac{1}{1-\alpha}}.$$

Using the expressions for s^* and $\tau^{p^*}(\bar{\Psi})$ in the above expressions immediately implies that $s^* \cdot (1 - \tau^{p^*}\bar{\Psi}) = s^{GR}$ and $k^*(\bar{\Psi}) = k^{GR}$. Furthermore, recall from Corollary 5 of Appendix E that for $\theta = 1$ and for a Pareto weight of $\omega_{-1} = 1$ on the initial old generation, the social planner maximizing steady state utility implements the saving rate s^{GR} along all periods of the transition. Therefore the Ramsey government setting s^* and $\tau^{p^*}(\bar{\Psi})$ from period 0 onward implements the socially optimal allocation. We summarize these results in the next

Proposition 15. *In the deterministic economy, for $\theta = 1$ and $\omega_{-1} = 1$, setting the optimal tax rates maximizing steady state utility*

$$\tau^{p^*}(\bar{\Psi}) = \frac{\frac{\beta}{1+\beta} - \frac{\alpha}{1-\alpha}}{1-\kappa} - \frac{\kappa}{1-\kappa} \;\; \textit{and} \;\; \tau^*(\bar{\Psi}) = 0 \tag{79}$$

in period 0 and holding them constant induces a transition path that implements the social optimum, with the golden rule saving rate $s^{GR} = \frac{\alpha}{(1-\kappa)(1-\alpha)}$. The economy converges monotonically to the golden rule steady state capital stock $k^* = k^{GR} = \alpha^{\frac{1}{1-\alpha}}$.

We also observe from the above that if the initial deterministic laissez-faire economy has a capital stock below the golden rule, $k_0^{CE} < k^{GR}$, then the optimal long-run steady state welfare maximizing social security contribution rate is negative. In contrast, if the competitive equilibrium capital stock is above the golden rule, $k_0^{CE} > k^{GR}$, then the optimal contribution rate is positive.

Since in the deterministic economy the Ramsey government implements the golden rule capital stock, since the optimal Ramsey net saving rate s^* is independent of income risk and since the optimal contribution rate τ^{p^*} to the pension system is strictly increasing in income risk we have the following

Corollary 8. *In the economy where η is risky, the optimal Ramsey long run capital stock satisfies $k^* < k^{GR}$.*

Related to Proposition 4 in the main text, we now establish that there is a threshold risk level such that for risk above that threshold we have $\tau^{p^*} > 0$, and for risk below the threshold, $\tau^{p^*} < 0$. That threshold lies in the intermediate risk range of Proposition 4.

To see this, recall that we have established that without income risk $\tau^{p^*}(\bar{\Psi})$ implements the golden rule capital stock, and the associated optimal tax on capital is $\tau^* = 0$. Since by the assumption on parameters maintained in Proposition 4 the laissez-faire competitive equilibrium capital stock is below the golden rule, $k(\bar{\Psi}) < k^{GR}$, implementing the golden rule capital stock without income risk requires $\tau^{p^*}(\bar{\Psi}) < 0$. Thus the starting point is the economy without risk and with optimal policy $\tau^{p^*} < 0, \tau^* = 0$. Now increase income risk.

We have established above that both τ^{p^*} and τ^* are strictly increasing in income risk. Thus there exists some threshold risk level $\hat{\Gamma}$ for which $\tau^{p^*} = 0$. Recall that the first order condition for τ^p (see equation (73)) is

$$\frac{1+\alpha\beta}{1-\alpha}\frac{1}{1-\tau^p} = \beta(1-\alpha)(1-\kappa)\Gamma(\tau^p).$$

which for $\tau^{p^*} = 0$ defines the risk threshold $\hat{\Gamma}$ explicitly as

$$\hat{\Gamma} = \frac{1+\alpha\beta}{\beta(1-\alpha)^2(1-\kappa)} \tag{80}$$

To show that this threshold $\hat{\Gamma}$ lies in the intermediate risk interval of Proposition 4, $\left(\frac{1+\beta}{(1-\alpha)\beta}, \frac{1}{\beta\left(1-\alpha-\frac{1}{\Gamma}\right)}\right)$, first investigate the lower bound of the interval. Notice that

$$\begin{aligned}
& \hat{\Gamma} > \frac{1+\beta}{(1-\alpha)\beta} \\
\Leftrightarrow \qquad & \frac{1+\alpha\beta}{\beta(1-\alpha)^2(1-\kappa)} > \frac{1+\beta}{(1-\alpha)\beta} \\
\Leftrightarrow \qquad & \frac{1+\alpha\beta}{(1-\alpha)(1-\kappa)} > 1+\beta \\
\Leftrightarrow \qquad & s^{GR} = \frac{\alpha}{(1-\alpha)(1-\kappa)} > \frac{\alpha(1+\beta)}{1+\alpha\beta} = s^*
\end{aligned}$$

and note that s^{GR} is defined as a gross saving rate whereas s^* is defined as a net saving rate. However, for $\tau^{p^*} = 0$, the gross and the net saving rates are identical. The inequality above follows from the proof of proposition 4: For the intermediate risk case we established there that $s^* < s^{GR}$, a result which carries over to the current analysis of social security as long as $\tau^{p^*} = 0$.

Now consider the upper bound. Notice that

$$
\begin{aligned}
& \hat{\Gamma} < \frac{1}{\beta\left(1-\alpha-\frac{1}{\Gamma}\right)} \\
\Leftrightarrow \qquad & \frac{1+\alpha\beta}{\beta(1-\alpha)^2(1-\kappa)} < \frac{1}{\beta\left(1-\alpha-(\kappa(1-\alpha)+\alpha)\right)} \\
\Leftrightarrow \qquad & \frac{1+\alpha\beta}{(1-\alpha)^2(1-\kappa)} < \frac{1}{(1-\alpha)(1-\kappa)-\alpha} \\
\Leftrightarrow \qquad & \frac{1}{(1-\alpha)(1-\kappa)} < \frac{1}{(1+\alpha\beta)\left((1-\kappa)-\frac{\alpha}{1-\alpha}\right)},
\end{aligned}
$$

and since $s^{GR} = \frac{\alpha}{(1-\alpha)(1-\kappa)} < 1 \;\Leftrightarrow\; (1-\kappa) - \frac{\alpha}{1-\alpha} > 0$ we further get

$$
\begin{aligned}
\Leftrightarrow \qquad & (1-\alpha)(1-\kappa) > (1+\alpha\beta)\left((1-\kappa)-\frac{\alpha}{1-\alpha}\right) \\
\Leftrightarrow \qquad & (1-\alpha)(1-\kappa) > (1+\alpha\beta)(1-\kappa) - \frac{\alpha(1+\alpha\beta)}{1-\alpha} \\
\Leftrightarrow \qquad & (1-\kappa) - \alpha(1-\kappa) > (1-\kappa) + \alpha\beta(1-\kappa) - \frac{\alpha(1+\alpha\beta)}{1-\alpha} \\
\Leftrightarrow \qquad & \frac{\alpha(1+\alpha\beta)}{1-\alpha} > \alpha\beta(1-\kappa) + \alpha(1-\kappa) \\
\Leftrightarrow \qquad & s^{GR} = \frac{\alpha}{(1-\alpha)(1-\kappa)} > \frac{\alpha(1+\beta)}{(1+\alpha\beta)} = s^*
\end{aligned}
$$

and again the above inequality follows from proposition 4. Therefore the threshold satisfies $\hat{\Gamma} \in \left(\frac{1+\beta}{(1-\alpha)\beta}, \frac{1}{\beta\left(1-\alpha-\frac{1}{\Gamma}\right)}\right)$, that is, lies in the intermediate risk interval of proposition 4 in the main text. With this characterization of $\hat{\Gamma}$ we can state the next proposition, which serves as a generalization of proposition 4. It characterizes the jointly optimal pension contribution and capital tax rate, and also covers the case when a nonnegativity constraint on pension contributions and thus pension benefits is imposed.

Proposition 16. *Let $\theta = 1$ so that the Ramsey government maximizes steady state welfare. Denote by s^{CE} the saving rate in the laissez-faire competitive equilibrium and by s^{GR} the gross saving rate that implements the golden rule capital stock. Further denote by s^* the optimal Ramsey net saving rate, $s^* = \frac{a^*}{w(1-\tau^{p^*})}$, where τ^{p^*} is the optimal Ramsey pension contribution rate. Finally denote by τ^* the optimal Ramsey capital tax rate.*

1. *Let income risk be **large**, $\Gamma > \frac{1}{\beta\left((1-\alpha)-\frac{1}{\Gamma}\right)}$. Then $s^{CE} > s^{GR} > s^*$, and $\tau^* > 0$,*

and $\tau^{p^*} > 0$.

2. *Let income risk be **fairly large**, $\Gamma \in \left(\frac{1+\alpha\beta}{\beta(1-\alpha)^2(1-\kappa)}, \frac{1}{\beta\left((1-\alpha)-\frac{1}{\Gamma}\right)}\right)$. Then $s^{CE} < s^{GR}$ and $\tau^* > 0$, and $\tau^{p^*} > 0$, and thus $s^* < s^{CE}$.*

3. *Let income risk be **fairly small**, $\Gamma \in \left(\frac{1+\beta}{(1-\alpha)\beta}, \frac{1+\alpha\beta}{\beta(1-\alpha)^2(1-\kappa)}\right)$. Then $s^{CE} < s^{GR}$ and $\tau^* > 0$. If the social security contribution rate is unrestricted, then $\tau^{p^*} < 0$. If it is subject to a nonnegativity constraint, then $\tau^{p^*} = 0$ and $s^* < s^{CE}$.*

4. *Let income risk be **small**, $\Gamma \in \left(\bar{\Gamma}, \frac{1+\beta}{(1-\alpha)\beta}\right)$. Then $s^{CE} < s^{GR}$. If the social security contribution rate is unrestricted, then $\tau^* > 0$ and $\tau^{p^*} < 0$. If it is subject to a nonnegativity constraint, then $\tau^* < 0, \tau^{p^*} = 0$, and $s^* < s^{CE}$.*

The interesting interval is thus the interval where risk is *fairly small*, where the optimal capital tax is positive but the pension contribution rate is negative (or zero, if constrained to be nonnegative). To provide some intuition for this finding, notice that the optimal pension contribution rate turns positive in the stochastic economy at a level of risk that is below the risk level where the laissez-faire competitive equilibrium economy's capital stock is equal to the golden rule capital stock because the pension system serves two purposes: it provides optimal intergenerational redistribution and it partially insures against idiosyncratic income risk. This dual role can directly be inferred from the first-order condition (73). Given the contribution rate and the implied remaining idiosyncratic consumption risk (which cannot be fully eliminated by social security benefits), the capital income tax implements the optimal Ramsey saving rate, offsetting the negative pecuniary externality from increasing saving rates induced by income risk of households, exactly as in the model without social security. If the pension contribution rate is restricted to zero, then the tax on capital also targets inter-generational redistribution, as in the benchmark model. So why does the optimal tax rate on capital turn positive for a lower threshold of risk with, compared to without social security? Without social security as an inter-generational redistribution instrument, capital taxation partially fills the role of providing inter-generational redistribution in addition to addressing the pecuniary externality, and in the case the competitive equilibrium capital stock is below the golden rule, this force pushes down the tax on capital (to encourage capital accumulation) relative to the case where social security tackles the desired intergenerational redistribution (through a negative contribution rate, if permitted).

Finally, denote by $k^*(\tau^*, \tau^p = 0)$ the optimal Ramsey steady state capital stock in the economy without a pension system and by $k^*(\tau^*, \tau^{p^*})$ the steady state capital stock in the economy with a pension system. From the optimal saving rate and the optimal pension contribution rate characterized in Proposition 16 we obtain the next

Corollary 9. *The optimal long run capital stock in the economy with and the economy without social security are related as follows:*

1. *For large income risk,* $\Gamma > \frac{1+\alpha\beta}{\beta(1-\alpha)^2(1-\kappa)}$*, we have* $k^*(\tau^*, \tau^{p^*}) < k^*(\tau^*, \tau^p = 0)$
2. *For small income risk,* $\Gamma \leq \frac{1+\alpha\beta}{\beta(1-\alpha)^2(1-\kappa)}$*, we have* $k^*(\tau^*, \tau^{p^*}) \geq k^*(\tau^*, \tau^p = 0)$

This result immediately follows from the fact that the optimal net savings rates satisfy $s^*(\tau^*, \tau^p = 0) = s^*(\tau^*, \tau^{p^*})$ and the steady state capital stock follows from the saving rate as $k^* = (s^*(1-\tau^p)(1-\kappa)(1-\alpha))^{\frac{1}{1-\alpha}}$. For $\Gamma > \frac{1+\alpha\beta}{\beta(1-\alpha)^2(1-\kappa)}$, proposition 16 implies that $\tau^{p^*} > 0$ (and thus $k^*(\tau^*, \tau^{p^*}) < k^*(\tau^*, \tau^p = 0)$), and for $\Gamma < \frac{1+\alpha\beta}{\beta(1-\alpha)^2(1-\kappa)}$ we have $\tau^{p^*} \leq 0$ (and thus $k^*(\tau^*, \tau^{p^*}) \geq k^*(\tau^*, \tau^p = 0)$) where the equality is strict if the constraint $\tau^{p^*} \geq 0$ applies.

Transition Under Optimal Steady State Policy Under the optimal long-run steady state welfare maximizing policy τ^*, τ^{p^*} implemented in period 0, the economy converges to the long steady state with the dynamics of the capital stock along the transition given by

$$k_{t+1} = s^*(1-\tau^{p^*})(1-\kappa)(1-\alpha)k_t^\alpha$$

and transfers to the initial old generation of

$$b_0 = \tau^{p^*}(1-\kappa)(1-\alpha)k_0^\alpha$$

so that initial consumption of the old is

$$\begin{aligned}
c_0^o &= a_0 R_0 + \kappa\eta w_0 + b_0 \\
&= k_0 \alpha k_0^{\alpha-1} + \kappa\eta(1-\alpha)k_0^\alpha + \tau^{p^*}(1-\kappa)(1-\alpha)k_0^\alpha \\
&= \left(\alpha + (1-\alpha)\left(\kappa\eta + \tau^{p^*}(1-\kappa)\right)\right)k_0^\alpha.
\end{aligned}$$

G.1.5 Illustration

Panels (a) and (b) of Figure 3 illustrate Proposition 16 by plotting the optimal social security contribution rate and the optimal capital tax rate against the extent of income risk. It does so both for the case in which there is a nonnegativity constraint on τ^p and the case where the social security contribution rate is unconstrained. Panels (c) and (d) in the figure show the policy instruments for optimal government debt to which we turn in Section G.2.

Panels (d) and (b) first demonstrate that both the social security contribution rate and the capital tax rate are increasing in income risk, strictly so for τ^* and also for τ^{p^*} unless the latter is constrained to be nonnegative, in which case $\tau^{p^*} = 0$ if income risk is small, see Panel (b). The vertical lines separate the x-axis into the four intervals characterized in proposition 16. In the last two intervals (income risk fairly large and large), both tax rates are positive, strictly increasing in risk and the nonnegativity constraint on τ^p is not binding. Below the threshold associated with $\hat{\Gamma}$ the payroll tax is either constrained at zero and the tax on capital is rising in income risk and turns from negative to positive at the first threshold characterized in the original proposition 4. Alternatively, there is no constraint, in which case τ^* is unambiguously positive and rising in income risk, and τ^{p^*} is also strictly increasing in income risk but negative for small and fairly small (in the nomenclature of proposition 16) income risk. Finally, we observe that the impact of increased income risk on the tax rate on capital is smaller when the social security contribution rate is free to adjust (i.e. can be negative) than when it is constrained to be nonnegative, see the respective slopes of the two lines.

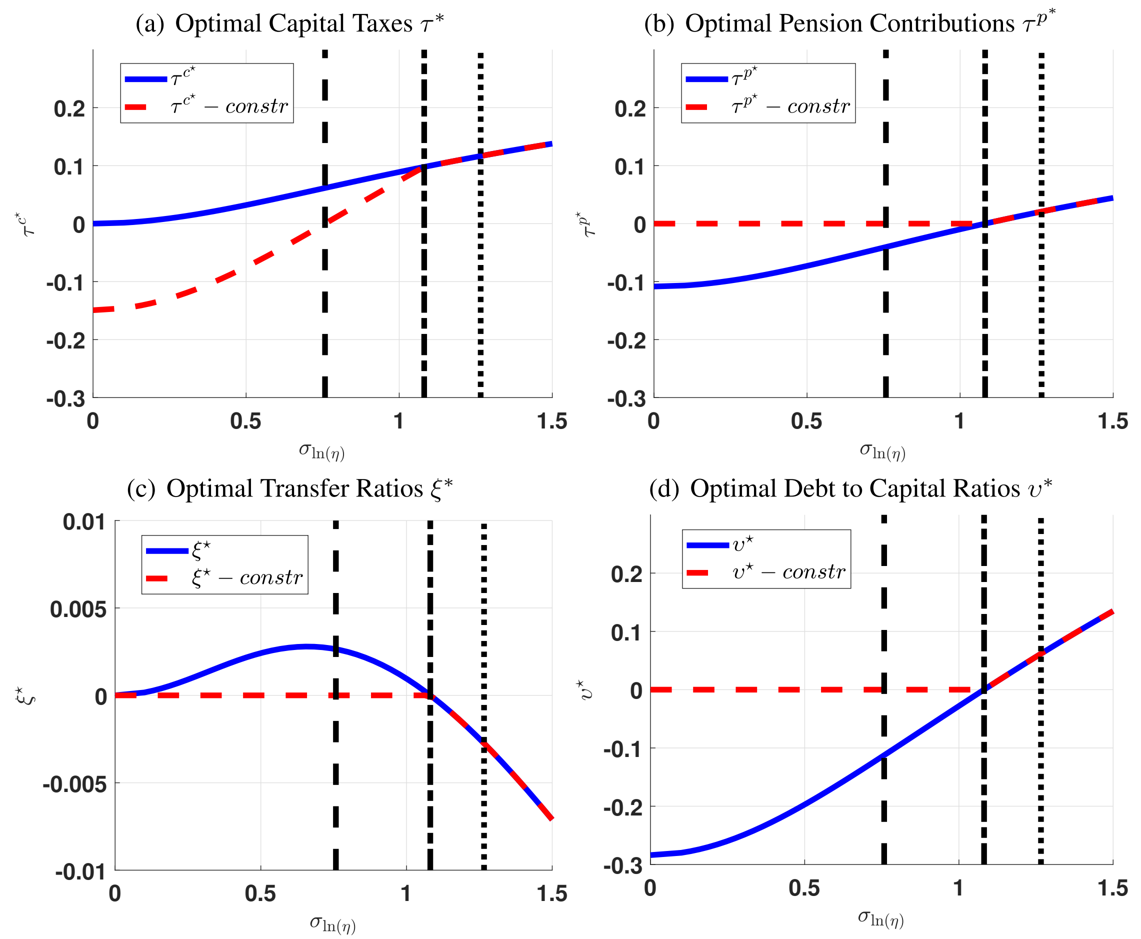
G.2 Equivalence of Social Security and Government Debt

In this subsection we establish equivalence of the optimal Ramsey allocations for a general social discount function $\boldsymbol{\omega}$ when the government has assess to a PAYG social security system analyzed in the previous section, and when, alternatively, it has access to government debt. We first characterize the policy instruments and allocations in the economy with debt and subsequently prove the equivalence by showing that a given allocation implemented by the pension-taxation policy can be implemented by the debt-taxation policy and vice-versa, and by arguing that the solution to the Ramsey maximization problem is unique.

G.2.1 The Economy with Government Debt

In period 0 the initial government debt position in the laissez-faire competitive equilibrium is assumed to be $b_0 = 0$. The government then pays transfers Z_0 to the initial old households and finances these transfers by issuing government debt b_1. In all other periods, the government finances transfers to the old of Z_t with government debt, in addition to the

Figure 3: Optimal Tax on Capital, Optimal Pension Contributions & Optimal Debt Instruments



Notes: Panel (a) shows the optimal capital income tax τ^*, Panel (b) the optimal pension contribution rate τ^{p^*}, Panel (c) the optimal transfer to wage ratio ξ^* (on a different scale than the other panels), and Panel (d) the optimal debt to capital ratio υ^* as a function of income risk, measured by the standard deviation, $\sigma_{\ln\eta}$. Solid blue lines are for the unconstrained solution, red dashed lines for the constrained solution with $\tau^{p^*} \geq 0$, respectively $\upsilon^* \geq 0$. The vertical lines separate the four risk intervals characterized in Proposition 16.

transfers T_t financed by capital income taxes. Thus, the government budget constraint is, in each period,

$$b_{t+1} = \begin{cases} Z_0 & \text{for } t = 0 \\ R_t b_t + T_t - \tau_t R_t a_t + Z_t = R_t b_t + Z_t & \text{for } t > 0, \end{cases}$$

where we maintain the assumption from the benchmark model that the government sets transfers T_t equal to revenues from capital taxation, $T_t = \tau_t R_t a_t$. The household budget constraints in both periods of life read as

$$\begin{aligned} c_t^y + a_{t+1} &= (1-\kappa)w_t \\ c_t^o &= a_t R_t(1-\tau_t) + \kappa\eta w_t + T_t + Z_t \end{aligned}$$

and equilibrium in the asset market now requires that private assets equal the capital stock and the stock of outstanding government debt:

$$a_t = k_t + b_t.$$

G.2.2 Analysis

Define the gross saving rate as

$$\tilde{s}_t = \frac{a_{t+1}}{(1-\kappa)w_t}.$$

Also define the ratio of government debt to the capital stock as $\upsilon_t = \frac{b_t}{k_t}$. With this definition the law of motion of the capital stock in general equilibrium can be derived from

$$a_{t+1} = \tilde{s}_t(1-\alpha)(1-\kappa)k_t^\alpha = k_{t+1} + b_{t+1} = (1+\upsilon_{t+1})k_{t+1}$$

as

$$k_{t+1} = \frac{\tilde{s}_t}{1+\upsilon_{t+1}}(1-\alpha)(1-\kappa)k_t^\alpha. \tag{81}$$

Finally, define the transfer rate $\xi_t = \frac{Z_t}{(1-\kappa)(1-\alpha)k_t^\alpha}$ and use it in the government budget constraint to obtain an alternative representation of the law of motion of the capital stock:

$$\begin{aligned} b_{t+1} &= b_t R_t + Z_t \\ &= b_t \alpha k_t^{\alpha-1} + \xi_t(1-\kappa)(1-\alpha)k_t^\alpha \\ \Leftrightarrow \qquad \upsilon_{t+1}k_{t+1} &= \left[\alpha\upsilon_t + \xi_t(1-\kappa)(1-\alpha)\right]k_t^\alpha \\ \Leftrightarrow \qquad k_{t+1} &= \left[\alpha\frac{\upsilon_t}{\upsilon_{t+1}} + \frac{\xi_t}{\upsilon_{t+1}}(1-\kappa)(1-\alpha)\right]k_t^\alpha. \end{aligned} \tag{82}$$

Comparing (81) to (82) gives the law of motion for the debt to capital ratio $\upsilon_{t+1} = \frac{b_{t+1}}{k_{t+1}}$ as a function of $\upsilon_t, \xi_t, \tilde{s}_t$:

$$
\begin{aligned}
& \frac{\tilde{s}_t}{1+\upsilon_{t+1}}(1-\kappa)(1-\alpha) = \alpha\frac{\upsilon_t}{\upsilon_{t+1}} + \frac{\xi_t}{\upsilon_{t+1}}(1-\kappa)(1-\alpha) \\
\Leftrightarrow \quad & \frac{\upsilon_{t+1}}{1+\upsilon_{t+1}} = \frac{1}{\tilde{s}_t}\left(\xi_t + \frac{\alpha}{(1-\kappa)(1-\alpha)}\upsilon_t\right) \\
\Leftrightarrow \quad & \upsilon_{t+1} = \frac{\xi_t + \frac{\alpha}{(1-\kappa)(1-\alpha)}\upsilon_t}{\tilde{s}_t - \left(\xi_t + \frac{\alpha}{(1-\kappa)(1-\alpha)}\upsilon_t\right)} \qquad (83)
\end{aligned}
$$

Finally, turn to the solution of the household model. Using (81) we can rewrite consumption of young and old households as

$$
\begin{aligned}
c_t^y &= (1-\tilde{s}_t)(1-\alpha)(1-\kappa)k_t^\alpha = \frac{1-\tilde{s}_t}{\tilde{s}_t}(1+\upsilon_{t+1})k_{t+1} \\
c_{t+1}^o &= \left[\alpha + (1-\alpha)\left(\kappa\eta + (1-\kappa)\left(\xi_{t+1} + \frac{\alpha\upsilon_{t+1}}{(1-\alpha)(1-\kappa)}\right)\right)\right]k_{t+1}^\alpha .
\end{aligned}
$$

Using this in the competitive equilibrium household Euler equation with log utility

$$
1 = \alpha\beta k_{t+1}^{\alpha-1}(1-\tau_{t+1})\int \frac{c_t^y}{c_{t+1}^o(\eta)}d\Psi(\eta)
$$

yields

$$
\begin{aligned}
1 &= \alpha\beta k_{t+1}^{\alpha-1}(1-\tau_{t+1})\int \frac{\frac{1-\tilde{s}_t}{\tilde{s}_t}(1+\upsilon_{t+1})k_{t+1}}{\left(\alpha + (1-\alpha)\left(\kappa\eta + \left(\xi_{t+1} + \frac{\alpha\upsilon_{t+1}}{(1-\alpha)(1-\kappa)}\right)\right)\right)k_{t+1}^\alpha}d\Psi(\eta) \\
&= \alpha\beta(1-\tau_{t+1})(1+\upsilon_{t+1})\hat{\Gamma}(\alpha,\kappa,\Psi;\xi_{t+1},\upsilon_{t+1})\frac{1-\tilde{s}_t}{\tilde{s}_t},
\end{aligned}
$$

where the constant summarizing the impact of income risk is now given by

$$
\Gamma(\alpha,\kappa,\Psi;\xi_{t+1},\upsilon_{t+1}) = \int\left[\alpha + (1-\alpha)\left(\kappa\eta + \left(\xi_{t+1} + \frac{\alpha\upsilon_{t+1}}{(1-\alpha)(1-\kappa)}\right)\right)\right]^{-1}d\Psi(\eta) \qquad (84)
$$

and thus the private saving rate is a function of the capital tax rate, the transfer rate and the ratio of debt to the capital stock as well as exogenous parameters:

$$\tilde{s}_t(\tau_{t+1}, \upsilon_{t+1}) = \frac{1}{1 + \left[\alpha\beta(1-\tau_{t+1})(1+\upsilon_{t+1})\Gamma(\alpha, \kappa, \Psi; \xi_{t+1}, \upsilon_{t+1})\right]^{-1}}. \tag{85}$$

Therefore, the implementation result for the tax rate on capital now takes the following form: given the period t debt to capital ratio υ_t and period t choices of the government ξ_t and a saving rate $\tilde{s}_t$, we obtain υ_{t+1} from equation (83) and can thus compute the tax rate τ_{t+1} implementing the private saving in competitive equilibrium from equation (85).

G.2.3 Proof of Equivalence

We first establish that a given allocation implemented by policy instruments of a pension-taxation policy can equivalently be implemented with policy instruments of a debt-taxation policy. Likewise we show that a given allocation implemented by policy instruments of a debt-taxation policy can be implemented with policy instruments of a pension-taxation policy. Formally, this equivalence is stated in the next

Proposition 17. *1. Consider an allocation $\{c_t^y, c_t^o, k_{t+1}\}_{t=0}^{\infty}$ implemented with a pension-taxation policy $\{\tau_t^p, \tau_t\}_{t=0}^{\infty}$ with associated saving rate $\{s_t\}_{t=0}^{\infty}$. The same allocation can be implemented by a debt-taxation policy $\{\xi_t, \tau_t\}_{t=0}^{\infty}$ with associated saving rate and debt to capital ratio $\{\tilde{s}_t, \upsilon_{t+1}\}_{t=0}^{\infty}$.*

2. Consider an allocation $\{c_t^y, c_t^o, k_{t+1}\}_{t=0}^{\infty}$ implemented with a debt-taxation policy $\{\xi_t, \tau_t\}_{t=0}^{\infty}$ with associated saving rate and debt to capital ratio $\{\tilde{s}_t, \upsilon_{t+1}\}_{t=0}^{\infty}$. The same allocation can be implemented by a pension-taxation policy $\{\tau_t^p, \tau_t\}_{t=0}^{\infty}$ with associated saving rate $\{s_t\}_{t=0}^{\infty}$.

Proof. Recall that the allocations and their dependency on policy instruments in the debt-taxation policy are given by

$$c_t^y = (1-\tilde{s}_t)(1-\alpha)(1-\kappa)k_t^\alpha \tag{86a}$$

$$c_t^o = \left[\alpha + (1-\alpha)\left(\kappa\eta + (1-\kappa)\left(\xi_t + \frac{\alpha\upsilon_t}{(1-\alpha)(1-\kappa)}\right)\right)\right]k_t^\alpha \tag{86b}$$

$$k_{t+1} = \frac{\tilde{s}_t}{1+\upsilon_{t+1}}(1-\alpha)(1-\kappa)k_t^\alpha \tag{86c}$$

whereas in the pension-taxation policy they are given by

$$c_t^y = (1 - s_t)(1 - \tau_t^p)(1 - \alpha)(1 - \kappa)k_t^\alpha \tag{87a}$$

$$c_t^o = (\alpha + (1 - \alpha)(\kappa\eta + (1 - \kappa)\tau_t^p))\, k_t^\alpha \tag{87b}$$

$$k_{t+1} = s_t(1 - \tau_t^p)(1 - \kappa)(1 - \alpha)k_t^\alpha. \tag{87c}$$

1. To establish part 1 of the proposition consider the following forward iteration in time from $t = 0, \ldots, \infty$, starting at $k_0, \upsilon_0 = 0$. In any period $t \geq 0$ for a given k_t, υ_t:

   (a) From (86b) and (87b) the consumption allocation of the period t-old implemented by a pension-taxation policy can be equivalently implemented by a debt-taxation policy through

   $$\xi_t = \tau_t^{p^*} - \frac{\alpha \upsilon_t^*}{(1 - \kappa)(1 - \alpha)} \tag{88}$$

   which for $t = 0$ gives $\xi_0 = \tau_0^p$.

   (b) From (86a) and (87a) the consumption allocation of the period t young implemented by a pension-taxation policy can be equivalently implemented by a debt-taxation policy through:

   $$\tilde{s}_t = s_t + \tau_t^p(1 - s_t). \tag{89}$$

   (c) Equivalence implies a path of government debt. In particular, this path can be inferred from the pension-taxation policy by using (88) in (83) to get

   $$\upsilon_{t+1} = \frac{1}{s_t\left(\frac{1}{\tau_t^p} - 1\right)}. \tag{90}$$

   (d) Finally, notice from the households' first-order condition that with substitutions (88)-(90) we obtain

$$
\begin{aligned}
1 &= \beta\alpha(1-\tau_{t+1})\frac{1-\tilde{s}_t}{\tilde{s}_t}(1+\upsilon_{t+1}) \\
&\quad \cdot \int \frac{1}{\alpha + (1-\alpha)\left(\kappa\eta + (1-\kappa)\left(\xi_{t+1} + \upsilon_{t+1}\frac{\alpha}{(1-\alpha)(1-\kappa)}\right)\right)} d\Psi(\eta) \\
&= \beta\alpha(1-\tau_{t+1})\frac{1-\tilde{s}_{t+1}}{\tilde{s}_{t+1}}(1+\upsilon_{t+1})\Gamma(\tau^p_{t+1}) \\
&= \beta\alpha(1-\tau_{t+1})\frac{1-s_{t+1}}{s_{t+1}}\Gamma(\tau^p_{t+1}) \qquad (91)
\end{aligned}
$$

2. To establish part 2 of the proposition we proceed analogously by inverting (89) to obtain τ^p_t, by inverting (88) to obtain s_t, and by using s_t, τ^p_{t+1} in (91) to obtain τ_{t+1}.

□

Finally, we can verify that the evolution of the capital stock is the same under both policies by comparing equations (86c) and (87c) to obtain

$$
\begin{aligned}
& \frac{\tilde{s}_t}{1+\upsilon_{t+1}} = s_t(1-\tau^p_t) \\
\Leftrightarrow \qquad & \upsilon_{t+1} = \frac{\tilde{s}_t}{s_t(1-\tau^p_t)} - 1 = \frac{s_t(1-\tau^p_t)+\tau^p_t}{s_t(1-\tau^p_t)} - 1 = \frac{\tau^p_t}{s_t(1-\tau^p_t)},
\end{aligned}
$$

which is the same as (90).

Therefore, we have thus shown that the same set of allocations can be implemented by either of the two policy instruments. Furthermore, since maximizing the strictly concave objective function (71) subject to either the linear constraints (86) or the linear constraints (87) are convex maximization problems, the respective solutions are unique. Thus, the optimal allocation implemented by the one policy (social security and capital taxes) can be implemented by the respective other policy (government debt and capital taxes). □

G.2.4 Characterization of the Optimal Ramsey Debt-Taxation Policy

We now want to characterize the optimal debt-taxation policy ξ^*_t, τ^*_t with associated optimal saving rate and optimal debt to capital ratio $\tilde{s}^*_t, \upsilon^*_{t+1}$. Observe from (89) and (90) that υ^*_{t+1} and $\tilde{s}^*_t$ are increasing in income risk, because $\tau^{p^*}_t$ is increasing in income risk and because s^*_t is constant in income risk. As a consequence, we see from (88) that it is ambiguous how ξ^*_t

varies with income risk. Also observe from (90) that $\upsilon^*_{t+1} \gtreqless 0$ if $\tau^{p^*}_t \gtreqless 0$ and therefore the qualitative behavior of debt is the same as of the pension payments.

Suppose next that the Ramsey government's discount function is geometric so that $\omega_t = \theta^t$ for some $\theta \in (0,1]$. Notice from (89)–(91) that then the debt policy instruments are constant over time, $\xi^*_t = \xi^*, \tau^*_t = \tau^*, \upsilon^*_{t+1} = \upsilon^*$, because, as established above, the optimal contribution and saving rates in the pension system are constant, $\tau^{p^*}_t = \tau^{p^*}, s^*_t = s^*$.

Finally, consider a Ramsey government maximizing utility in steady state, hence $\theta = 1$. Under this assumption we now characterize how ξ^* varies with income risk. From the analysis of the deterministic economy above recall that

$$\frac{\alpha}{(1-\alpha)(1-\kappa)} = s^*(1 - \tau^{p^*}(\bar{\Psi}))$$

and next use (88) and (90) to rewrite ξ^* as

$$\xi^* = \tau^{p^*}\left(1 - \frac{1-\tau^{p^*}(\bar{\Psi})}{1-\tau^{p^*}}\right). \tag{92}$$

Expressing ξ^* in terms of τ^{p^*} leads us to the following cases concerning the dependence of optimal debt policy on income risk. First, consider the case that the deterministic laissez-faire competitive equilibrium has a capital stock below the golden rule. Then, at $\tau^{p^*}(\bar{\Psi}) < 0$ we have $\xi^* = 0$, and at $\hat{\Gamma}$ (see equation (80), we have $\tau^{p^*}(\hat{\Gamma}) = 0$ and thus $\xi^*(\hat{\Gamma}) = 0$. Furthermore, since for all $\Gamma \in (\bar{\Gamma}, \hat{\Gamma})$ $\tau^{p^*}(\bar{\Psi}) < \tau^{p^*} < 0$ we find that $\xi^* > 0$ for $\Gamma \in (\bar{\Gamma}, \hat{\Gamma})$, whereas for all $\Gamma > \hat{\Gamma}$ we have $\tau^{p^*}(\bar{\Psi}) < 0 < \tau^{p^*}$ and thus $\xi^* < 0$ for $\Gamma > \hat{\Gamma}$. Since in the deterministic economy $\xi^*(\bar{\Psi}) = 0$, the government finances some initial transfers to the period 0 old of Z_0 and then rolls over this debt into the future.

In the stochastic economy, however, the Ramsey government pays additional positive transfers to the period 0 old as long as risk is below the threshold level where social security turns positive and levies lump-sum taxes on the old for risk beyond that threshold. Second, in case the deterministic competitive equilibrium economy has a capital stock already above the golden rule, then for all $\Gamma > \bar{\Gamma}$ the optimal ξ^* is negative and falling in income risk.

G.2.5 Illustration

For a numerical illustration we return to Figure 3. Panels (c) and (d) of this figure show the optimal transfer to wage ratio ξ^* and the optimal debt to capital ratios υ^* as a function of income risk. As with social security, it does so both for the case in which there is a nonnegativity constraint on υ and the case where the debt level is unconstrained.

First turn to Panel (d) which illustrates that the optimal debt to capital ratio has the same properties as the optimal pension contribution rate. If the constraint $\upsilon \geq 0$ does not apply, the debt level is negative as long as income risk is small or fairly small, and it is increasing in risk, turning positive for large income risk. Finally, panel (c) shows the non-monotonicity of the transfer ratio (on a different scale than the other panels of the figure), which is positive if income risk is small and fairly small and (increasingly) negative for fairly large and large income risk.

G.3 Bequest Motive

In this section we provide the detailed analysis of the model with survival risk and warm-glow bequest motives. Assume now that households survive to the second period with probability $\varsigma \in (0,1)$. In the second period of life they receive flow utility from own consumption in case of survival and from bequests, including interest net of taxes, in case of death.

G.3.1 Households

We assume that bequest utility takes the same functional form (log utility) as utility from consumption with utility weight parameter $\varphi > 0$. Accordingly the objective is

$$\max_{c_t^y, c_{t+1}^o, a_{t+1}^o} \ln(c_t^y) + \beta \mathbb{E}_t \left[\varsigma \ln(c_{t+1}^o) + (1-\varsigma)\varphi \ln\left(a_{t+1}^o R_{t+1}(1-\tau_{t+1})\right)\right].$$

and maximization is subject to the per period budget constraints

$$\begin{aligned} c_t^y + a_{t+1}^o &= (1-\kappa)w_t + a_t^y + T_t^y =: x_t \\ c_{t+1}^o &= a_{t+1}^o R_{t+1}(1-\tau_{t+1}) + \frac{\kappa}{\varsigma}\eta w_{t+1} + T_{t+1}^o \end{aligned}$$

where a_t^y are initial assets from warm-glow bequests and x_t is cash in hand of young households. We denote transfers from the government to the young and old by T_t^y, T_{t+1}^o, respectively. We further make the following

Assumption 5. *The total effective utility weight on bequests satisfies*

$$\beta(1-\varsigma)\varphi < 1.$$

G.3.2 Capital Market Equilibrium

Scaling of labor productivity in the second period by ς achieves that labor in the economy again aggregates to one

$$L_t = (1-\kappa) + \frac{\kappa}{\varsigma}\varsigma \int \eta d\Psi(\eta) = 1$$

so that $k_{t+1} = \frac{K_{t+1}}{L_{t+1}} = K_{t+1}$ still applies. The capital market clearing condition reads as

$$K_{t+1} = k_{t+1} = a^o_{t+1} = s_t x_t.$$

where s_t is the private saving rate out of cash in hand.

G.3.3 Bequests

The aggregate amount of bequests distributed to the period $t+1$ young generation is

$$B_{t+1} = (1-\varsigma)a^o_{t+1}R_{t+1}(1-\tau_{t+1}).$$

Since the size of the young population is of measure 1 and they receive all bequests, initial assets of this generation are given by

$$a^y_{t+1} = (1-\varsigma)a^o_{t+1}R_{t+1}(1-\tau_{t+1}).$$

G.3.4 Government

Total government tax revenue is

$$T_{t+1} = a^o_{t+1}R_{t+1}\tau_{t+1}.$$

By assumption, tax revenues are redistributed to the young and old $T_{t+1} = T^y_{t+1} + \varsigma T^o_{t+1}$ according to the rule

$$\begin{aligned} T^y_{t+1} &= (1-\varsigma)a^o_{t+1}R_{t+1}\tau_{t+1} \\ \varsigma T^o_{t+1} &= \varsigma a^o_{t+1}R_{t+1}\tau_{t+1}. \end{aligned}$$

G.3.5 Household Maximization

Recall that cash in hand of young households is defined as $x_t = (1-\kappa)w_t + a_t^y + T_t^y$. Using the budget constraints we can rewrite the objective of a household born in t as

$$\max_{a_{t+1}^o} \ln(x_t - a_{t+1}^o) +$$
$$\beta \left(\mathbb{E}_t \left[\varsigma \ln \left(a_{t+1}^o R_{t+1}(1-\tau_{t+1}) + \frac{\kappa}{\varsigma}\eta w_{t+1} + T_{t+1}^o \right) \right] + (1-\varsigma)\varphi \left[\ln\left(a_{t+1}^o\right) + \ln\left(R_{t+1}(1-\tau_{t+1})\right) \right] \right)$$

and from the last term we observe that τ_{t+1} must be strictly less than one, $\tau_{t+1} < 1$, for the household maximization problem to be well-defined and a competitive equilibrium to exist. The first order condition with respect to a_{t+1}^o is given by

$$-\frac{1}{c_t^y} + \beta \left(\mathbb{E}_t \left[\varsigma \frac{1}{c_{t+1}^o} \right] R_{t+1}(1-\tau_{t+1}) + (1-\varsigma)\varphi \frac{1}{a_{t+1}^o} \right) = 0$$

and thus

$$\begin{aligned}
1 &= \beta \left(\mathbb{E}_t \left[\varsigma \frac{c_t^y R_{t+1}(1-\tau_{t+1})}{c_{t+1}^o} \right] + (1-\varsigma)\varphi \frac{c_t^y}{a_{t+1}^o} \right) \\
&= \beta \left(\mathbb{E}_t \left[\varsigma \frac{((1-s_t)x_t)\, R_{t+1}(1-\tau_{t+1})}{c_{t+1}^o} \right] + (1-\varsigma)\varphi \frac{(1-s_t)x_t}{s_t x_t} \right) \\
&= \beta \left(\mathbb{E}_t \left[\varsigma \frac{((1-s_t)x_t)\, R_{t+1}(1-\tau_{t+1})}{a_{t+1}^o R_{t+1}(1-\tau_{t+1}) + \frac{\kappa}{\varsigma}\eta w_{t+1} + T_{t+1}^o} \right] + (1-\varsigma)\varphi \frac{(1-s_t)}{s_t} \right) \\
&= \beta \left(\mathbb{E}_t \left[\varsigma \frac{((1-s_t)x_t)\, R_{t+1}(1-\tau_{t+1})}{k_{t+1}\alpha k_{t+1}^{\alpha-1} + \frac{\kappa}{\varsigma}\eta(1-\alpha)k_{t+1}^{\alpha}} \right] + (1-\varsigma)\varphi \frac{(1-s_t)}{s_t} \right) \\
&= \beta \left(\mathbb{E}_t \left[\varsigma \frac{\frac{1-s_t}{s_t} k_{t+1}\alpha k_{t+1}^{\alpha-1}(1-\tau_{t+1})}{k_{t+1}\alpha k_{t+1}^{\alpha-1} + \frac{\kappa}{\varsigma}\eta(1-\alpha)k_{t+1}^{\alpha}} \right] + (1-\varsigma)\varphi \frac{(1-s_t)}{s_t} \right) \\
&= \beta \frac{1-s_t}{s_t} \left(\mathbb{E}_t \left[\varsigma \frac{(1-\tau_{t+1})}{\alpha + (1-\alpha)\frac{\kappa}{\varsigma}\eta} \right] + (1-\varsigma)\varphi \right) \\
&= \beta \frac{1-s_t}{s_t} \left(\varsigma(1-\tau_{t+1})\Gamma(\alpha,\kappa,\varsigma,\Psi) + (1-\varsigma)\varphi \right) \\
&= \frac{1-s_t}{s_t} \Lambda(\tau_{t+1},\alpha,\beta,\kappa,\varsigma,\varphi,\Psi)
\end{aligned}$$

and thus

$$s_t = \frac{1}{1 + \Lambda(\cdot)^{-1}},$$

where $\Lambda(\tau_{t+1}, \alpha, \beta, \kappa, \varsigma, \varphi, \Psi) = \beta\left[\varsigma(1 - \tau_{t+1})\Gamma(\alpha, \kappa, \varsigma, \Psi) + (1 - \varsigma)\varphi\right]$ and $\Gamma(\alpha, \kappa, \varsigma, \Psi) = \int \frac{1}{\alpha + (1-\alpha)\frac{\kappa}{\varsigma}\eta} d\Psi(\eta)$. The benchmark model is obtained for $\varsigma = 1$. Therefore, the saving rate is constant $s_t = s$ for all t if and only if the capital income tax is constant $\tau_{t+1} = \tau$ for all t. Furthermore, the comparative statics results from the main paper apply to this extension unchanged, i.e., the competitive equilibrium saving rate s_t increases in income risk, and it falls in the capital tax rate τ_{t+1}. Also note that s_t increases in the bequest utility parameter φ, and since

$$\frac{\partial \Lambda}{\partial \varsigma} = \beta(1 - \tau_{t+1}) \left(\Gamma + \varsigma \underbrace{\frac{\partial \Gamma}{\partial \varsigma}}_{>0} \right) - \beta\varphi$$

the saving rate increases in survival risk ς only if the bequest utility parameter φ is sufficiently low. Otherwise, leaving warm-glow bequests is so valuable, in utility terms, that a higher likelihood of death *increases* savings incentives.

As noted above τ_{t+1} must be strictly less than one for the maximization problem of the household to be well-defined and a competitive equilibrium to exist. This implies a lower bound on the set of implementable saving rates which we can derive from the private household first-order condition, by solving for $1 - \tau_{t+1}$

$$\begin{aligned}
1 &= \beta \frac{1 - s_t}{s_t} \left[\varsigma(1 - \tau_{t+1})\Gamma(\alpha, \kappa, \varsigma, \Psi) + (1 - \varsigma)\varphi\right] \\
\Leftrightarrow \qquad 1 - \tau_{t+1} &= \frac{1}{\varsigma\Gamma} \left(\frac{s_t}{\beta(1 - s_t)} - (1 - \varsigma)\varphi \right) \\
&= \frac{1}{\varsigma\Gamma} \frac{s_t - \beta(1 - \varsigma)\varphi(1 - s_t)}{\beta(1 - s_t)} \\
&= \frac{1}{\varsigma\Gamma} \frac{s_t(1 + \beta(1 - \varsigma)\varphi) - \beta(1 - \varsigma)\varphi}{\beta(1 - s_t)}
\end{aligned}$$

and by next noting that

$$
\begin{aligned}
& 1-\tau_{t+1} > 0 \\
\Leftrightarrow \qquad & \frac{1}{\varsigma\Gamma}\frac{s_t(1+\beta(1-\varsigma)\varphi)-\beta(1-\varsigma)\varphi}{\beta(1-s_t)} > 0 \\
\Leftrightarrow \qquad & s_t > \frac{1}{1+(\beta(1-\varsigma)\varphi)^{-1}}.
\end{aligned}
\tag{93}
$$

Finally, note that to implement a saving rate approaching 1 we require a tax rate

$$
\lim_{s_t\to 1} 1-\tau_{t+1} = \lim_{s_t\to 1}\frac{1}{\varsigma\Gamma}\frac{s_t(1+\beta(1-\varsigma)\varphi)-\beta(1-\varsigma)\varphi}{\beta(1-s_t)} = +\infty
$$

and thus a $\tau_{t+1} = -\infty$ is required. This leads to the following proposition, the counterpart to proposition 1 of the main text:

Proposition 18. *For all $k_t > 0$ and all $\tau_t \in (-\infty, 1)$ the unique saving rate is given by*

$$
s_t = \frac{1}{1+\Lambda(\cdot)^{-1}} \in \left(\frac{1}{1+(\beta(1-\varsigma)\varphi)^{-1}}, 1\right)
$$

where $\Lambda(\tau_{t+1}, \alpha, \beta, \kappa, \varsigma, \varphi, \Psi) = \beta\left[\varsigma(1-\tau_{t+1})\Gamma(\alpha,\kappa,\varsigma,\Psi) + (1-\varsigma)\varphi\right]$ and $\Gamma(\alpha,\kappa,\varsigma,\Psi) = \int \frac{1}{\alpha+(1-\alpha)\frac{\kappa}{\varsigma}\eta} d\Psi(\eta)$.

G.3.6 Maximizing Steady State Utility

In order to obtain a sharp characterization of the optimal solution to the Ramsey problem with warm-glow bequests we focus on the case of $\theta = 1$ in which case the Ramsey government maximizes steady state welfare. To do so, we now rewrite the Ramsey problem in terms of the steady state capital stock $k(s)$, which in turn is determined by the steady state saving rate s. To this purpose note that

$$
\begin{aligned}
a_t^y + T_t^y &= (1-\varsigma)a_t^o R_t(1-\tau_t) + (1-\varsigma)a_t^o R_t \tau_t \\
&= (1-\varsigma)a_t^o R_t
\end{aligned}
$$

and thus in general equilibrium

$$
a_t^y + T_t^y = (1-\varsigma)a_t^o R_t = (1-\varsigma)k_t \alpha k_t^{\alpha-1} = (1-\varsigma)\alpha k_t^{\alpha}
$$

and thus consumption of the young and old in general equilibrium is

$$
\begin{aligned}
c_t^y &= (1-s_t)x_t = (1-s_t)\left((1-\kappa)(1-\alpha)+(1-\varsigma)\alpha\right)k_t^\alpha \\
c_{t+1}^o(\eta) &= \left(\alpha+(1-\alpha)\frac{\kappa}{\varsigma}\eta\right)k_{t+1}^\alpha.
\end{aligned}
$$

Similarly, we can write bequeathed wealth, including net-of-tax interest, as

$$
\begin{aligned}
a_{t+1}^o R_{t+1}(1-\tau_{t+1}) &= k_{t+1}\alpha k_{t+1}^{\alpha-1}(1-\tau_{t+1}) \\
&= \alpha k_{t+1}^\alpha(1-\tau_{t+1}).
\end{aligned}
$$

Recall from the implementation result that

$$
1-\tau_{t+1} = \frac{1}{\varsigma\Gamma}\frac{s_t(1+\beta(1-\varsigma)\varphi)-\beta(1-\varsigma)\varphi}{\beta(1-s_t)}
$$

and thus bequeathed wealth in terms of the saving rate in period t and the capital stock in period $t+1$ is

$$
\begin{aligned}
a_{t+1}^o R_{t+1}(1-\tau_{t+1}) &= \alpha k_{t+1}^\alpha(1-\tau_{t+1}) \\
&= \alpha\frac{1}{\varsigma\Gamma}\frac{s_t(1+\beta(1-\varsigma)\varphi)-\beta(1-\varsigma)\varphi}{\beta(1-s_t)}k_{t+1}^\alpha.
\end{aligned}
$$

From the implementation result in Proposition 18 it follows that $s_t(1+\beta(1-\varsigma)\varphi)-\beta(1-\varsigma)\varphi > 0$ and thus $a_{t+1}^o R_{t+1}(1-\tau_{t+1}) > 0$.

The link between the saving rate and the capital stock is

$$
\begin{aligned}
k_{t+1} &= s_t x_t \\
&= s_t\left((1-\kappa)(1-\alpha)+(1-\varsigma)\alpha\right)k_t^\alpha,
\end{aligned}
$$

and thus the steady state capital stock, as a function of the steady state saving rate s, is

$$
k(s) = \left[s\left((1-\kappa)(1-\alpha)+(1-\varsigma)\alpha\right)\right]^{\frac{1}{1-\alpha}}.
$$

We can then rewrite consumption when young and old and bequeathed wealth in terms of the steady state capital stock and the steady state saving rate s as

$$\begin{aligned}c^y &= (1-s)\left((1-\kappa)(1-\alpha)+(1-\varsigma)\alpha\right)k(s)^\alpha\\ c^o(\eta) &= \left(\alpha+(1-\alpha)\frac{\kappa}{\varsigma}\eta\right)k(s)^\alpha\\ a^oR(1-\tau) &= \alpha\frac{1}{\varsigma\Gamma}\frac{s(1+\beta(1-\varsigma)\varphi)-\beta(1-\varsigma)\varphi}{\beta(1-s)}k(s)^\alpha\end{aligned}$$

The social welfare function maximizing steady state utility in terms of the saving rate s is therefore given by

$$\begin{aligned}W(s) &= \ln(c^y)+\beta\varsigma\int\ln(c^o(\eta))d\Psi(\eta)+\beta(1-\varsigma)\varphi\ln\left(a^oR(1-\tau)\right)\\ &= \ln\left[(1-s)\left((1-\kappa)(1-\alpha)+(1-\varsigma)\alpha\right)k(s)^\alpha\right]+\\ &\quad \beta\varsigma\int\ln\left[\left(\alpha+(1-\alpha)\frac{\kappa}{\varsigma}\eta\right)k(s)^\alpha\right]d\Psi(\eta)+\\ &\quad \beta(1-\varsigma)\varphi\ln\left[\alpha\frac{1}{\varsigma\Gamma}\frac{s(1+\beta(1-\varsigma)\varphi)-\beta(1-\varsigma)\varphi}{\beta(1-s)}k(s)^\alpha\right]\\ &= \Xi+\ln(1-s)+\alpha\ln(k(s))+\alpha\beta\varsigma\ln(k(s))+\\ &\quad \alpha\beta(1-\varsigma)\varphi\ln(k(s))+\beta(1-\varsigma)\varphi\ln(s(1+\beta(1-\varsigma)\varphi)-\beta(1-\varsigma)\varphi)-\beta(1-\varsigma)\varphi\ln(1-s)\\ &= \Xi+(1-\beta(1-\varsigma)\varphi)\ln(1-s)+\alpha\left(1+\beta\left(\varsigma+(1-\varsigma)\varphi\right)\right)\ln(k(s))+\\ &\quad \beta(1-\varsigma)\varphi\ln\left(s(1+\beta(1-\varsigma)\varphi)-\beta(1-\varsigma)\varphi\right)\\ &= \tilde{\Xi}+(1-\beta(1-\varsigma)\varphi)\ln(1-s)+\frac{\alpha\left(1+\beta\left(\varsigma+(1-\varsigma)\varphi\right)\right)}{1-\alpha}\ln(s)+\\ &\quad \beta(1-\varsigma)\varphi\ln\left(s(1+\beta(1-\varsigma)\varphi)-\beta(1-\varsigma)\varphi\right),\end{aligned}$$

for some constants Ξ and $\tilde{\Xi}$. From the last term of the objective function we observe that the optimal saving rate lies in the interval $s\in\left(\frac{1}{1+(\beta(1-\varsigma)\varphi)^{-1}},1\right)$. The first order condition with respect to s is given by

$$f(s) = -\left(1-\beta(1-\varsigma)\varphi\right)\frac{1}{1-s}+\frac{\alpha\left(1+\beta\left(\varsigma+(1-\varsigma)\varphi\right)\right)}{1-\alpha}\frac{1}{s}+\frac{\beta(1-\varsigma)\varphi(1+\beta(1-\varsigma)\varphi)}{s(1+\beta(1-\varsigma)\varphi)-\beta(1-\varsigma)\varphi}=0$$

Using assumption 5 note that $f(s)$ is continuous and $\frac{\partial f(s)}{\partial s}<0$, $\lim_{s\to 1}=-\infty$, as well as $\lim_{s\to\frac{1}{1+(\beta(1-\varsigma)\varphi)^{-1}}}f(s)=\infty$, and thus by the intermediate value there exists a unique solution

$$s^*\left(\alpha,\beta,\kappa,\varsigma,\varphi\right)\in\left(\frac{1}{1+\left(\beta(1-\varsigma)\varphi\right)^{-1}},1\right),$$

which is independent of income risk. Also recall from the implementation result in Proposition 18 that this optimal saving rate can be implemented by some tax rate $\tau^*\in(-\infty,1)$, which is increasing in income risk. Thus our main results from the benchmark model without mortality risk and warm-glow bequests go through qualitatively unchanged, even though we can no longer solve for the optimal Ramsey saving rate in closed form.

Finally, we can establish additional comparative statics results with respect to the new parameter φ measuring the importance of warm-glow bequests. To do so rewrite $f(s)$ as

$$f(s)=-\left(1-\beta(1-\varsigma)\varphi\right)\frac{1}{1-s}+\frac{\alpha\left(1+\beta\left(\varsigma+(1-\varsigma)\varphi\right)\right)}{1-\alpha}\frac{1}{s}+\frac{\beta(1-\varsigma)\varphi}{s-\frac{1}{1+[\beta(1-\varsigma)\varphi]^{-1}}}=0$$

and note that since $s>\frac{1}{1+[\beta(1-\varsigma)\varphi]^{-1}}$ we have $\frac{\partial f(s)}{\partial\varphi}>0$ and thus by the implicit function theorem

$$\frac{\partial s}{\partial\varphi}=-\frac{\frac{\partial f(s)}{\partial\varphi}}{\frac{\partial f(s)}{\partial s}}>0.$$

Therefore, s^* is increasing in the bequest utility weight parameter φ (as is the optimal competitive equilibrium saving rate s^{CE}).

G.4 One-Sided Altruism

Finally, in this subsection we discuss a model where private intergenerational transfers are motivated by one-sided altruism of parents towards their children. Thus, rather than valuing bequests directly in the utility functions parents value the lifetime utility of their children and potentially give bequests in order to raise that lifetime utility. We aim to show that this model shares strong similarities to an Aiyagari (1994) style model with infinitely lived agents facing uninsurable idiosyncratic income risk whose optimal fiscal policy implications are explored by the references cited in the introduction.

To this end, consider an economy that again extends from $t=0,1,\ldots$. Young and old households are intergenerationally linked through one-sided altruism whose strength is governed by the parameter $\delta\geq 0$, which measures the relative weight on the lifetime utility of the offspring in the lifetime utility function of the parental generation.

G.4.1 Budget Constraints

We assume that bequests b_t flow from the current period old to the current period young within the period so that no interest payments accrue on bequests. Since b_t denotes the private transfers received by generation t when young, accordingly b_{t+1} is the private transfer this generation pays as bequest to the currently young in $t+1$. We follow the literature (e.g., Bernheim (1989)) and assume that inter-generational transfers cannot be negative, $b_t \geq 0$. We also augment the model with a standard borrowing constraint. Since now, through intergenerational linkages, we obtain an endogenous wealth distribution (see below), households are no longer ex-ante identical when young and this borrowing constraint is potentially binding for some households. We set the borrowing constraint to $-\bar{A}_t \geq -\bar{A}_t^{NB}$, where $-\bar{A}_t^{NB} = -\frac{\kappa\underline{\eta}}{R_{t+1}(1-\tau_{t+1})}$ is the natural debt limit, with $\underline{\eta} = \min \eta$. The budget constraints for cohort t read as

$$c_t^y + a_{t+1} = (1-\kappa)w_t + b_t = x_t^y \tag{94a}$$

$$c_{t+1}^o + b_{t+1} = a_{t+1}R_{t+1}(1-\tau_{t+1}) + \kappa\eta_{t+1}w_{t+1} + T_{t+1} = x_{t+1}^o \tag{94b}$$

$$a_{t+1} \geq -\bar{A}_t \tag{94c}$$

$$b_{t+1} \geq 0, \tag{94d}$$

where the timing of action is such that consumption c_{t+1}^o and transfers b_{t+1} take place after the income shock η_{t+1} has been realized. We define by x_t^y cash-in-hand of young and by x_t^o cash-in-hand of old households and note that the law of motion for cash-in-hand is given by

$$x_{t+1}^o = \left(x_t^y - c_t^y\right) R_{t+1}(1-\tau_{t+1}) + \kappa\eta_{t+1}w_{t+1} + T_{t+1}. \tag{95}$$

Adding the budget constraints of the young and old households in period t we obtain

$$c_t + a_{t+1} = a_t R_t(1-\tau_t) + \left(1-\kappa+\kappa\eta_t\right) w_t + T_t$$

where $c_t = c_t^y + c_t^o$ is the total consumption of a dynasty in period t. Thus, the budget constraint of the period t dynastic household is equivalent to a standard budget constraint in an Aiyagari (1994) style model with idiosyncratic income risk where the income shock is $\epsilon_t = (1-\kappa) + \kappa\eta_t$.

G.4.2 Preferences

We denote preferences of cohort $t = -1, \ldots, \infty$ by $\tilde{V}_t$. Each cohort t takes as given future cohorts' optimal decision rules $c^y_{t+s}(x^y_{t+s})$, $c^o_{t+s}(x^o_{t+s})$, $a^o_{t+s+1}(x^y_{t+s})$, $b_{t+s+1}(x^o_{t+s+1})$, for $s > t$. Preferences of cohort t are given by

$$\begin{aligned}\tilde{V}_t &= u(c^y_t) + E_t\left[\beta u(c^o_{t+1}) + \delta\left(u(c^y_{t+1}(x^y_{t+1})) + \beta u(c^o_{t+2}(x^o_{t+2}))\right)\right. \\ &\quad \left. + \delta^2\left(u(c^y_{t+2}(x^y_{t+2})) + \beta u(c^o_{t+3}(x^o_{t+3}))\right) + \ldots\right] \\ &= u(c^y_t) + E_t\left[\delta\left(u(c^y_{t+1}(x^y_{t+1})) + \frac{\beta}{\delta}u(c^o_{t+1})\right) + \right. \\ &\quad \left. \delta^2\left(u(c^y_{t+2}(x^y_{t+2})) + \frac{\beta}{\delta}u(c^o_{t+2}(x^o_{t+2}))\right) + \ldots\right], \end{aligned} \tag{96}$$

where expectations in t are taken with respect to the sequence of shocks $\{\eta_s\}_{s=t+1}^{\infty}$. Observe that in any period $s > t$ the relative utility weight between the old and young is $\frac{\beta}{\delta}$.

We assume that the initial old cohort alive in period 0 have the same preferences but its consumption-savings decision at period -1 has already been made and thus remaining per period 0 utility constitutes a constant that cannot be affected by the policy instruments available to the Ramsey government. We spell out the maximization problem of the initial old explicitly in the next subsection.

G.4.3 The Dynastic Competitive Equilibrium

We focus on a sequential competitive equilibrium where in period 0 all dynastic households are identical. This is achieved by setting $\eta_0 = 1$ so that the initial old are (ex-post) identical. Thus, the distribution of cash-in-hand of the old $\Phi^o_0(x_0)$ is degenerate with unit mass at $x^o_0 = a_0 R_0(1-\tau_0) + \kappa\eta_0 w_0 + T_0$, and, by market clearing in the capital market, we have $a_0 = k_0$. As a consequence, initial consumption c^o_0 and initial transfers of the old households to the period 0 young households $b_0 \geq 0$ are singletons, and emerge from maximizing

$$\tilde{V}_{-1} = \left[u(c^y_0(x^y_0)) + \frac{\beta}{\delta}u(c^o_0)\right] + \delta E_0\left[u(c^y_1(x^y_1)) + \frac{\beta}{\delta}u(c^o_1(x^o_1)) + \ldots\right] \tag{97}$$

subject to the constraints

$$c_0^o + b_0 = a_0 R_0(1-\tau_0) + \kappa w_0 + T_0 = x_0^o \quad \text{and} \quad b_0 \geq 0 \tag{98}$$

taking as given future cohorts' optimal decision rules $c_s^y(x_s^y)$, $c_s^o(x_s^o)$, $a_{s+1}^o(x_s^y)$, $b_{s+1}(x_{s+1}^o)$, for all $s > 0$. Since there is no transfer heterogeneity among the initial old, the period 0 young are ex-ante identical and the endogenously determined distribution of the initial young $\Phi_0^y(x_0^y)$ is degenerate, with a unit mass of cash-in hand equal to $x_0^y = (1-\kappa)w_0 + b_0$. Equipped with these initial conditions we set the stage for the social welfare function defined below, which, as in Davila et al. (2012) and the optimal Ramsey policy literature in Bewley-style models, evaluates welfare in a sequential equilibrium from an ex-ante perspective where all households are identical.

For a given policy, a sequential dynastic competitive equilibrium is defined as follows:

Definition 4. *Given the initial condition $k_0 = a_0$, and an associated degenerate initial distribution $\Phi^o(x_0)$ with unit mass at $x_0 = a_0 R_0(1-\tau_0) + \kappa w_0 + T_0$ and a sequence of tax policies $\{\tau_t\}_{t=0}^{\infty}$ a dynastic competitive equilibrium is an allocation $\{c_t^y, c_t^o, L_t, a_{t+1}, b_{t+1}, x_{t+1}, k_{t+1}\}_{t=0}^{\infty}$, cross-sectional measures $\{\{\Phi_t^j(x_t)\}_{j\in\{y,o\}}\}_{t=0}^{\infty}$, prices $\{R_t, w_t\}_{t=0}^{\infty}$ and transfers $\{T_t\}_{t=0}^{\infty}$ such that*

1. *given prices $\{R_t, w_t\}_{t=0}^{\infty}$ and government policies $\{\tau_t, T_t\}_{t=0}^{\infty}$, for each $t \geq 0$,*
   - *(a) consumption, savings and transfer decisions $(c_t^y(x_t^y), c_{t+1}^o(x_{t+1}^o), a_{t+1}(x_t^y), b_{t+1}(x_{t+1}^o))$ maximize (96) subject to (94), and households take as given optimal decision rules at $s > t$, $(c_{t+s}^y(x_{t+s}^y), c_{t+s}^o(x_{t+s}^o), a_{t+s+1}^o(x_{t+s}^y), b_{t+s+1}(x_{t+s+1}^o))$;*
   - *(b) consumption and transfers of the initial old ex-post identical households $(c_0^o(x_0^o), b_0(x_0^o))$ follow from maximizing (97) subject to (98) taking as given future cohorts' optimal decision rules at $s > 0$, $c_s^y(x_s^y), c_s^o(x_s^o), a_{s+1}^o(x_s^y), b_{s+1}(x_{s+1}^o)$;*
2. *prices satisfy equations (3a) and (3b);*
3. *the government budget constraint is satisfied in every period: for all $t \geq 0$*

$$T_t = \tau_t R_t k_t$$

4. *markets clear*

$$\begin{aligned}
L_t &= L = 1 \\
k_{t+1} &= \int a_{t+1}(x_t^y) d\Phi_t^y(x_t^y) \\
C_t^j &= \int c_t^j(x_t^j) d\Phi_t^j(x_t^j), \text{ for } j \in \{y, o\} \\
C_t^y + C_t^o + k_{t+1} &= k_t^\alpha
\end{aligned}$$

5. *the cross sectional measures evolve as*

$$\begin{aligned}
\Phi_{t+1}^o(x_{t+1}^o) &= H^o(\Phi_t^y(x_t^y)) \\
\Phi_t^y(x_t^y) &= H^y(\Phi_t^o(x_t^o)),
\end{aligned}$$

where the law of motion H^o is generated by the cash-in-hand transition (95) and the stochastic i.i.d. shocks $\eta_{t+1} \sim \Psi(\eta_{t+1})$, and the law of motion H^y is generated by the transfer decision $b_t(x_t^o)$.

In a dynastic competitive equilibrium the consumption-savings-transfer problem of any cohort t is solved by backward induction, starting from a final steady state. Denote by $a_{t+2}(x_{t+1}^y(b_{t+1}))$ the savings decision function of cohort $t+1$, for a given amount of transfers b_{t+1}, which we make explicit by writing $x_{t+1}^y(b_{t+1})$. Use (94) in (96) to get

$$\tilde{V}_t = u(x_t^y - a_{t+1}) + \delta E_t \left[u\left(\underbrace{(1-\kappa)w_{t+1} + b_{t+1} - a_{t+2}(x_{t+1}^y(b_{t+1}))}_{=c_{t+1}^y(x_{t+1}^y)} \right) + \frac{\beta}{\delta} u \left(\underbrace{x_{t+1}^o - b_{t+1}}_{=c_{t+1}^o(x_{t+1}^o)} \right) + \dots \right],$$

which shows how a period t cohort influences, through its transfer decision b_{t+1}, the consumption-savings decision of its successor's generation when young $c_{t+1}^y(x_{t+1}(b_{t+1}))$. As in a standard consumption savings model with a borrowing constraint, constraints (94c) and (94d) induce a precautionary savings motive beyond the standard prudence motive, because a binding constraint (94c) in period $t+1$ will reduce c_{t+1}^y, and a binding constraint (94d) will reduce c_{t+1}^o, relative to the optimal interior paths. These occasionally binding constraints, together with the standard prudence argument, will induce households to save more in period t in the presence of idiosyncratic income risk, in turn inducing the pecuniary externality from changing factor prices w_{t+1}, R_{t+1} in general equilibrium emphasized in the main text.

G.4.4 Social Welfare Function

As discussed above, we follow Davila et al. (2012) and evaluate welfare from an ex-ante perspective. Consider a Ramsey government that weighs explicitly the utility of the period zero young and all future generations through Pareto weights $\rho_t \geq 0$ and that—different from Bernheim (1989)—, also puts welfare weight $\rho_{-1} \geq 0$ on the initial old generation's per period utility:

$$SWF = \rho_{-1} u(c_0^o) + \rho_0 \tilde{V}_0 + E_0 \left[\sum_{t=1}^{\infty} \rho_t \tilde{V}_t \right]$$

where we note that each term in the above infinite sum takes the form

$$\begin{aligned} E_0[\tilde{V}_t] &= E_0 \left[u(c_t^y) + E_t \left[\delta \left(u(c_{t+1}^y(x_{t+1}^y)) + \frac{\beta}{\delta} u(c_{t+1}^o) \right) + \delta^2 \left(u(c_{t+2}^y(x_{t+2}^y)) + \frac{\beta}{\delta} u(c_{t+2}^o(x_{t+2}^o)) \right) + \ldots \right] \right] \\ &= E_0 \left[u(c_t^y) + \delta \left(u(c_{t+1}^y(x_{t+1}^y)) + \frac{\beta}{\delta} u(c_{t+1}^o) \right) + \delta^2 \left(u(c_{t+2}^y(x_{t+2}^y)) + \frac{\beta}{\delta} u(c_{t+2}^o(x_{t+2}^o)) \right) + \ldots \right] \end{aligned}$$

by the law of iterated expectations. We can thus rewrite the social welfare function as

$$\begin{aligned} SWF &= \rho_{-1} u(c_0^o) + E_0 \left[\rho_0 \left(u(c_0^y) + \beta u(c_1^o) + \delta \left(u(c_1^y) + \beta u(c_2^o) \right) + \delta^2 \left(u(c_2^y) + \beta u(c_3^o) \right) + \ldots \right) + \right. \\ &\quad \left. + \rho_1 \left(u(c_1^y) + \beta u(c_2^o) + \delta \left(u(c_2^y) + \beta u(c_3^o) \right) + \ldots \right) + \ldots \right] \\ &= \rho_{-1} u(c_0^o) + \rho_0 V_0 + E_0 \left[(\rho_0 \delta + \rho_1) V_1 + \left(\rho_0 \delta^2 + \rho_1 \delta + \rho_2 \right) V_2 + \ldots \right] \\ &= \rho_{-1} u(c_0^o) + \omega_0 V_0 + E_0 \left[\sum_{t=1}^{\infty} \omega_t V_t \right] \end{aligned} \tag{99}$$

V_t is expected lifetime utility of generation t, $\omega_t = \sum_{s=0}^{t} \rho_s \delta^{t-s}$ and $\sum_{t=0}^{\infty} \omega_t < \infty$ is assumed.

First assume that inter-generational transfers are not operational so that $b_t = 0$ in all t. Then all households in all periods t start with zero bequests, are ex-ante identical and the borrowing constraint (94c) is not binding. Since the social welfare function (99) is the same as the one in (4) all results in the main text on the Ramsey optimum can therefore, not surprisingly, be reinterpreted as emerging in a dynastic competitive market economy where intergenerational transfers are not operative.

Now return to the general case where intergenerational transfers are potentially operative so that $b_t \geq 0$ for all $t \geq 0$. Assume first that $\rho_t = 1$ for $t = 0$ and $\rho_t = 0$ for all $t > 0$. Then the Ramsey government maximizes the same objective as the dynastic period 0 household in competitive equilibrium. In this case $\omega_t = \delta^t$ and the social welfare function is recursive, as in the benchmark model of the main text. The Ramsey government internalizes two effects not taken into account by dynastic households making private consumption-saving decisions in the competitive equilibrium. First as in the benchmark model there is a pecuniary externality from increasing savings on the equilibrium wage and interest rate. The increase in the wage raises the stochastic income component in old age, the lower interest rate may lead to increased borrowing (if the substitution and the human capital wealth effect dominate the income effect) and thus more frequently binding borrowing constraints (94c) and (94d). Second, households do not internalize the distributional effects their decisions have, through changing factor prices, on the endogenously evolving wealth distribution. These are precisely the same mechanisms emphasized in Davila et al. (2012)'s study of the constrained planner problem of the Aiyagari (1994) model.

If, in addition $\rho_t > 0$ for $t > 0$ then the Ramsey government puts additional weight on future generations and thus, in addition to these two mechanisms, the future generations effect from the OLG benchmark model of the main text is operative. In this case we can engineer welfare weights ρ_t such that the social welfare function again has a recursive representation, but now with $\omega_t = \theta^t$. Concretely, this construction is given as follows: using that $\omega_t = \sum_{s=0}^{t} \rho_s \delta^{t-s}$ we obtain:

$$\begin{aligned}
&\omega_0 = \theta^0 = 1 = \rho_0 && \Leftrightarrow && \rho_0 = 1 \\
&\omega_1 = \theta = \rho_0 \delta + \rho_1 && \Leftrightarrow && \rho_1 = \theta - \delta \\
&\omega_2 = \theta^2 = \delta^2 + \rho_1 \delta + \rho_2 && \Leftrightarrow && \rho_2 = \theta^2 - \delta^2 - \delta(\theta - \delta) = \theta(\theta - \delta) \\
&\omega_3 = \theta^3 = \delta^3 + \rho_1 \delta^2 + \rho_2 \delta + \rho_3 && \Leftrightarrow && \rho_3 = \theta^3 - \delta^3 - \delta^2(\theta - \delta) - \delta\left(\theta - \delta\theta\right) = \theta^2\left(\theta - \delta\right)
\end{aligned}$$

and thus $\rho_t = 1$, for $t = 0$ and $\rho_t = \theta^{t-1}\left(\delta - \theta\right)$ for $t > 0$. Notice that $\rho_t \geq 0$ for $t > 0$ if and only if $\theta \geq \delta$. Thus, this is a valid social discount function if and only if the planner exhibits weakly more patience than the dynastic household, and we summarize the cases as:

$$\omega_t = \begin{cases} \delta^t & \text{for } \theta = \delta \\ \theta^t & \text{for } \theta > \delta, \end{cases}$$

In the first case the Ramsey government—in addition to valuing the initial old generation through weight ρ_{-1}—shares the same objective as the initial dynastic cohort at $t = 0$. In the second case, the Ramsey government values future generations more heavily and the future generations effect is operative.

To summarize the discussion in this section, we note that in the dynastic model described here the same mechanisms shape optimal Ramsey allocations as they do in the constrained planner problem of Davila et al. (2012) if the Ramsey government weighs only the period 0 utility of the period 0 dynastic household from an ex-ante perspective. If the Ramsey government places additional welfare weights on future generations, then the additional future generations effect from the main text emerges, pushing down the optimal capital income taxes. An analytical solution of this model is infeasible even with log-utility, and a full numerical exploration is left for future research, and, in the case of $\theta = \delta$, such an analysis directly relates to the papers on optimal policy in the Aiyagari (1994)-style models cited in the main text.

H Capital Stock Dynamics and Capital *Income* Taxes

In this appendix we make precise the relation between the capital taxes τ_t studied thus far, and the implied optimal capital *income* taxes τ_t^k. These are related by the equation[43]

$$1 + (R_t - 1)(1 - \tau_t^k) = R_t(1 - \tau_t)$$

and thus

$$\tau_t^k = \frac{R_t}{R_t - 1}\tau_t,$$

where the gross return is given by $R_t = \alpha \left(k_t\right)^{\alpha - 1}$. As long as $R_t > 1$ for all t, capital taxes and capital income taxes have the same sign. To give a sufficient condition for this, note that the saving rate, together with the law of motion for the capital stock

$$k_{t+1} = s_t(1-\kappa)(1-\alpha)k_t^\alpha = \frac{\alpha(\theta+\beta)(1-\kappa)(1-\alpha)}{1+\alpha\beta}k_t^\alpha$$

[43]This equation assumes that the government does not permit the expensing of investment from capital income taxes. Abel (2007) shows that if such expensing is allowed, capital income taxes are nondistortionary (under appropriate ancillary assumptions). Since the Ramsey government optimally distorts the capital accumulation decision of private households in this paper, one implication of our results is that it is not optimal for the government to permit full expensing of investment in our environment (under the maintained other restrictions on the tax instruments).

and the initial condition k_0 determine the entire time path for the capital stock. That sequence $\{k_t\}_{t=1}^{\infty}$ is independent of the amount of income risk and converges monotonically to the steady state

$$k^* = \left[\frac{\alpha(\theta+\beta)(1-\kappa)(1-\alpha)}{1+\alpha\beta}\right]^{\frac{1}{1-\alpha}},$$

either from above if $k_0 > k^*$ or from below, if $k_0 < k^*$. A sufficient condition for $R_t > 1$ for all t can then be given as:

Assumption 6. *The initial capital stock and the model parameters satisfy $k_0 < \alpha^{\frac{1}{1-\alpha}}$ and*

$$\frac{1+\alpha\beta}{(\theta+\beta)(1-\kappa)(1-\alpha)} > 1.$$

This assumption assures that net returns are strictly positive at all times in the Ramsey equilibrium, since $R_0 = \alpha\,(k_0)^{\alpha-1} > 1$ and $R^* = \alpha\,(k^*)^{\alpha-1} > 1$, (and because the sequence of R_t along the transition is monotone) and thus the Ramsey allocation can be supported by capital income taxes of the same sign as the corresponding wealth taxes. Under assumption 6 therefore all interpretations and qualitative results extend without change to capital income taxes.

I Robustness to Other Assumptions

I.1 Idiosyncratic Return Risk

We denote return shocks by ϱ_{t+1} and assume that they are iid. We assume that the cdf of ϱ is given by $\Upsilon(\varrho)$ and denote the corresponding pdf by $\upsilon(\varrho)$. We again assume that a LLN applies so that Υ is both the population distribution of ϱ as well as the individual cdf of return shocks. We make the following

Assumption 7. *The shock ϱ takes positive values Υ-almost surely and*

$$\int \varrho d\Upsilon = 1.$$

Furthermore, shocks η and ϱ are independent[44] *and therefore*

$$\int_{\varrho}\int_{\eta}\varrho\eta d\Upsilon(\varrho)d\Psi(\eta)=\int\varrho d\Upsilon(\varrho)\cdot\int\eta d\Psi(\eta)$$

almost surely.

The budget constraints now write as

$$\begin{aligned}
&a_{t+1}+c_t^y=(1-\kappa)w_t\\
&c_{t+1}^o(\eta,\varrho)=a_{t+1}R_{t+1}\varrho_{t+1}(1-\tau_{t+1})+\eta_{t+1}\kappa w_{t+1}+T_{t+1}(\varrho)
\end{aligned}$$

and we assume that transfer payments are contingent on the rate of return realization,

$$T_{t+1}(\varrho)=a_{t+1}R_{t+1}\varrho_{t+1}\tau_{t+1}.$$

I.1.1 General Equilibrium

Proposition 19. *The structure of the competitive equilibrium is unchanged, but now idiosyncratic risk summarized by Γ is expressed in terms of the distribution $\Pi(\delta_{t+1})$ of the random variable $\delta_{t+1}=\frac{\eta_{t+1}}{\varrho_{t+1}}$ instead of $\Psi(\eta_{t+1})$.*

Proof. The first-order condition for log utility is now

$$\begin{aligned}
1&=\beta R_{t+1}(1-\tau_{t+1})\int\int\varrho_{t+1}\frac{c_t^y}{c_{t+1}^o(\eta)}d\Psi(\eta)d\Upsilon(\varrho)\\
&=\alpha\beta k_{t+1}^{\alpha-1}(1-\tau_{t+1})\int\int\varrho_{t+1}\frac{(1-s_t)(1-\kappa)(1-\alpha)k_t^\alpha}{s_t(1-\kappa)(1-\alpha)k_t^\alpha\alpha k_{t+1}^{\alpha-1}\varrho_{t+1}+\eta_{t+1}\kappa(1-\alpha)k_{t+1}^\alpha}d\Psi(\eta)d\Upsilon(\varrho)\\
&=\alpha\beta(1-\tau_{t+1})\frac{1-s_t}{s_t}\int\int\left(\alpha+\kappa(1-\alpha)\frac{\eta_{t+1}}{\varrho_{t+1}}\right)^{-1}d\Psi(\eta)d\Upsilon(\varrho)\\
&=\alpha\beta(1-\tau_{t+1})\frac{1-s_t}{s_t}\int\left(\alpha+\kappa(1-\alpha)\delta_{t+1}\right)^{-1}d\Pi(\delta)\\
&=\alpha\beta(1-\tau_{t+1})\frac{1-s_t}{s_t}\Gamma(\alpha,\kappa;\delta,\Pi).
\end{aligned}$$

and thus the general equilibrium saving rate is the same as before, with Γ expressed in terms of random variable δ and its cdf $\Pi(\delta)$. □

[44] Independence is assumed for simplicity of notation but can be relaxed for the result.

I.1.2 Ramsey Problem

Proposition 20. *The structure of the optimal Ramsey problem is unchanged, but with idiosyncratic risk now expressed in terms of the random variable* $\delta_{t+1} = \frac{\eta_{t+1}}{\varrho_{t+1}}$ *instead of* η_{t+1}.

Proof. The steps are identical to the ones in the derivation in equation (35). The objective function of the Ramsey government in the current period can be written as

$$\begin{aligned}
W(k) &= \ln\left((1-s)(1-\kappa)(1-\alpha)k^{\alpha}\right) + \beta \int\int \ln\left(\kappa\eta w(s) + R(s)\varrho k'(s)\right) d\Upsilon(\varrho) d\Psi(\eta) \\
&= \ln\left((1-s)(1-\kappa)(1-\alpha)k^{\alpha}\right) + \beta \int\int \ln\left(\varrho\left(\kappa\frac{\eta}{\varrho}(1-\alpha)+\alpha\right)k'(s)\right) d\Upsilon(\varrho) d\Psi(\eta) \\
&= \ln(1-s) + \alpha\beta\ln(s) + (1+\alpha\beta)\ln((1-\kappa)(1-\alpha)) + \alpha(1+\alpha\beta)\ln(k) \\
&+ \beta\int \ln(\varrho) d\Upsilon(\varrho) + \beta\int \ln\left(\kappa\delta(1-\alpha)+\alpha\right) d\Pi(\delta).
\end{aligned}$$

Note that the risk terms in the last line simply add maximization-irrelevant constants to the period objective of the Ramsey government. □

I.2 Ex-Ante Heterogeneity

Permanent productivity is denoted by ν and we assume that the cdf of ν is given by $\Phi(\nu)$. We assume that a LLN applies so that Φ is both the population distribution of permanent productivity ν as well as the ex-ante cdf over ν for each household. We make the following

Assumption 8. *The shock* ν *takes positive values* Φ*-almost surely and*

$$\int \nu d\Phi = 1.$$

Furthermore, shocks η *and* ν *are independent, thus*

$$\int_{\nu}\int_{\eta} \nu\eta d\Phi(\nu) d\Psi(\eta) = \int \nu d\Phi(\nu) \cdot \int \eta d\Psi(\eta) = 1.$$

The budget constraints of each household of productivity type i is now given by

$$\begin{aligned}
a_{t+1}(\nu) + c_t^y(\nu) &= (1-\kappa)\nu w_t \\
c_{t+1}^o(\nu,\eta) &= a_{t+1}R_{t+1}(1-\tau_{t+1}) + \eta_{t+1}\nu\kappa w_{t+1} + T_{t+1}(\nu),
\end{aligned}$$

where

$$T_{t+1}(\nu) = a_{t+1}(\nu) R_{t+1} \tau_{t+1}$$

In all periods t we have $L_t = \int\int ((1-\kappa)\nu + \kappa\nu\eta_t)\, d\Psi(\eta) d\Phi(\nu) = 1$ and thus the capital stock in period $t+1$, K_{t+1}, is equal to the capital intensity $k_{t+1} = \frac{K_{t+1}}{L_{t+1}}$. Denote by

$$s_t(\nu) = \frac{a_{t+1}(\nu)}{(1-\kappa)\nu w_t}$$

the saving rate of household of type ν. The capital intensity in period $t+1$ is then

$$k_{t+1} = \int a_{t+1}(\nu) d\Phi(\nu) = (1-\kappa)(1-\alpha) k_t^{\alpha} \int s_t(\nu) \nu d\Phi(\nu).$$

I.2.1 General Equilibrium

Proposition 21. *The general equilibrium saving rates $s_t(\nu)$ are identical for all agents: $s_t(\nu) = s_t$ for all ν.*

Proof. If $s(\nu) = s_t$ then since $\int \nu d\Phi(\nu) = 1$ the law of motion of the capital stock is

$$k_{t+1} = s_t (1-\kappa)(1-\alpha) k_t^{\alpha}.$$

The first-order condition with log utility of each household is now

$$\begin{aligned}
1 &= \beta R_{t+1}(1-\tau_{t+1}) \int\int \frac{c_t^y(\nu)}{c_{t+1}^o(\eta,\nu)} d\Psi(\eta) d\Phi(\nu) \\
&= \alpha\beta k_{t+1}^{\alpha-1}(1-\tau_{t+1}) \int\int \frac{(1-s_t)\nu(1-\kappa)(1-\alpha)k_t^{\alpha}}{s_t(1-\kappa)(1-\alpha)k_t^{\alpha}\alpha k_{t+1}^{\alpha-1} + \eta_{t+1}\kappa(1-\alpha)k_{t+1}^{\alpha}} d\Psi(\eta) d\Phi(\nu) \\
&= \alpha\beta k_{t+1}^{\alpha-1}(1-\tau_{t+1}) \int \frac{\frac{(1-s_t)}{s_t} k_{t+1}}{k_{t+1}\alpha k_{t+1}^{\alpha-1} + \eta_{t+1}\kappa(1-\alpha)k_{t+1}^{\alpha}} d\Psi(\eta) \\
&= \alpha\beta(1-\tau_{t+1}) \frac{1-s_t}{s_t} \Gamma.
\end{aligned}$$

Thus the optimal saving rate is independent of permanent productivity ν. $\square$

I.2.2 Ramsey Problem

Proposition 22. *Permanent ex-ante heterogeneity in productivity ν does not affect the optimal choice of s.*

Proof. The objective of the Ramsey planner is now given by

$$\begin{aligned} W(k) = \max_{s\in(0,1)} & \int \ln\left((1-s)\nu(1-\kappa)(1-\alpha)k^{\alpha}\right) d\Phi(\nu) + \\ & \beta \int\int \ln\left(\kappa\eta\nu w(s) + R(s)s\nu(1-\kappa)(1-\alpha)k^{\alpha}\right) d\Phi(\nu) d\Psi(\eta), \\ = (1+\beta) & \int \ln(\nu) d\Phi(\nu) + \max_{s\in(0,1)} \ln\left((1-s)(1-\kappa)(1-\alpha)k^{\alpha}\right) + \\ & \beta \int \ln\left(\kappa\eta w(s) + R(s)s(1-\kappa)(1-\alpha)k^{\alpha}\right) d\Psi(\eta) \end{aligned}$$

and thus heterogeneity with respect to ν does not affect the optimization. □

I.3 Time Varying Technological Progress and Population Growth

Denote by A_t the level of technology (labor productivity) and assume that it evolves deterministically according to $A_t = (1+g_t)A_{t-1}$, where the growth rate of technology g_t is allowed to be time-varying. The population growth rate $n \geq 0$ is assumed to be constant over time, so that the size of the young population evolves according to $N_t^y = (1+n)N_{t-1}^y$. With these modifications, aggregate production is

$$Y_t = F(K_t, A_t L_t) = K_t^{\alpha} \left(A_t L_t\right)^{1-\alpha},$$

where L_t is aggregate labor supply given by

$$L_t = (1-\kappa)N_t^y + \kappa N_t^o = \left((1-\kappa)(1+n) + \kappa\right) N_{t-1}^y.$$

Define the capital intensity in terms of efficiency units of labor as $k_t = \frac{K_t}{A_t L_t}$. Then, under the maintained assumption of Cobb-Douglas production, $Y_t = K_t^{\alpha} \left(A_t L_t\right)^{1-\alpha}$ we get $y_t = \frac{Y_t}{A_t L_t} = k_t^{\alpha}$ and thus wages (per effective unit of labor) and interest rates are

$$\begin{aligned} w_t &= (1-\alpha)k_t^{\alpha} A_t \\ R_t &= \alpha k_t^{\alpha-1}. \end{aligned}$$

The law of motion of the capital intensity can be derived as

$$\begin{aligned} K_{t+1} &= a_{t+1}N_t^y = s_t(1-\kappa)(1-\alpha)k_t^\alpha A_t N_t^y \\ \Leftrightarrow \qquad k_{t+1} &= s_t \frac{(1-\kappa)(1-\alpha)}{(1+g_{t+1})\left((1-\kappa)(1+n)+\kappa\right)} k_t^\alpha. \end{aligned}$$

I.3.1 General Equilibrium

Proposition 23. *A time varying rate of technological progress g_t does not affect the saving rate in the competitive general equilibrium, whereas an increase of the constant population growth rate n increases the saving rate.*

Proof. Start from the FOC, equation (6), given by

$$1 = \beta(1-\tau_{t+1}) \int \frac{1-s_t}{s_t(1-\tau_{t+1}) + \frac{\kappa w_{t+1}}{(1-\kappa)w_t R_{t+1}}\eta_{t+1} + \frac{T_{t+1}}{(1-\kappa)w_t R_{t+1}}} d\Psi(\eta_{t+1})$$

and use that

$$\tau_{t+1}s_t = \frac{T_{t+1}}{(1-\kappa)w_t R_{t+1}}$$

to obtain

$$1 = \beta(1-\tau_{t+1}) \int \frac{1-s_t}{s_t + \frac{\kappa w_{t+1}}{(1-\kappa)w_t R_{t+1}}\eta_{t+1}} d\Psi(\eta_{t+1})$$

Next, rewrite $\frac{w_{t+1}}{w_t R_{t+1}}$ as

$$\begin{aligned} \frac{w_{t+1}}{w_t R_{t+1}} &= \frac{k_{t+1}^\alpha A_{t+1}}{k_t^\alpha A_t \alpha k_{t+1}^{\alpha-1}} = (1+g_{t+1})\frac{1}{\alpha}\frac{k_{t+1}}{k_t^\alpha} \\ &= (1+g_{t+1})\frac{1}{\alpha} s_t(1-\kappa)(1-\alpha)\frac{1}{(1+g_{t+1})\left((1-\kappa)(1+n)+\kappa\right)} \\ &= \frac{1}{\alpha} s_t (1-\kappa)(1-\alpha)\frac{1}{(1-\kappa)(1+n)+\kappa}. \end{aligned}$$

Observe that the time varying growth rate g_{t+1} cancels out, and we can rewrite the FOC as

$$\begin{aligned}1 &= \alpha\beta(1-\tau_{t+1})\frac{1-s_t}{s_t}\int \frac{1}{\alpha+\kappa(1-\alpha)\frac{1}{(1-\kappa)(1+n)+\kappa}\eta_{t+1}}d\Psi(\eta_{t+1})\\ &= \alpha\beta(1-\tau_{t+1})\frac{1-s_t}{s_t}\check{\Gamma}.\end{aligned}$$

where $\check{\Gamma} := \int \frac{1}{\alpha+\kappa(1-\alpha)\frac{1}{(1-\kappa)(1+n)+\kappa}\eta_{t+1}}d\Psi(\eta_{t+1})$. □

I.3.2 Ramsey Optimum

Proposition 24. *A time varying rate of technological progress g_t as well as a constant population growth rate n leave the optimal Ramsey saving rate unchanged.*

Proof. With log utility, cohort t lifetime utility is given by

$$\begin{aligned}V_t(k_t,s_t,A_t) &= \ln(A_t)+\ln\left((1-s_t)(1-\kappa)k_t^{\alpha}\right)+\alpha\beta\ln\left((1+g_{t+1})k_{t+1}(s_t)\right)+\beta\ln\left(\Gamma_2\right)\\ &= \ln(A_t)+\alpha\beta\ln(1+g_{t+1})+\tilde{V}_t(k_t,s_t),\end{aligned}$$

where $\Gamma_2 = \int\left((1-\alpha)\kappa\eta_{t+1}+\alpha\right)^{1-\sigma}d\Psi(\eta_{t+1})$. Next, assume that the government maximizes the discounted sum of utility of cohorts t weighted by the population size of that cohort so that the objective is to maximize

$$W_0 = \sum_{t=0}^{\infty}\omega_t N_t^y V_t(k_t,s_t,A_t) = \chi + \sum_{t=0}^{\infty}\omega_t N_t^y \tilde{V}_t(k_t,s_t),$$

where χ is a maximization irrelevant constant. Finally, normalizing $N_0 = 1$ we get

$$W_0 = \sum_{t=0}^{\infty}\tilde{\omega}_t\tilde{V}_t(k_t,s_t)$$

where $\tilde{\omega}_t = \omega_t(1+n)^t$. Also note that

$$k_{t+1}(s_t) = s_t\frac{(1-\kappa)(1-\alpha)}{(1+g_{t+1})\left((1-\kappa)(1+n)+\kappa\right)}k_t^{\alpha}.$$

and thus

$$
\begin{aligned}
\tilde{V}_t(k_t, s_t) &= \ln\left(((1-s_t)(1-\kappa)k_t^\alpha)\right) + \alpha\beta\ln\left(k_{t+1}(s_t)\right) + \beta\ln\left(\Gamma_2\right) \\
&= \ln\left(((1-s_t)(1-\kappa)k_t^\alpha)\right) + \alpha\beta\ln\left(s_t\frac{(1-\alpha)(1-\kappa)}{(1+g_{t+1})\left((1-\kappa)(1+n)+\kappa\right)}k_t^\alpha\right) + \beta\ln\left(\Gamma_2\right) \\
&= \chi_t + \ln\left(1-s_t\right) + \alpha(1+\alpha\beta)\ln\left(k_t\right) + \alpha\beta\ln\left(s_t\right)
\end{aligned}
$$

and thus time varying technological progress and population growth only add a maximization irrelevant (time varying) additive parameter. Also since

$$
\begin{aligned}
\ln(k_{t+1}) &= \ln(1-\alpha) + \ln(1-\kappa) + \alpha\ln(k_t) + \ln(s_t) - \ln\left((1+g_{t+1})\left((1-\kappa)(1+n)+\kappa\right)\right) \\
&= \varkappa_{t+1} + \sum_{\tau=0}^{t}\alpha^\tau\ln(s_{t-\tau}) + \alpha^{t+1}\ln(k_0) \\
&= \tilde{\varkappa}_{t+1} + \sum_{\tau=0}^{t}\alpha^\tau\ln(s_{t-\tau})
\end{aligned}
$$

we can substitute out $\ln\left(k_t\right)$ in the cohort t utility function (as before), which adds additional maximization irrelevant time varying terms. $\square$

I.3.3 The Bounds of Proposition 4 with Technological Progress and Population Growth

We focus on a steady state where the rate of technological progress is a constant g.

Golden Rule. Maximizing steady state utility is equivalent to maximizing per capita consumption. The per capita resource constraint, noticing that in the social planner's optimum $c_t^o(\eta) = c_t^o$, is

$$
\frac{c_t^y N_t^y + c_t^o N_t^o}{N_t} = \frac{F(K_t, L_t) - K_{t+1}}{N_t}.
$$

Now observe that in steady state where $k_{t+1} = k_t = k$ we have

$$\begin{aligned}
N_t^y &= (1+n)N_{t-1}^y, \quad N_t^o = N_{t-1}^y \\
N_t &= N_t^y + N_t^o = (2+n)N_{t-1}^y \\
L_t &= ((1-\kappa)(1+n)+\kappa)\, N_{t-1}^y \\
F(K_t, L_t) &= k^\alpha A_t L_t = k^\alpha A_t \left((1-\kappa)(1+n)+\kappa\right) N_{t-1}^y \\
K_{t+1} &= k A_{t+1} L_{t+1} = k(1+n)(1+g)\left((1-\kappa)(1+n)+\kappa\right) N_{t-1}^y
\end{aligned}$$

and thus maximizing per capita consumption is equivalent to

$$\max_k \{\tilde{c}_t^y(1+n) + \tilde{c}_t^o\} = \max_k \{(k^\alpha - k(1+n)(1+g))\left((1-\kappa)(1+n)+\kappa\right)\}$$

where $\tilde{c}_t = \frac{c_t}{A_t}$ is detrended consumption. The first-order condition gives

$$\alpha k^{\alpha-1} = (1+n)(1+g)$$

and thus the golden-rule capital stock is

$$k^{GR} = \left(\frac{\alpha}{(1+n)(1+g)}\right)^{\frac{1}{1-\alpha}}$$

with the standard intuitive explanation that, with population growth and technological progress, more efficient workers have to be equipped each period with an increasing capital stock to hold constant capital per efficient worker. The golden rule interest rate is thus

$$R^{GR} = \alpha k^{GR^{\alpha-1}} = (1+n)(1+g).$$

Finally, from the law of motion of the capital stock we have

$$k' = s\frac{(1-\kappa)(1-\alpha)}{(1+g)\left((1-\kappa)(1+n)+\kappa\right)}k^\alpha$$

and thus the steady state capital stock for given saving rate is

$$k^* = \left(s^*\frac{(1-\kappa)(1-\alpha)}{(1+g)\left((1-\kappa)(1+n)+\kappa\right)}\right)^{\frac{1}{1-\alpha}}.$$

Setting $k^* = k^{GR}$ then gives the golden rule saving rate as

$$s^{GR} = \frac{\alpha\left((1-\kappa)(1+n)+\kappa\right)}{(1-\kappa)(1-\alpha)(1+n)}.$$

Competitive Equilibrium and Overaccumulation of Capital Since $R^* = \alpha k^{*\alpha-1}$, the steady state interest rate for given saving rate is

$$R^* = \frac{\alpha(1+g)\left((1-\kappa)(1+n)+\kappa\right)}{s^*(1-\kappa)(1-\alpha)}.$$

Now use that

$$s^* = \frac{(1-\tau)\alpha\beta\check{\Gamma}}{1+(1-\tau)\alpha\beta\check{\Gamma}},$$

as defined above, to get

$$R^*(\tau,\check{\Gamma}) = \frac{(1+g)\left((1-\kappa)(1+n)+\kappa\right)}{(1-\kappa)(1-\alpha)}\left(\alpha + \frac{1}{(1-\tau)\beta\check{\Gamma}}\right)$$

and thus in the laissez-faire steady state we have

$$R^*(\tau=0,\check{\Gamma}) = \frac{(1+g)\left((1-\kappa)(1+n)+\kappa\right)}{(1-\kappa)(1-\alpha)}\left(\alpha + \frac{1}{\beta\check{\Gamma}}\right).$$

Since the laissez-faire equilibrium economy has overaccumulated capital if $R^*(\tau=0,\check{\Gamma}) < (1+n)(1+g)$ we obtain overaccumulation if

$$\begin{aligned}
& \frac{(1+g)\left((1-\kappa)(1+n)+\kappa\right)}{(1-\kappa)(1-\alpha)}\left(\alpha + \frac{1}{\beta\check{\Gamma}}\right) < (1+n)(1+g) \\
\Leftrightarrow \qquad & \beta > \frac{1}{\left(\frac{(1-\kappa)(1-\alpha)(1+n)}{(1-\kappa)(1+n)+\kappa} - \alpha\right)\check{\Gamma}}
\end{aligned}$$

Recall that

$$\check{\Gamma} = \int \frac{1}{\alpha + \kappa(1-\alpha)\frac{1}{(1-\kappa)(1+n)+\kappa}\eta_{t+1}} d\Psi(\eta_{t+1}).$$

and thus in the deterministic economy we have

$$\bar{\bar{\check{\Gamma}}} = \frac{1}{\alpha + \kappa(1-\alpha)\frac{1}{(1-\kappa)(1+n)+\kappa}}$$

Now rewrite the bound on β above to get

$$\begin{aligned}
\beta &> \frac{1}{\left(\frac{(1-\kappa)(1-\alpha)(1+n)}{(1-\kappa)(1+n)+\kappa} - \alpha\right)\check{\Gamma}} \\
&= \frac{1}{\left(\frac{-\kappa(1-\alpha)+(1-\alpha)(1+n)-\kappa(1-\alpha)n}{(1-\kappa)(1+n)+\kappa} - \alpha\right)\check{\Gamma}} \\
&= \frac{1}{\left(\frac{(1-\alpha)(1+n(1-\kappa))}{1+n(1-\kappa)} - \left(\alpha + \frac{\kappa(1-\alpha)}{(1-\kappa)(1+n)+\kappa}\right)\right)\check{\Gamma}} \\
&= \frac{1}{\left((1-\alpha)\check{\Gamma} - \check{\Gamma}/\bar{\bar{\check{\Gamma}}}\right)} := \Theta_1\left(\check{\Gamma}, \bar{\bar{\check{\Gamma}}}\right)
\end{aligned}$$

Since the structure of the Ramsey problem has not changed, we continue to find that the optimal saving rate for $\theta = 1$ is

$$s^* = \frac{\alpha(1+\beta)}{1+\alpha\beta}$$

and thus the tax rate implementing it satisfies

$$1 - \tau = \frac{1+\beta}{(1-\alpha)\beta\check{\Gamma}}$$

and thus we have $\tau > 0$, if and only if

$$\frac{1+\beta}{(1-\alpha)\beta\check{\Gamma}} < 1$$

or if and only if

$$\Theta_2\left(\check{\Gamma}\right) := \frac{1}{(1-\alpha)\check{\Gamma} - 1} < \beta.$$

Stating the inequalities in terms of $\check{\Gamma}$ the regions corresponding to Proposition 4 become

1. $\check{\Gamma} > \frac{1}{\left((1-\alpha)-1/\bar{\bar{\check{\Gamma}}}\right)\beta}$: $k > k^{GR}, \tau > 0$

2. $\check{\Gamma} \in \left(\frac{1+\beta}{(1-\alpha)\beta}, \frac{1}{\left((1-\alpha)-1/\bar{\bar{\Gamma}}\right)\beta}\right]$: $k \leq k^{GR}, \tau > 0$

3. $\check{\Gamma} \in \left(\bar{\bar{\Gamma}}, \frac{1+\beta}{(1-\alpha)\beta}\right]$: $k \leq k^{GR} \tau < 0$

Recall that

$$\bar{\bar{\Gamma}} = \frac{1}{\alpha + \kappa(1-\alpha)\frac{1}{(1-\kappa)(1+n)+\kappa}}$$

and thus an increase of n increases $\bar{\bar{\Gamma}}$ increasing the lower bound of the third interval. By increasing $\bar{\bar{\Gamma}}$ it also reduces $\frac{1}{\left((1-\alpha)-1/\bar{\bar{\Gamma}}\right)\beta}$ and thus the interesting interval (the case 2 of intermediate risk) gets smaller. Finally, positive population growth reduces the sensitivity of $\check{\Gamma}$ with respect to increasing risk.

J General Intertemporal Elasticity of Substitution ρ and Risk Aversion σ

In this appendix we provide the detailed analysis of a more general utility function with intertemporal elasticity of substitution ρ and risk aversion σ summarized in Section 6.3.6 of the main text. Most of the analysis focuses on *steady states*, but we establish that our closed form results for the entire *transition* go through unchanged for an IES $\rho = 1$. We first characterize the competitive equilibrium for a given tax policy, prior to stating and analyzing the Ramsey problem.

J.1 Competitive Equilibrium for Given Tax Policy

The Euler equation with the more general utility function now reads as:

$$1 = \beta(1-\tau_{t+1})R_{t+1}\left[\int\left(\frac{c^o_{t+1}(\eta_{t+1})}{c^y_t}\right)^{1-\sigma} d\Psi(\eta_{t+1})\right]^{\frac{\sigma-\frac{1}{\rho}}{1-\sigma}} \int\left(\frac{c^o_{t+1}(\eta_{t+1})}{c^y_t}\right)^{-\sigma} d\Psi(\eta_{t+1}).$$

and, using the expressions for consumption in both periods and the law of motion of the capital stock, as in the previous analysis we can rewrite the first-order condition as

$$1 \quad = \quad \alpha\beta\left((1-\kappa)(1-\alpha)\right)^{(\alpha-1)\left(1-\frac{1}{\rho}\right)}(1-\tau_{t+1})k_t^{\alpha(\alpha-1)\left(1-\frac{1}{\rho}\right)} {s_t}^{(\alpha-1)\left(1-\frac{1}{\rho}\right)}\left(\frac{1-s_t}{s_t}\right)^{\frac{1}{\rho}}\tilde{\Gamma}.$$

In steady state the Euler equation reads as

$$1 = \alpha\beta\left((1-\kappa)(1-\alpha)\right)^{(\alpha-1)\left(1-\frac{1}{\rho}\right)}(1-\tau)k^{\alpha(\alpha-1)\left(1-\frac{1}{\rho}\right)}s^{(\alpha-1)\left(1-\frac{1}{\rho}\right)}\left(\frac{1-s}{s}\right)^{\frac{1}{\rho}}\tilde{\Gamma}$$

where

$$k = \left[(1-\kappa)(1-\alpha)s\right]^{\frac{1}{1-\alpha}} \tag{100}$$

is the steady state capital stock and the constant $\tilde{\Gamma}$ is given by

$$\tilde{\Gamma} = v^{\left(\sigma-\frac{1}{\rho}\right)}\int\left(\kappa\eta_{t+1}(1-\alpha)+\alpha\right)^{-\sigma}d\Psi(\eta_{t+1}) \tag{101}$$

and v is the certainty equivalent of η defined as

$$v = \begin{cases}\left[\int\left(\alpha+(1-\alpha)\kappa\eta\right)^{1-\sigma}d\Psi(\eta)\right]^{\frac{1}{1-\sigma}} & \text{for } \sigma = 1\\ \exp\left(\int\ln\left(\alpha+(1-\alpha)\kappa\eta\right)d\Psi(\eta)\right) & \text{otherwise.}\end{cases} \tag{102}$$

Inserting the steady state capital from equation (100) into the Euler equation delivers

$$1 = (1-\tau)\alpha\beta\left((1-\kappa)(1-\alpha)\right)^{\left(\frac{1}{\rho}-1\right)}\frac{(1-s)^{\frac{1}{\rho}}}{s}\tilde{\Gamma}. \tag{103}$$

This result is the generalization of the log-case where $\rho = \sigma = 1$, and where the steady state Euler equation was given as

$$1 \quad = \quad (1-\tau)\alpha\beta\left(\frac{1-s}{s}\right)\Gamma$$

Thus our previous analysis for log-utility is just a special case. Also note that if $\rho = 1$ but $\sigma \neq 1$, then the steady state Euler equation is given by

$$1 = (1-\tau)\alpha\beta\left(\frac{1-s}{s}\right)^{\frac{1}{\rho}}\tilde{\Gamma}$$

but the risk factor $\tilde{\Gamma}$ with $\sigma \neq 1$ differs from the risk factor Γ with $\sigma = 1$.

$$\tilde{\Gamma} = \frac{\int\left(\alpha+(1-\alpha)\kappa\eta\right)^{-\sigma}d\Psi(\eta)}{\int\left(\alpha+(1-\alpha)\kappa\eta\right)^{1-\sigma}d\Psi(\eta)} \neq \int\left(\kappa\eta(1-\alpha)+\alpha\right)^{-1}d\Psi(\eta) = \Gamma$$

J.1.1 Precautionary Savings Behavior in the Competitive Equilibrium

In order to aid with the interpretation of the optimal Ramsey tax rate it is useful to establish conditions under which, for a fixed tax rate, the saving rate in competitive equilibrium is increasing in income risk.

Proposition 25. *If $\tilde{\Gamma}$ is strictly increasing in income risk, then for any given tax rate $\tau \in (-\infty, 1)$ the steady state saving rate $s^{CE}(\tau)$ in competitive equilibrium is strictly increasing in income risk. If $\tilde{\Gamma}$ is strictly decreasing in income risk, then so is $s^{CE}(\tau)$.*

Proof. Rewrite equation (103) as

$$f(s) = (1-\tau)\alpha\beta\left((1-\kappa)(1-\alpha)\right)^{\left(\frac{1}{\rho}-1\right)} \frac{(1-s)^{\frac{1}{\rho}}}{s} - \frac{1}{\tilde{\Gamma}}.$$

Then a saving rate $s^{CE}(\tau)$ that satisfies $f(s^{CE}(\tau)) = 0$ is a steady state equilibrium saving rate. We readily observe that f is continuous and strictly decreasing in s, with

$$\begin{aligned} \lim_{s \to 0} f(s) &= \infty \\ f(1) &= -\frac{1}{\tilde{\Gamma}} < 0 \end{aligned}$$

and thus for each $\tau \in (-\infty, 1)$ there is a unique $s = s^{CE}(\tau)$ that satisfies $f(s^{CE}(\tau)) = 0$. Inspection of f immediately reveals that $s^{CE}(\tau)$ is strictly increasing in $\tilde{\Gamma}$, from which the comparative statics results follow. □

Corollary 10. *For any given $\tau \in (-\infty, 1)$, the steady state saving rate $s^{CE}(\tau)$ increases in income risk if either $\rho \leq 1$, or $1 < \rho < \frac{1}{\sigma}$.*

Proof. Follows directly from the previous proposition and Lemma 2 in the main text (and proved in the next section) characterizing the behavior of $\tilde{\Gamma}$ with respect to income risk. □

Proposition 25 establishes a sufficient condition for the private saving rate to increase in income risk. But, for $\rho > \frac{1}{\sigma} > 1$ it is possible that the combination of individual savings behavior and general equilibrium factor price movements lead to the result that, for *fixed* government policy, the equilibrium saving rate is decreasing in income risk.[45] We will

[45] Also observe that a parameter constellation $1 < \rho < \frac{1}{\sigma}$ pairs a high IES with a preference for a *late* resolution of risk in a multi-period (more than two periods) model. Interestingly, the competitive equilibrium saving rate may therefore decrease in income risk precisely when we pair a high IES with a preference constellation for *early* resolution of risk.

show below that this in turn is a necessary condition for the optimal Ramsey tax rate to decrease in income risk.

J.2 The Ramsey Problem

As in Section 4, equation (12), we can write lifetime utility of a generation born in period t, in general equilibrium, as a function of the beginning of the period capital stock k_t and the saving rate s_t chosen by the Ramsey government and implemented by the appropriate choice of the capital tax τ_{t+1}.

J.2.1 Ramsey Problem for Unit IES

Now we use the formulation of lifetime utility in equation (33). Then it is straightforward to show that for $\rho = 1$ the analysis of the Ramsey problem proceeds exactly as for log utility ($\rho = \sigma = 1$), by making the problem recursive and using the method of undetermined coefficients:

$$
\begin{aligned}
W(k) &= \Theta_0 + \Theta_1 \ln(k) \\
&= \max_{s\in[0,1]} \Big\{ \ln((1-s)(1-\kappa)\,(1-\alpha)\,k^{\alpha}) \\
&\quad + \frac{\beta}{1-\sigma} \ln \int (\kappa\eta w(s) + R(s)s(1-\kappa)(1-\alpha)k^{\alpha})^{1-\sigma}\, d\Psi(\eta) + \theta W(k') \Big\} \\
&= \max_{s\in[0,1]} \Big\{ \ln((1-s)(1-\kappa)\,(1-\alpha)\,k^{\alpha}) \\
&\quad + \frac{\beta}{1-\sigma} \ln \int ([\kappa\eta(1-\alpha)+\alpha]\,[s(1-\kappa)(1-\alpha)k^{\alpha}]^{\alpha})^{1-\sigma}\, d\Psi(\eta) + \theta W(s(1-\kappa)(1-\alpha)k^{\alpha}) \Big\} \\
&= \alpha\,[1+\theta\Theta_1+\alpha\beta]\ln(k) + \ln[(1-\kappa)\,(1-\alpha)] + \theta\Theta_0 + \theta\Theta_1 \ln((1-\kappa)(1-\alpha)) \\
&\quad + \beta\alpha \ln[(1-\kappa)(1-\alpha)] + \frac{\beta \ln \int [\kappa\eta(1-\alpha)+\alpha]^{1-\sigma}\, d\Psi(\eta)}{1-\sigma} \\
&\quad + \max_{s\in[0,1]} \{\ln(1-s) + \alpha\beta \ln(s) + \theta\Theta_1 \ln(s)\}
\end{aligned}
$$

As in Appendix B.2, comparing the terms involving k gives the constant $\Theta_1 = \frac{\alpha(1+\alpha\beta)}{(1-\alpha\theta)}$, and taking the first order condition with respect to s and solving it delivers the optimal saving rate as stated in the main text:

$$s = \frac{\alpha(\beta+\theta)}{1+\alpha\beta}.$$

This result clarifies that the closed form solution, and the fact that the optimal saving rate is constant over time and independent of the level of capital, is driven by the assumption that $IES = \rho = 1$ (and obtained for arbitrary risk aversion), whereas the size of the capital tax needed to implement the optimal Ramsey allocation evidently does depend on risk aversion σ, see Section J.1.

J.2.2 Steady State Analysis of Ramsey Problem for Arbitrary IES $\rho \neq 1$

The Ramsey government maximizing steady state lifetime utility has the objective function:

$$
\begin{aligned}
V(s) &= \frac{(c_t^y)^{1-\frac{1}{\rho}} + \beta \left\{ \left[\int c_{t+1}^o (\eta_{t+1})^{1-\sigma} d\Psi \right]^{\frac{1}{1-\sigma}} \right\}^{1-\frac{1}{\rho}}}{1-\frac{1}{\rho}} \\
&= \frac{((1-\kappa)(1-s)(1-\alpha)k^{\alpha})^{1-\frac{`1}{\rho}}}{1-\frac{1}{\rho}} \\
&\quad + \frac{\beta \left[s(1-\kappa)(1-\alpha)k^{\alpha} \right]^{\alpha\left(1-\frac{1}{\rho}\right)} \left\{ \left[\int \left\{ [\kappa\eta(1-\alpha)+\alpha] \right\}^{1-\sigma} d\Psi \right]^{\frac{1}{1-\sigma}} \right\}^{1-\frac{1}{\rho}}}{1-\frac{1}{\rho}} \\
&= \frac{((1-\kappa)(1-\alpha))^{1-\frac{`1}{\rho}}}{1-\frac{1}{\rho}} (1-s)^{\left(1-\frac{1}{\rho}\right)} k^{\alpha\left(1-\frac{1}{\rho}\right)} + \frac{\beta \left[(1-\kappa)(1-\alpha)\right]^{\alpha\left(1-\frac{1}{\rho}\right)} \tilde{\Gamma}_2}{1-\frac{1}{\rho}} s^{\alpha\left(1-\frac{1}{\rho}\right)} k^{\alpha^2\left(1-\frac{1}{\rho}\right)}
\end{aligned}
$$

where

$$
\tilde{\Gamma}_2 = \left[\int \left\{ [\kappa\eta(1-\alpha)+\alpha] \right\}^{1-\sigma} d\Psi \right]^{\frac{1-\frac{1}{\rho}}{1-\sigma}}.
$$

Exploiting that in steady state

$$
k = ((1-\kappa)(1-\alpha)s)^{\frac{1}{1-\alpha}}
$$

yields

$$
\begin{aligned}
V(s) &= \frac{((1-\kappa)(1-\alpha))^{1-\frac{`1}{\rho}}}{1-\frac{1}{\rho}} (1-s)^{\left(1-\frac{1}{\rho}\right)} ((1-\kappa)(1-\alpha)s)^{\frac{\alpha\left(1-\frac{1}{\rho}\right)}{1-\alpha}} \\
&\quad + \frac{\beta \left[(1-\kappa)(1-\alpha)\right]^{\alpha\left(1-\frac{1}{\rho}\right)} \tilde{\Gamma}_2}{1-\frac{1}{\rho}} (s)^{\alpha\left(1-\frac{1}{\rho}\right)} ((1-\kappa)(1-\alpha)s)^{\frac{\alpha^2\left(1-\frac{1}{\rho}\right)}{1-\alpha}} \\
&= \tilde{\phi} \left((1-s)^{\left(1-\frac{1}{\rho}\right)} + \beta\tilde{\zeta}\tilde{\Gamma}_2 \right) s^{\frac{\alpha\left(1-\frac{1}{\rho}\right)}{1-\alpha}},
\end{aligned}
$$

where

$$
\begin{aligned}
\tilde{\phi} &= \frac{((1-\kappa)(1-\alpha))^{\frac{1-\frac{1}{\rho}}{1-\alpha}}}{1-\frac{1}{\rho}} \\
\tilde{\zeta} &= \left(\frac{1}{(1-\kappa)(1-\alpha)}\right)^{(1-\frac{1}{\rho})} > 0 \\
\tilde{\Gamma}_2 &= \left(\left[\int \{[\kappa\eta(1-\alpha)+\alpha]\}^{1-\sigma}\, d\Psi\right]^{\frac{1}{1-\sigma}}\right)^{1-\frac{1}{\rho}} > 0.
\end{aligned}
$$

Thus the steady state analysis in the main text carries through to Epstein-Zin-Weil utility mostly unchanged, but with the constant that maps earnings risk into the optimal saving rate now being affected both by risk aversion and the IES.

Taking the first order condition, the optimal steady state saving rate is defined implicitly by the equation

$$
\frac{s}{(1-s)^{\frac{1}{\rho}}} = \frac{\alpha}{1-\alpha}(1-s)^{\left(1-\frac{1}{\rho}\right)} + \beta\frac{\alpha}{1-\alpha}\tilde{\zeta}\tilde{\Gamma}_2 \tag{104}
$$

and rewriting this equation yields

$$
LHS(s) = s = \frac{\alpha}{1-\alpha}\left[(1-s) + \beta\tilde{\zeta}\tilde{\Gamma}_2(1-s)^{\frac{1}{\rho}}\right] = RHS(s). \tag{105}
$$

We observe that the left hand side is linearly increasing in s, with $LHS(0) = 0$ and $LHS(1) = 1$ and the right hand side is strictly decreasing in s, with $RHS(0) > 0$ and $RHS(1) = 0$. Since both sides are continuous in s, from the intermediate value theorem it follows that there is a unique $s^* \in (0,1)$ solving the first order condition of the Ramsey problem (105). Since $RHS(s)$ is strictly increasing in $\tilde{\Gamma}_2$, the Ramsey saving rate is strictly increasing in $\tilde{\Gamma}_2$. We then have

Proposition 26. *Suppose that $\theta = 1$ and thus the Ramsey government maximizes steady state welfare. There exists a unique optimal Ramsey saving rate $s^* \in (0,1)$ solving equation (105). This saving rate is strictly increasing in the risk constant $\tilde{\Gamma}_2$ and can be implemented with a capital tax rate τ^* determined by the competitive equilibrium Euler equation:*

$$
1 = (1-\tau^*)\alpha\beta\left((1-\kappa)(1-\alpha)\right)^{\left(\frac{1}{\rho}-1\right)}\frac{(1-s^*)^{\frac{1}{\rho}}}{s^*}\tilde{\Gamma}. \tag{106}
$$

For future reference we rewrite equation (105) as

$$\frac{(1-s)^{\frac{1}{\rho}}}{s} = \frac{\frac{1-\alpha}{\alpha} - \frac{(1-s)}{s}}{\beta\tilde{\zeta}\tilde{\tilde{\Gamma}}_2} = \frac{\frac{1}{\alpha} - \frac{1}{s}}{\beta\tilde{\zeta}\tilde{\tilde{\Gamma}}_2}. \tag{107}$$

J.2.3 Comparative Statics with Respect to Income Risk

In Appendix K.1 we prove the following result relating the extent of income risk to the constants $\tilde{\Gamma}, \tilde{\Gamma}_2$ which are in turn crucial for determining comparative statics results.

Lemma 2. *An increase in income risk (a mean-preserving spread of η) increases $\tilde{\Gamma}_2$ if and only if $\rho \leq 1$ and increases $\tilde{\Gamma}$ if $\rho \leq 1$, or $\rho > 1$ and $\sigma < 1/\rho$.*

Note that the condition that characterizes the relation between income risk and $\tilde{\Gamma}_2$ is necessary and sufficient whereas the two alternative conditions that characterize the relation between income risk and $\tilde{\Gamma}$ are only sufficient.[46] We provide further intuition for this result below when discussing implementation of the optimal Ramsey policy. We now derive the comparative statics of s^* and τ^* with respect to income risk discussed in the main text.

Risk and the Optimal Saving Rate The comparative static results of the steady state Ramsey saving rate with respect to income risk is stated in the next:

Proposition 27. *An increase in income risk increases the optimal steady state Ramsey saving rate s^* if and only if $\rho < 1$ and decreases it if and only if $\rho > 1$.*

The proof of this result follows directly from Lemma 2 and Proposition 26. Thus the direction of the change in s with respect to income risk is exclusively determined by the IES ρ, with the log-case acting as a watershed. Of course how strongly the saving rate responds to an increase in income risk is also controlled by risk aversion through the term $\tilde{\Gamma}_2$. What is the intuition for this result? Suppose the economy is in the steady state associated with a given extent of income risk and the optimal Ramsey tax policy, and now consider an increase in income risk. The Ramsey government can always neutralize the response of private households' savings behavior, by appropriate adjustment of the tax rate on capital

[46]The dependency of precautionary savings on both risk aversion and the IES with recursive preferences was demonstrated by Kimball and Weil (2009), and the sufficient conditions provided in the Lemma are stated in their Propositions 5 and 6.

to implement the new optimal saving rate.[47]

The question is then how the saving rate desired by the Ramsey government itself changes. Households (and thus the Ramsey government) obtain utility from safe consumption when young and risky consumption when old, and the desire for smoothing utility from safe consumption when young and the certainty equivalent of consumption when old is determined by the IES ρ. As risk increases, old age consumption is now a less effective way to generate utility, and the certainty equivalent of old-age consumption declines, holding the consumption allocation constant. Whether the Ramsey government wants to raise or lower old-age consumption (by increasing or reducing the saving rate) depends on how much households value a smooth life cycle utility profile. In the log-case the two forces exactly balance out and the Ramsey saving rate does not respond to income risk at all. In contrast, if households strongly desire a smooth path of (the certainty equivalent of) consumption, then the Ramsey government compensates for the loss of old-age certainty equivalent consumption from larger income risk by saving at a higher rate, and s increases with income risk if the IES ρ is small. The reverse is true for a high IES.

Risk and the Optimal Tax Rate Finally, we can also determine the impact of income risk on optimal steady state capital taxes. From equation (106) the optimal Ramsey tax rate is given by

$$1 = (1-\tau^*)\alpha\beta\left((1-\kappa)(1-\alpha)\right)^{\left(\frac{1}{\rho}-1\right)}\frac{(1-s^*)^{\frac{1}{\rho}}}{s^*}\tilde{\Gamma}. \tag{108}$$

We observe that income risk affects the optimal tax rate in two ways. First, for a given target saving rate s^*, the direct impact of income risk depends on how $\tilde{\Gamma}$ (and thus the private saving rate) responds to an increase in risk. Second, a change in income risk changes the optimal saving rate s^* through $\tilde{\Gamma}_2$, as characterized in the previous proposition.

Proposition 28. *If $\rho \leq 1$, then an increase in income risk increases the optimal tax rate on capital. Similarly, if $\rho > 1$ and $\sigma \leq 1/\rho$, then an increase in income risk increases the optimal tax rate on capital. If $\rho > 1$ and $\sigma > 1/\rho$, an increase in income risk might lead to a strict reduction in the optimal tax rate τ on capital. A necessary condition for this to occur is that the competitive equilibrium saving rate for given τ is strictly decreasing in*

[47] We saw this explicitly in the decomposition of the first order condition of the Ramsey government in Section 4.1, where the risk term Γ from the competitive equilibrium optimality condition dropped out because the government chooses, through taxes and the associated changes in factor prices, to exactly offset the impact of higher risk on private household savings decisions. In the logic of that section, an increase in Γ increases $PE(s)$ but reduces $GE(s)$ by precisely the same factor.

income risk.

The intuition for the last part of the proposition is that, if $\rho > \max\{1, 1/\sigma\}$, then private households might *decrease* their saving rate too much in general equilibrium in response to an increase in income risk since they do no internalize the impact of the decline of the saving rate on the capital stock and thus on wages of future generations. For the capital tax to decrease in income risk this future generations effect has to be sufficiently strong. To see this formally, in the next paragraph we first derive the decomposition of the first-order condition for the optimal saving rate into the terms $PE(s), GE(s)$ and $FG(s)$ for the general EZW utility function, and then we use this decomposition to write equation (108) as

$$1 = \underbrace{(1-\tau^*)\frac{\tilde{\Gamma}}{\tilde{\Gamma}_2}}_{\text{from } PE(s)+GE(s)} - \underbrace{(1-\tau^*)\frac{\alpha}{s^*}\frac{\tilde{\Gamma}}{\tilde{\Gamma}_2}}_{\text{from } FG(s)}.$$

Since $\tilde{\Gamma}/\tilde{\Gamma}_2$ is increasing in income risk, the optimal capital tax rate τ^* can only decrease in income risk when the last term, the future generations effect, is large. This effect calls for a tax rate that decreases with income risk since s^* is decreasing in risk for $\rho > 1$.

J.3 Details of Proposition 28

J.3.1 Implementation

We start with a discussion of the optimal tax rate in the steady state. The optimal steady state capital tax rate τ^* satisfies, from equation (103)

$$1 = (1-\tau^*)\alpha\beta\left((1-\kappa)(1-\alpha)\right)^{\left(\frac{1}{\rho}-1\right)}\frac{(1-s^*)^{\frac{1}{\rho}}}{s^*}\tilde{\Gamma}. \tag{109}$$

We observe that the optimal tax rate is strictly increasing in $\tilde{\Gamma}$ and strictly decreasing in the Ramsey saving rate s^* that is to be implemented. Further, recall that the Ramsey saving rate s^* itself satisfies the first order condition (107)

$$\frac{(1-s^*)^{\frac{1}{\rho}}}{s^*} = \frac{\frac{1}{\alpha}-\frac{1}{s^*}}{\beta\tilde{\zeta}\tilde{\Gamma}_2} \tag{110}$$

and is impacted by income risk through $\tilde{\Gamma}_2$. Plugging (110) into (109) and exploiting the definition of $\tilde{\zeta}$ yields

$$1 = (1-\tau^*)\left(1-\frac{\alpha}{s^*}\right)\frac{\tilde{\Gamma}}{\tilde{\Gamma}_2}. \tag{111}$$

Lemma 2 establishes that $\frac{\tilde{\Gamma}}{\tilde{\Gamma}_2}$ is strictly increasing in income risk, and Proposition 27 in the main text establishes that an increase in income risk increases s^* if and only if $\rho < 1$ and decreases it if and only if $\rho > 1$. To sign the overall impact of income risk on the capital tax rate it is therefore useful to consider the following cases:

Case $\rho \leq 1$. This case gives clean results. From equation (111), since $\frac{\tilde{\Gamma}}{\tilde{\Gamma}_2}$ is strictly increasing in income risk, and since s^* is increasing in income risk for $\rho \leq 1$, strictly so if $\rho < 1$, it follows that τ^* is strictly increasing in risk.

Case $\rho > 1$ and $\sigma \leq 1/\rho$. In this case $\tilde{\Gamma}$ is strictly increasing in risk (Lemma 2) and s^* is strictly decreasing in risk (see Proposition 27). It then directly follows from equation (111) that τ^* is strictly increasing in income risk as well.

Case $\rho > 1$ and $\sigma > 1/\rho$. Since $\rho > 1$, the Ramsey saving rate s^* is strictly decreasing in income risk (which by itself calls for a tax rate that is strictly increasing in income risk), by equation (109). However, now the direct impact of income risk on taxes through the term $\tilde{\Gamma}$ might call for lower taxes since $\tilde{\Gamma}$ might now be decreasing in income risk. If $\tilde{\Gamma}$ is weakly increasing in income risk, then so is τ^*. Thus a necessary condition for τ^* to decrease with income risk is for $\tilde{\Gamma}$ to be strictly decreasing with income risk. This in turn is a necessary and sufficient condition for the private saving rate in competitive equilibrium to decrease with income risk (see Proposition 25). Thus the Ramsey tax rate τ^* is strictly decreasing in income risk *only if* the private saving rate $s^{CE}(\tau)$ is strictly decreasing in income risk (for any given tax rate τ).

Finally, one might conjecture that, since $\rho > 1$ and $\sigma > 1/\rho$ is required for the capital tax to decrease in income risk, that as long as both parameters are large enough the result will materialize. This conjecture turns out to be false, as an investigation of the most extreme case $\rho = \sigma = \infty$ shows. In this case lifetime utility is given by

$$V_t = c_t^y + \beta \underline{c}_{t+1}^o \tag{112}$$

where $\underline{c}^o_{t+1}$ is consumption in old age if the lowest possible labor productivity realization $\eta = \underline{\eta}_{t+1}$ materializes. In this case one can solve analytically for the optimal interior Ramsey saving and tax rate, and show that the optimal tax rate is the higher the lower is $\underline{\eta}_{t+1}$ and thus the higher is income risk.[48]

Proposition 13 above provides a fairly general implementation result for expected utility along the transition. The next proposition extends this result to EZW utility.

Proposition 29. *If the utility function is of the EZW form, then in general equilibrium we have* $s_\tau = \frac{\partial s_t}{\partial \tau_{t+1}} < 0$ *and unambiguous implementation.*

Proof. Recall from Section J.1 that the first-order condition in any period t of the transition is

$$1 = \quad \alpha\beta\left((1-\kappa)(1-\alpha)\right)^{(\alpha-1)\left(1-\frac{1}{\rho}\right)}(1-\tau_{t+1})k_t^{\alpha(\alpha-1)\left(1-\frac{1}{\rho}\right)}{s_t}^{(\alpha-1)\left(1-\frac{1}{\rho}\right)}\left(\frac{1-s_t}{s_t}\right)^{\frac{1}{\rho}}\tilde{\Gamma}.$$

Observe that an increase in the tax rate decreases the RHS. Collect terms on the saving rate as

$${s_t}^{(\alpha-1)\left(1-\frac{1}{\rho}\right)}\left(\frac{1-s_t}{s_t}\right)^{\frac{1}{\rho}} = {s_t}^{(\alpha-1)\left(1-\frac{1}{\rho}\right)-\frac{1}{\rho}}\left(1-s_t\right)^{\frac{1}{\rho}}$$

and notice that for any $\rho > 0$ term $(1-s_t)^{\frac{1}{\rho}}$ decreases in the saving rate. In response to an increase of the tax rate this force drives the saving rate down. To get unambiguous implementation, we thus require that the exponent

$$(\alpha-1)\left(1-\frac{1}{\rho}\right)-\frac{1}{\rho} < 0 \quad \Leftrightarrow \quad \frac{1}{\rho} > 0 > 1-\frac{1}{\alpha}$$

which holds for all $\alpha \in (0,1)$. $\square$

J.3.2 Decomposition of the FOC into $PE(s)$, $GE(s)$ and $FG(s)$

Now we decompose the first order condition of the Ramsey problem into three terms:

[48] In this case it is possible that the Ramsey government will want to implement a saving rate of $s = 1$ since households have linear preferences over consumption when young and minimum (across η) consumption when old. As long as $\underline{\eta}$ is sufficiently small, however, the Ramsey government prefers to implement an interior saving rate.

Proposition 30. *For $\theta = 1$, $\sigma \neq \frac{1}{\rho}$, terms $PE(s), GE(s), FG(s)$ are given by*

$$\begin{aligned}
PE(s) &= -\frac{1}{1-s}\left(\frac{1-s}{s}\right)^{1-\frac{1}{\rho}} k(s)^{1-\frac{1}{\rho}} + \frac{\alpha\beta}{s}\tilde{\Gamma} k(s)^{\alpha\left(1-\frac{1}{\rho}\right)} \\
GE(s) &= \frac{\alpha\beta}{s} k(s)^{\alpha\left(1-\frac{1}{\rho}\right)} \left(\tilde{\Gamma}_2 - \tilde{\Gamma}\right) \\
FG(s) &= \frac{\alpha}{s(1-\alpha)}\left(\frac{1-s}{s}\right)^{1-\frac{1}{\rho}} k(s)^{1-\frac{1}{\rho}} + \frac{\alpha^2\beta}{s(1-\alpha)} k(s)^{\alpha\left(1-\frac{1}{\rho}\right)}\tilde{\Gamma}_2
\end{aligned}$$

where $k(s) = (s(1-\kappa)(1-\alpha))^{\frac{1}{1-\alpha}}$ is the steady state capital stock.

Therefore,

$$PE(s) + GE(s) = -\frac{1}{1-s}\left(\frac{1-s}{s}\right)^{1-\frac{1}{\rho}} k(s)^{1-\frac{1}{\rho}} + \frac{\alpha\beta}{s} k(s)^{\alpha\left(1-\frac{1}{\rho}\right)}\tilde{\Gamma}_2. \tag{113}$$

and

$$PE(s)+GE(s)+FG(s) = \left(\frac{1-s}{s}\right)^{1-\frac{1}{\rho}} k(s)^{1-\frac{1}{\rho}} \left(\frac{\alpha}{s(1-\alpha)} - \frac{1}{1-s}\right) + \frac{1}{s} k(s)^{\alpha\left(1-\frac{1}{\rho}\right)} \frac{\alpha\beta}{(1-\alpha)}\tilde{\Gamma}_2. \tag{114}$$

Thus, compared to the expressions for these three effects we derived in Section 4.1, the partial equilibrium precautionary savings effect still cancels out the current generations general equilibrium effect ($\tilde{\Gamma}$ cancels out when adding up $PE(s)$ and $GE(s)$). However, additionally risk enters through $\tilde{\Gamma}_2$. With $\rho < 1$ an increase of risk increases $\tilde{\Gamma}_2$ thereby pushing up the desired saving rate of the Ramsey planner. The reason is that an increase of risk decreases the utility value of second period consumption of current generations (effect in $GE(s)$) and of all future generations (effect in $FG(s)$). With a low IES, it is optimal to compensate this with higher savings; vice versa for a high IES where the Ramsey planner rather prefers increased first-period consumption, respectively current generations consumption, over future consumption in response to an increase in risk.

Proof of Proposition 30. Calculating the respective terms yields

$$PE(s) = (1-\kappa)(1-\alpha)k^{\alpha}\Big[-\left((1-s)(1-\kappa)\left(1-\alpha\right)k^{\alpha}\right)^{-\frac{1}{\rho}} +$$

$$\alpha k'(s)^{\alpha-1}\beta\left(\int\left(\kappa\eta(1-\alpha)+\alpha\right)^{1-\sigma}d\Psi\right)^{\frac{\sigma-\frac{1}{\rho}}{1-\sigma}} k'(s)^{\alpha\left(\sigma-\frac{1}{\rho}\right)}\int\left(\kappa\eta(1-\alpha)+\alpha\right)^{-\sigma}d\Psi k'(s)^{-\alpha\sigma}\Big]$$

$$= -\frac{1}{1-s}\left(\frac{1-s}{s}\right)^{1-\frac{1}{\rho}} k(s)^{1-\frac{1}{\rho}} + \frac{\alpha\beta}{s}\tilde{\Gamma}k(s)^{\alpha\left(1-\frac{1}{\rho}\right)}.$$

and for

$$GE(s) = \beta\left(\int c^{o}(\eta)^{1-\sigma}d\Psi\right)^{\frac{\sigma-\frac{1}{\rho}}{1-\sigma}}\int\left(c^{o}(\eta)^{-\sigma}\right)\left[\kappa\eta w'(s) + (1-\kappa)(1-\alpha)k^{\alpha}R'(s)s\right]d\Psi(\eta)$$

$$= \beta\Gamma_2^{\frac{\sigma-\frac{1}{\rho}}{1-\sigma}}k'(s)^{\alpha\left(\sigma-\frac{1}{\rho}\right)}\int\left(\kappa\eta(1-\alpha)+\alpha\right)^{-\sigma}k'(s)^{-\alpha\sigma}\alpha(1-\alpha)s^{-1}$$

$$\cdot\left[\kappa\eta k'(s)^{\alpha} - (1-\kappa)(1-\alpha)k^{\alpha}k'(s)^{\alpha-1}s\right]d\Psi$$

$$= \frac{\alpha\beta}{s}k'(s)^{\alpha\left(1-\frac{1}{\rho}\right)}\Gamma_2^{\frac{\sigma-\frac{1}{\rho}}{1-\sigma}}\int\left(\kappa\eta(1-\alpha)+\alpha\right)^{-\sigma}\left[\kappa\eta(1-\alpha)+\alpha-1\right]d\Psi$$

$$= \frac{\alpha\beta}{s}k(s)^{\alpha\left(1-\frac{1}{\rho}\right)}\left(\tilde{\Gamma}_2-\tilde{\Gamma}\right).$$

When maximizing steady state utility, $FG(s)$ is equivalent to the derivative of the utility function with respect to the current period capital stock. Therefore:

$$FG(s) = u_{c^y}c^y_{k(s)}k(s)_s + \beta\left(\int c^{o}(\eta)^{1-\sigma}d\Psi\right)^{\frac{\sigma-\frac{1}{\rho}}{1-\sigma}}\int\left(c^{o}(\eta)^{-\sigma}\right)c^{o}_{k'(s)}k'(s)_{k(s)}k(s)_s d\Psi,$$

where

$$u_{c^y}c^y_{k(s)}k(s)_s = \left((1-s)(1-\kappa)(1-\alpha)k(s)^{\alpha}\right)^{-\frac{1}{\rho}}(1-s)(1-\kappa)(1-\alpha)\alpha k(s)^{\alpha-1}(1-\kappa)k(s)^{\alpha}$$

$$= \frac{\alpha}{s(1-\alpha)}\left(\frac{1-s}{s}\right)^{1-\frac{1}{\rho}}k'(s)^{1-\frac{1}{\rho}}$$

and

$$\left(\int c^{o}(\eta)^{1-\sigma}d\Psi\right)^{\frac{\sigma-\frac{1}{\rho}}{1-\sigma}} = \Gamma_2^{\frac{\sigma-\frac{1}{\rho}}{1-\sigma}}k'(s)^{\alpha\left(\sigma-\frac{1}{\rho}\right)}$$

and

$$\beta \int c^{0-\sigma} c^{o}_{k'(s)} k'(s)_{k(s)} k(s)_s d\Psi =$$

$$\beta \int \left(\kappa\eta(1-\alpha)+\alpha\right)^{-\sigma} k'(s)^{-\alpha\sigma} \left(\kappa\eta(1-\alpha)+\alpha\right) d\Psi \alpha k'(s)^{\alpha-1} k'(s)\alpha(1-\kappa)k(s)^{\alpha-1}$$

$$= \frac{\alpha^2\beta}{s(1-\alpha)} k'(s)^{\alpha(1-\sigma)}\Gamma_2.$$

Therefore:

$$FG(s) = \frac{\alpha}{s(1-\alpha)}\left(\frac{1-s}{s}\right)^{1-\frac{1}{\rho}} k(s)^{1-\frac{1}{\rho}} + \frac{\alpha^2\beta}{s(1-\alpha)} k(s)^{\alpha\left(1-\frac{1}{\rho}\right)}\tilde{\Gamma}_2.$$

□

J.3.3 Decomposition of τ^*

Given this result, the optimal tax rate τ^* can be decomposed as stated in the previous paragraph:

Corollary 11. *τ^* can only be decreasing in risk if the effect of $FG(s)$ is sufficiently strong.*

Proof. We know that the FOC for s^* follows from

$$PE(s) + GE(s) + FG(s) = 0$$

Now set $FG(s) = 0$. Rewrite from (113)

$$PE(s) + GE(s) = 0 \;\Leftrightarrow\; \frac{s}{(1-s)^{\frac{1}{\rho}}} = \alpha\beta\tilde{\zeta}\tilde{\Gamma}_2,$$

which uses $k(s) = (s(1-\kappa)(1-\alpha))^{\frac{1}{1-\alpha}}$ and $\tilde{\zeta} = ((1-\alpha)(1-\kappa))^{\frac{1}{\rho}-1}$. Using the above in (108) gives

$$1 = (1-\tau^*)\frac{\tilde{\Gamma}}{\tilde{\Gamma}_2}$$

and $\frac{\tilde{\Gamma}}{\tilde{\Gamma}_2}$ is unambiguously increasing in risk, see Section K.1. Using the above we can thus decompose equation (105) as described above:

$$1 = \underbrace{(1-\tau^*)\frac{\tilde{\Gamma}}{\tilde{\Gamma}_2}}_{\text{from } PE(s)+GE(s)} \underbrace{-(1-\tau^*)\frac{\alpha}{s^*}\frac{\tilde{\Gamma}}{\tilde{\Gamma}_2}}_{\text{from } FG(s)}.$$

□

J.4 Pareto Improving Transitions

Observe that specification (44) nests EZW preferences as a special case. Thus, Proposition 12 and Corollary 7 apply.

K Income Risk and $\Gamma, \Gamma_2, \tilde{\Gamma}, \tilde{\Gamma}_2$

K.1 General Case

In this section we prove Lemma 2 in Subsection J.2.3 through two separate Lemmas. For this, recall that the relevant expressions involving idiosyncratic income risk are given by:

$$\begin{aligned}
\Gamma &= \int (\kappa\eta(1-\alpha)+\alpha)^{-\sigma}\, d\Psi(\eta) \\
\Gamma_2 &= \int (\kappa\eta(1-\alpha)+\alpha)^{1-\sigma}\, d\Psi(\eta) \\
\tilde{\Gamma} &= \Gamma_2^{\frac{\sigma-\frac{1}{\rho}}{1-\sigma}}\Gamma = v^{\sigma-\frac{1}{\rho}}\Gamma \\
\tilde{\Gamma}_2 &= \Gamma_2^{\frac{\sigma-\frac{1}{\rho}}{1-\sigma}}\Gamma_2 = \Gamma_2^{\frac{1-\frac{1}{\rho}}{1-\sigma}} = v^{1-\frac{1}{\rho}} \\
\frac{\tilde{\Gamma}}{\tilde{\Gamma}_2} &= \frac{\Gamma}{\Gamma_2} \\
v &\equiv \begin{cases} \left[\int (\alpha+(1-\alpha)\kappa\eta)^{1-\sigma}\, d\Psi(\eta)\right]^{\frac{1}{1-\sigma}} & \text{for } \sigma \neq 1 \\ \exp\left[\int \ln(\alpha+(1-\alpha)\kappa\eta)\, d\Psi(\eta)\right] & \text{for } \sigma = 1 \end{cases}
\end{aligned}$$

Furthermore, as in the main text we use the notion of a mean-preserving spread in the random variable η when referring to an increase in risk, that is, formally, random variable η is replaced by $\tilde{\eta} = \eta + \nu$, where ν is a random variable with zero mean and positive variance (and Assumption 1 applies to $\tilde{\eta}$ as well).

Lemma 3. *The certainty equivalent v is decreasing in η-risk.*

Proof. If $\sigma > 1$ ($\sigma < 1$), then $(\alpha + (1-\alpha)\kappa\eta)^{1-\sigma}$ is convex and downward sloping (concave and upward sloping) in η. The certainty equivalent of a convex and downward sloping (respectively, concave and upward sloping) function is decreasing in risk. □

Lemma 4. *The comparative statics of the other risk terms with respect to a mean-preserving spread in η are given by:*

1. *Γ is increasing in η-risk.*
2. *Γ_2 is increasing (respectively, decreasing) in η-risk if $\sigma > 1$ (respectively $\sigma < 1$).*
3. *$\tilde{\Gamma}_2$ is increasing (decreasing) in η-risk if $\rho < 1$ ($\rho > 1$).*
4. *For $\rho < 1$, $\tilde{\Gamma}$ is increasing in η-risk. For $\rho > 1$ we have the following case distinction:*
   (a) *For $\frac{1}{\sigma} > \rho > 1$, $\tilde{\Gamma}$ unambiguously increases in income risk.*
   (b) *For $\rho > 1, \rho > \frac{1}{\sigma} > 0$, i.e., $\sigma < \infty$ the effect of η-risk on $\tilde{\Gamma}$ is ambiguous.*

Proof.

1. Γ is increasing in η-risk because $(\kappa\eta(1-\alpha)+\alpha)^{-\sigma}$ is a convex function in η (with the degree of convexity increasing in σ).
2. Γ_2 is increasing (decreasing) in η-risk if $\sigma > 1$ ($\sigma < 1$) because $(\kappa\eta(1-\alpha)+\alpha)^{1-\sigma}$ is a convex (concave) function of η.
3. $\tilde{\Gamma}_2$ is increasing (decreasing) in η-risk if $\rho < 1$ ($\rho > 1$) because the certainty equivalent v decreases in η-risk and because for $\rho < 1$ ($\rho > 1$) the exponent $1 - \frac{1}{\rho}$ is negative (positive).
4. For $\rho < 1$, $\tilde{\Gamma}$ is increasing in η-risk (sufficient condition). To see this, rewrite $\tilde{\Gamma}$ as

$$\tilde{\Gamma} = \frac{\Gamma}{\Gamma_2^{-\frac{-(1-\sigma)+(1-\frac{1}{\rho})}{1-\sigma}}} = \frac{\Gamma}{\Gamma_2^{1-\frac{1-\frac{1}{\rho}}{1-\sigma}}} = \frac{\Gamma}{\Gamma_2}\Gamma_2^{\frac{1-\frac{1}{\rho}}{1-\sigma}} = \frac{\Gamma}{\Gamma_2}v^{1-\frac{1}{\rho}} \tag{115}$$

   Notice that for $\sigma \leq 1$, $\frac{\Gamma}{\Gamma_2}$ is the ratio of the expectation of a strictly convex and a concave function. Hence, for $\sigma \leq 1$ the term $\frac{\Gamma}{\Gamma_2}$ is increasing in risk by Jensen's inequality. For $\sigma > 1$ term $\frac{\Gamma}{\Gamma_2}$ is the ratio of the expectation of two convex functions with the convexity of the function in the numerator, $(\kappa\eta(1-\alpha)+\alpha)^{-\sigma}$, being stronger than in the denominator, $(\kappa\eta(1-\alpha)+\alpha)^{1-\sigma}$ as long as $\sigma < \infty$. Therefore, also for $1 < \sigma < \infty$ term $\frac{\Gamma}{\Gamma_2}$ is increasing in risk. For $\sigma = \infty$ term $\frac{\Gamma}{\Gamma_2}$ is equal to 1. Finally, since the certainty equivalent v is decreasing in η-risk, term $v^{1-\frac{1}{\rho}}$ increases in η-risk if and only if $\rho < 1$.

5. For $\rho > 1$ we have the following case distinction, based on the representation of $\tilde{\Gamma} = v^{\sigma-\frac{1}{\rho}}\Gamma$:

   (a) For $\frac{1}{\sigma} > \rho > 1$, $\tilde{\Gamma}$ unambiguously increases in η-risk because v decreases in η-risk and $\sigma - \frac{1}{\rho} < 0$.

   (b) For $\rho > 1, \rho > \frac{1}{\sigma}$ the effect of η-risk on $\tilde{\Gamma}$ is ambiguous because v is decreasing in η-risk and $\sigma - \frac{1}{\rho} > 0$ so that $v^{\sigma-\frac{1}{\rho}}$ is decreasing in η-risk whereas Γ is increasing in η-risk. Rewriting $\tilde{\Gamma}$ as in equation (115) does not resolve this ambiguity because term $\frac{\Gamma}{\Gamma_2}$ is increasing in η-risk whereas $v^{1-\frac{1}{\rho}}$ is decreasing in η risk because $1 - \frac{1}{\rho} > 0$.

$\square$

K.2 Expressing Γ-Intervals from Proposition 4 in Terms of Variances

The bounds in Proposition 4 can be given in terms of the variances of the income shock η, to a second-order Taylor approximation of the integral defining Γ. This approximation around $\eta = 1$ gives

$$\Gamma(\alpha, \kappa, \sigma, \Psi) \approx \bar{\Gamma} + \frac{[\kappa(1-\alpha)]^2}{[\kappa(1-\alpha)+\alpha]^3}\sigma_\eta^2.$$

With this approximation the interval for intermediate risk, item 2 of Proposition 4, becomes $\sigma_\eta^2 \in \left(\underline{\sigma_\eta^2}, \overline{\sigma_\eta^2}\right)$ where

$$\begin{aligned}
\underline{\sigma_\eta^2} &= \frac{(\kappa(1-\alpha)+\alpha)^3}{(\kappa(1-\alpha))^2}\left(\frac{1+\beta}{(1-\alpha)\beta} - \bar{\Gamma}\right) \\
\overline{\sigma_\eta^2} &= \frac{(\kappa(1-\alpha)+\alpha)^3}{(\kappa(1-\alpha))^2}\left(\frac{1}{\left((1-\alpha)-\frac{1}{\bar{\Gamma}}\right)\beta} - \bar{\Gamma}\right)
\end{aligned}$$

and $\overline{\sigma_\eta^2} > \underline{\sigma_\eta^2} > 0$ under the maintained assumption that $\beta < \left[(1-\alpha)\bar{\Gamma} - 1\right]^{-1}$. Thus, all intervals defined in Proposition 4 can be expressed in terms of variances and are nonempty. Also note that if the distribution Ψ is log-normal and thus exclusively determined by its variance (given that the mean is pinned down by the assumption $E(\eta) = 1$), then no second order approximation is necessary in the above argument, but the mapping between the variance bounds and the Γ bounds is algebraically much more involved.