Model versus Data

Frank Schorfheide
University of Pennsylvania

Econ 722 – Part 1

February 13, 2019

State-space Representation of DSGE Model

- $n_y \times 1$ vector of observables:

$$y_t = M_y'[\log(X_t/X_{t-1}), \log lsh_t, \log \pi_t, \log R_t]'.$$

- $n_s \times 1$ vector of econometric state variables s_t

$$s_t = [\phi_t, \lambda_t, z_t, \epsilon_{R,t}, \widehat{x}_{t-1}]'$$

- DSGE model parameters:

$$\theta = [\beta, \gamma, \lambda, \pi^*, \zeta_p, \nu, \rho_\phi, \rho_\lambda, \rho_z, \sigma_\phi, \sigma_\lambda, \sigma_z, \sigma_R]'.$$

- Measurement equation:

$$y_t = \Psi_0(\theta) + \Psi_1(\theta)s_t.$$

- State-transition equation:

$$s_t = \Phi_1(\theta)s_{t-1} + \Phi_\epsilon(\theta)\epsilon_t, \quad \epsilon_t = [\epsilon_{\phi,t}, \epsilon_{\lambda,t}, \epsilon_{z,t}, \epsilon_{R,t}]'$$

State-Space Representation of DSGE Model

State-space representation:

$$
\begin{aligned}
y_t &= \Psi_0(\theta) + \Psi_1(\theta) s_t \\
s_t &= \Phi_1(\theta) s_{t-1} + \Phi_\epsilon(\theta)\epsilon_t
\end{aligned}
$$

System matrices:

$$
\Psi_0(\theta) = M'_y \begin{bmatrix} \log\gamma \\ \log(lsh) \\ \log \pi^* \\ \log(\pi^*\gamma/\beta) \end{bmatrix}, \quad x_\phi = -\frac{\kappa_p \psi_p/\beta}{1-\psi_p\rho_\phi}, \quad x_\lambda = -\frac{\kappa_p \psi_p/\beta}{1-\psi_p\rho_\lambda}, \quad x_z = \frac{\rho_z \psi_p}{1-\psi_p\rho_z}, \quad x_{\epsilon_R} = -\psi_p \sigma_R
$$

$$
\Psi_1(\theta) = M'_y \begin{bmatrix} x_\phi & x_\lambda & x_z+1 & x_{\epsilon_R} & -1 \\ 1+(1+\nu)x_\phi & (1+\nu)x_\lambda & (1+\nu)x_z & (1+\nu)x_{\epsilon_R} & 0 \\ \frac{\kappa_p}{1-\beta\rho_\phi}(1+(1+\nu)x_\phi) & \frac{\kappa_p}{1-\beta\rho_\lambda}(1+(1+\nu)x_\lambda) & \frac{\kappa_p}{1-\beta\rho_z}(1+\nu)x_z & +\kappa_p(1+\nu)x_{\epsilon_R} & 0 \\ \frac{\kappa_p/\beta}{1-\beta\rho_\phi}(1+(1+\nu)x_\phi) & \frac{\kappa_p/\beta}{1-\beta\rho_\lambda}(1+(1+\nu)x_\lambda) & \frac{\kappa_p/\beta}{1-\beta\rho_z}(1+\nu)x_z & (\kappa_p(1+\nu)x_{\epsilon_R}/\beta+\sigma_R) & 0 \end{bmatrix}
$$

$$
\Phi_1(\theta) = \begin{bmatrix} \rho_\phi & 0 & 0 & 0 & 0 \\ 0 & \rho_\lambda & 0 & 0 & 0 \\ 0 & 0 & \rho_z & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ x_\phi & x_\lambda & x_z & x_{\epsilon_R} & 0 \end{bmatrix}, \quad \Phi_\epsilon(\theta) = \begin{bmatrix} \sigma_\phi & 0 & 0 & 0 \\ 0 & \sigma_\lambda & 0 & 0 \\ 0 & 0 & \sigma_z & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}
$$

M'_y is an $n_y \times 4$ selection matrix that selects rows of Ψ_0 and Ψ_1.

DSGE Model Implications

- We want to understand the implications of the DSGE model.
- We could simulate data from the state-space representation

$$\begin{aligned} y_t &= \Psi_0(\theta) + \Psi_1(\theta) s_t \\ s_t &= \Phi_1(\theta) s_{t-1} + \Phi_\epsilon(\theta)\epsilon_t \end{aligned}$$

- using:

$$\epsilon_t \sim iidN(0, I).$$

- But some calculations are better done analytically.

DSGE Model Implications: Questions

- What is the correlation between consumption growth this quarter and four quarters ago?
- What is the correlation between inflation and interest rates?
- Does the labor share predict consumption growth one-year ahead?

Example: AR(1) Model

- Model:

$$y_t = \phi y_{t-1} + u_t, \quad u_t \sim N(0, \sigma_u^2)$$

- Mean:

$$\mathbb{E}[y_t] = \phi \mathbb{E}[y_{t-1}] + \mathbb{E}[u_t] \quad \Longrightarrow \quad \mathbb{E}[y_t] = 0.$$

- Variance:

$$\mathbb{V}[y_t] = \mathbb{E}[y_t] = \mathbb{E}\big[(\phi y_{t-1} + u_t)^2\big] = \phi^2 \mathbb{E}[y_{t-1}^2] + 2\phi \mathbb{E}[y_{t-1} u_t] + \mathbb{E}[u_t^2]$$

leads to

$$\gamma(0) = \mathbb{V}[y_t] = \frac{\sigma_u^2}{1-\phi^2}.$$

Example: AR(1) Model

- First-order autocovariance:

$$\gamma(1) = \mathbb{E}[y_t y_{t-1}] = \mathbb{E}[(\phi y_{t-1} + u_t) y_{t-1}] = \phi\gamma(0).$$

- h-th order autocovariance:

$$\gamma(h) = \mathbb{E}[y_t y_{t-h}] = \phi^h \gamma(0)$$

- Autocorrelation

$$\rho(h) = \frac{\gamma(h)}{\gamma(0)}.$$

DSGE Model Implications: Autocovariances

- State-space representation:

$$
\begin{aligned}
y_t &= \Psi_0(\theta) + \Psi_1(\theta) s_t \\
s_t &= \Phi_1(\theta) s_{t-1} + \Phi_\epsilon(\theta)\epsilon_t
\end{aligned}
$$

- Notation:

$$\Gamma_{yy}(h) = \mathbb{E}[y_t y_{t-h}], \quad \Gamma_{ss}(h) = \mathbb{E}[s_t s_{t-h}], \quad \text{and} \quad \Gamma_{ys}(h) = \mathbb{E}[y_t s'_{t-h}]$$

- Covariance matrix of s_t is solution to Lyapunov equation:

$$\Gamma_{ss}(0) = \Phi_1 \Gamma_{ss}(0) \Phi_1' + \Phi_\epsilon \Phi_\epsilon'.$$

- Autocovariance matrices for $h \neq 0$:

$$\Gamma_{ss}(h) = \Phi_1^h \Gamma_{ss}(0).$$

- Using the measurement equation, we deduce that

$$\Gamma_{yy}(h) = \Psi_1 \Gamma_{ss}(h) \Psi_1', \quad \Gamma_{ys}(h) = \Psi_1 \Gamma_{ss}(h).$$

DSGE Model Implications: Autocovariances

1. Fix a set of DSGE model parameters.
2. Solve model and compute matrices $\Psi_0(\theta)$, $\Psi_1(\theta)$, $\Phi_1(\theta)$, $\Phi_\epsilon(\theta)$ in state-space representation:

$$\begin{aligned} y_t &= \Psi_0(\theta) + \Psi_1(\theta) s_t \\ s_t &= \Phi_1(\theta) s_{t-1} + \Phi_\epsilon(\theta)\epsilon_t \end{aligned}$$

3. Compute autocovariances based on $\Psi_0(\theta)$, $\Psi_1(\theta)$, $\Phi_1(\theta)$, $\Phi_\epsilon(\theta)$.

DSGE Model Implications: Fix Parameters

Parameter	Value	Parameter	Value
β	1/1.01	γ	exp(0.005)
λ	0.15	π^*	exp(0.005)
ζ_p	0.65	ν	0
ρ_ϕ	0.94	ρ_λ	0.88
ρ_z	0.13		
σ_ϕ	0.01	σ_λ	0.01
σ_z	0.01	σ_R	0.01

DSGE Model Implications: Evaluate System Matrices

$$\Psi_0(\theta) = M_y' \begin{bmatrix} \log \gamma \\ \log(lsh) \\ \log \pi^* \\ \log(\pi^* \gamma/\beta) \end{bmatrix}, \quad x_\phi = -\frac{\kappa_p \psi_p/\beta}{1-\psi_p\rho_\phi}, \quad x_\lambda = -\frac{\kappa_p \psi_p/\beta}{1-\psi_p\rho_\lambda}, \quad x_z = \frac{\rho_z \psi_p}{1-\psi_p\rho_z}, \quad x_{\epsilon_R} = -\psi_p\sigma_R$$

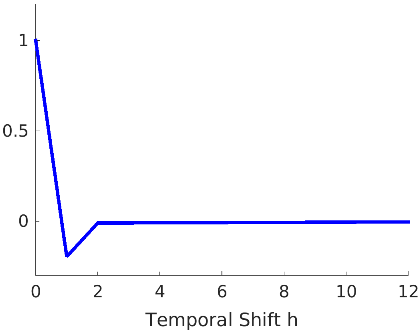
$$\Psi_1(\theta) = M_y' \begin{bmatrix} x_\phi & x_\lambda & x_z+1 & x_{\epsilon_R} & -1 \\ 1+(1+\nu)x_\phi & (1+\nu)x_\lambda & (1+\nu)x_z & (1+\nu)x_{\epsilon_R} & 0 \\ \frac{\kappa_p}{1-\beta\rho_\phi}(1+(1+\nu)x_\phi) & \frac{\kappa_p}{1-\beta\rho_\lambda}(1+(1+\nu)x_\lambda) & \frac{\kappa_p}{1-\beta\rho_z}(1+\nu)x_z & +\kappa_p(1+\nu)x_{\epsilon_R} & 0 \\ \frac{\kappa_p/\beta}{1-\beta\rho_\phi}(1+(1+\nu)x_\phi) & \frac{\kappa_p/\beta}{1-\beta\rho_\lambda}(1+(1+\nu)x_\lambda) & \frac{\kappa_p/\beta}{1-\beta\rho_z}(1+\nu)x_z & (\kappa_p(1+\nu)x_{\epsilon_R}/\beta+\sigma_R) & 0 \end{bmatrix}$$

$$\Phi_1(\theta) = \begin{bmatrix} \rho_\phi & 0 & 0 & 0 & 0 \\ 0 & \rho_\lambda & 0 & 0 & 0 \\ 0 & 0 & \rho_z & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ x_\phi & x_\lambda & x_z & x_{\epsilon_R} & 0 \end{bmatrix}, \quad \Phi_\epsilon(\theta) = \begin{bmatrix} \sigma_\phi & 0 & 0 & 0 \\ 0 & \sigma_\lambda & 0 & 0 \\ 0 & 0 & \sigma_z & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

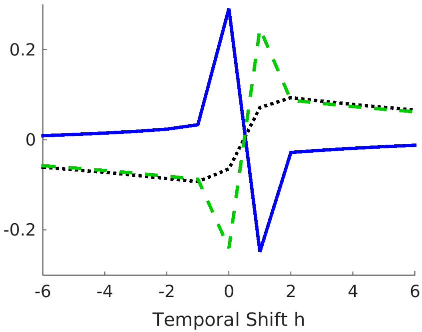
M_y' is an $n_y \times 4$ selection matrix that selects rows of Ψ_0 and Ψ_1.

Compute Autocovariances and Plot

$\text{Corr}\big(\log(X_t/X_{t-1}), \log(X_{t-h}/X_{t-h-1})\big)$



$\text{Corr}\big(\log(X_t/X_{t-1}), \log Z_{t-h}\big)$



Notes: Right panel: correlations of output growth with labor share (solid), inflation (dotted), and interest rates (dashed).

DSGE Model Implications: Forecast Error Variance Decomposition

- Fluctuations in the model are driven by shocks:
  - technology growth z_t;
  - mark-up λ_t;
  - preference ϕ_t;
  - monetary policy $\epsilon_{R,t}$;
  - government spending $\widehat{g}_t$.
- Shocks generate uncertainty about future macroeconomic outcomes
- Question: what is the contribution of monetary policy shocks to forecast errors in inflation rates?

DSGE Model Implications: Forecast Error Variance Decomposition

- The law of motion for s_t can be expressed as the infinite-order vector moving average (MA)

$$y_t = \Psi_0 + \Psi_1 \sum_{s=0}^{\infty} \Phi_1^s \Phi_\epsilon \epsilon_{t-s}.$$

- h-step-ahead forecast error is

$$e_{t|t-h} = y_t - \mathbb{E}_{t-h}[y_t] = \Psi_1 \sum_{s=0}^{h-1} \Phi_1^s \Phi_\epsilon \epsilon_{t-s}.$$

- h-step-ahead forecast error covariance matrix is

$$\mathbb{E}[e_{t|t-h} e'_{t|t-h}] = \Psi_1 \left(\sum_{s=0}^{h-1} \Phi_1^s \Phi_\epsilon \Phi'_\epsilon \Phi_1^{s'} \right) \Psi'_1 \quad \text{with} \quad \lim_{h \longrightarrow \infty} \mathbb{E}[e_{t|t-h} e'_{t|t-h}] = \Gamma_{ss}(0).$$

DSGE Model Implications: Forecast Error Variance Decomposition

- Recall $\mathbb{E}[\epsilon_t \epsilon_t'] = I$. Let $I^{(j)}$ be defined by setting all but the j-th diagonal element of the identity matrix I to zero:

$$I = \sum_{j=1}^{n_\epsilon} I^{(j)}.$$

- Express the contribution of shock j to the forecast error for y_t as

$$e_{t|t-h}^{(j)} = \Psi_1 \sum_{s=0}^{h-1} \Phi_1^s \Phi_\epsilon I^{(j)} \epsilon_{t-s}.$$

- The contribution of shock j to the forecast error variance of observation $y_{i,t}$ is

$$\text{FEVD}(i,j,h) = \frac{\left[\Psi_1 \left(\sum_{s=0}^{h-1} \Phi_1^s \Phi_\epsilon I^{(j)} \Phi_\epsilon' \Phi_1^{s'}\right) \Psi_1'\right]_{ii}}{\left[\Psi_1 \left(\sum_{s=0}^{h-1} \Phi_1^s \Phi_\epsilon \Phi_\epsilon' \Phi_1^{s'}\right) \Psi_1'\right]_{ii}},$$

where $[A]_{ij}$ denotes element (i,j) of a matrix A.

DSGE Model Implications: Fix Parameters

Parameter	Value	Parameter	Value
β	1/1.01	γ	exp(0.005)
λ	0.15	π^*	exp(0.005)
ζ_p	0.65	ν	0
ρ_ϕ	0.94	ρ_λ	0.88
ρ_z	0.13		
σ_ϕ	0.01	σ_λ	0.01
σ_z	0.01	σ_R	0.01

DSGE Model Implications: Evaluate System Matrices

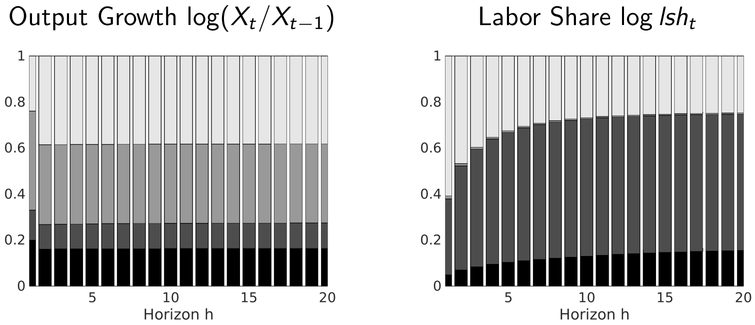
$$\Psi_0(\theta) = M_y' \begin{bmatrix} \log \gamma \\ \log(lsh) \\ \log \pi^* \\ \log(\pi^* \gamma/\beta) \end{bmatrix}, \quad x_\phi = -\frac{\kappa_p \psi_p/\beta}{1-\psi_p \rho_\phi}, \quad x_\lambda = -\frac{\kappa_p \psi_p/\beta}{1-\psi_p \rho_\lambda}, \quad x_z = \frac{\rho_z \psi_p}{1-\psi_p \rho_z}, \quad x_{\epsilon_R} = -\psi_p \sigma_R$$

$$\Psi_1(\theta) = M_y' \begin{bmatrix} x_\phi & x_\lambda & x_z + 1 & x_{\epsilon_R} & -1 \\ 1+(1+\nu)x_\phi & (1+\nu)x_\lambda & (1+\nu)x_z & (1+\nu)x_{\epsilon_R} & 0 \\ \frac{\kappa_p}{1-\beta\rho_\phi}(1+(1+\nu)x_\phi) & \frac{\kappa_p}{1-\beta\rho_\lambda}(1+(1+\nu)x_\lambda) & \frac{\kappa_p}{1-\beta\rho_z}(1+\nu)x_z & +\kappa_p(1+\nu)x_{\epsilon_R} & 0 \\ \frac{\kappa_p/\beta}{1-\beta\rho_\phi}(1+(1+\nu)x_\phi) & \frac{\kappa_p/\beta}{1-\beta\rho_\lambda}(1+(1+\nu)x_\lambda) & \frac{\kappa_p/\beta}{1-\beta\rho_z}(1+\nu)x_z & (\kappa_p(1+\nu)x_{\epsilon_R}/\beta + \sigma_R) & 0 \end{bmatrix}$$

$$\Phi_1(\theta) = \begin{bmatrix} \rho_\phi & 0 & 0 & 0 & 0 \\ 0 & \rho_\lambda & 0 & 0 & 0 \\ 0 & 0 & \rho_z & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ x_\phi & x_\lambda & x_z & x_{\epsilon_R} & 0 \end{bmatrix}, \quad \Phi_\epsilon(\theta) = \begin{bmatrix} \sigma_\phi & 0 & 0 & 0 \\ 0 & \sigma_\lambda & 0 & 0 \\ 0 & 0 & \sigma_z & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

M_y' is an $n_y \times 4$ selection matrix that selects rows of Ψ_0 and Ψ_1.

Compute Forecast Error Variance Decomposition and Plot



Notes: The stacked bar plots represent the cumulative forecast error variance decomposition. The bars, from darkest to lightest, represent the contributions of ϕ_t, λ_t, z_t, and $\epsilon_{R,t}$.

DSGE Model Implications: Impulse Response Functions

- What is the dynamic effect of a 25 basis point unanticipated reduction in the interest rate?
- What is the effect of an unanticipated increase in technology growth?

DSGE Model Implications: Impulse Response Functions

- **Definition:**

$$\text{IRF}(i,j,h|s_{t-1}) = \mathbb{E}\big[y_{i,t+h} \; \big| \; s_{t-1}, \epsilon_{j,t} = 1\big] - \mathbb{E}\big[y_{i,t+h} \; \big| \; s_{t-1}\big].$$

- Both expectations are conditional on the initial state s_{t-1} and integrate over current and future realizations of the shocks ϵ_t.
- First term also conditions on $\epsilon_{j,t} = 1$, whereas the second term averages of $\epsilon_{j,t}$.
- In a linearized DSGE model we have $\mathbb{E}[\epsilon_{t+h}|s_{t-1}] = 0$ for $h = 0, 1, \ldots$ and deduce

$$\text{IRF}(.,j,h) = \Psi_1 \frac{\partial}{\partial \epsilon_{j,t}} s_{t+h} = \Psi_1 \Phi_1^h [\Phi_\epsilon]_{.j},$$

where $[A]_{.j}$ is the j-th column of a matrix A. We dropped s_{t-1} from the conditioning set to simplify the notation.

DSGE Model Implications: Fix Parameters

Parameter	Value	Parameter	Value
β	$1/1.01$	γ	$\exp(0.005)$
λ	0.15	π^*	$\exp(0.005)$
ζ_p	0.65	ν	0
ρ_ϕ	0.94	ρ_λ	0.88
ρ_z	0.13		
σ_ϕ	0.01	σ_λ	0.01
σ_z	0.01	σ_R	0.01

DSGE Model Implications: Evaluate System Matrices

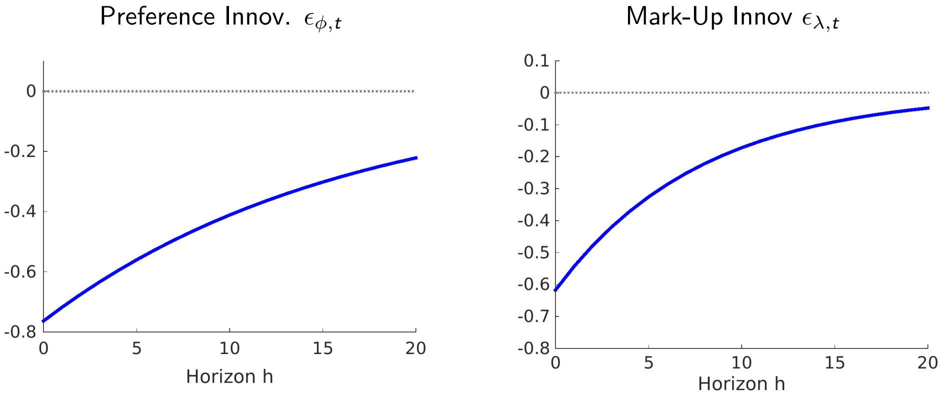
$$\Psi_0(\theta) = M_y' \begin{bmatrix} \log \gamma \\ \log(lsh) \\ \log \pi^* \\ \log(\pi^* \gamma/\beta) \end{bmatrix}, \quad x_\phi = -\frac{\kappa_p \psi_p/\beta}{1-\psi_p\rho_\phi}, \quad x_\lambda = -\frac{\kappa_p \psi_p/\beta}{1-\psi_p\rho_\lambda}, \quad x_z = \frac{\rho_z \psi_p}{1-\psi_p\rho_z}, \quad x_{\epsilon_R} = -\psi_p \sigma_R$$

$$\Psi_1(\theta) = M_y' \begin{bmatrix} x_\phi & x_\lambda & x_z + 1 & x_{\epsilon_R} & -1 \\ 1+(1+\nu)x_\phi & (1+\nu)x_\lambda & (1+\nu)x_z & (1+\nu)x_{\epsilon_R} & 0 \\ \frac{\kappa_p}{1-\beta\rho_\phi}(1+(1+\nu)x_\phi) & \frac{\kappa_p}{1-\beta\rho_\lambda}(1+(1+\nu)x_\lambda) & \frac{\kappa_p}{1-\beta\rho_z}(1+\nu)x_z & +\kappa_p(1+\nu)x_{\epsilon_R} & 0 \\ \frac{\kappa_p/\beta}{1-\beta\rho_\phi}(1+(1+\nu)x_\phi) & \frac{\kappa_p/\beta}{1-\beta\rho_\lambda}(1+(1+\nu)x_\lambda) & \frac{\kappa_p/\beta}{1-\beta\rho_z}(1+\nu)x_z & (\kappa_p(1+\nu)x_{\epsilon_R}/\beta + \sigma_R) & 0 \end{bmatrix}$$

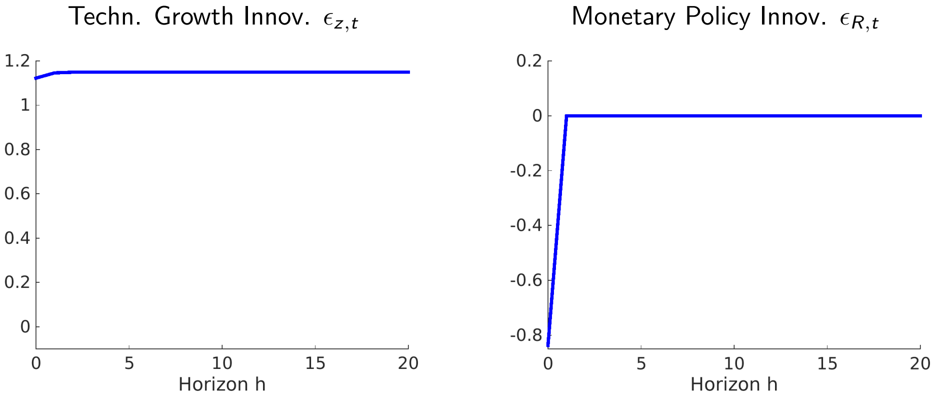
$$\Phi_1(\theta) = \begin{bmatrix} \rho_\phi & 0 & 0 & 0 & 0 \\ 0 & \rho_\lambda & 0 & 0 & 0 \\ 0 & 0 & \rho_z & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ x_\phi & x_\lambda & x_z & x_{\epsilon_R} & 0 \end{bmatrix}, \quad \Phi_\epsilon(\theta) = \begin{bmatrix} \sigma_\phi & 0 & 0 & 0 \\ 0 & \sigma_\lambda & 0 & 0 \\ 0 & 0 & \sigma_z & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

M_y' is an $n_y \times 4$ selection matrix that selects rows of Ψ_0 and Ψ_1.

Compute and Plot Impulse Responses of Log Output $100\log(X_{t+h}/X_t)$



Compute and Plot Impulse Responses of Log Output $100 \log(X_{t+h}/X_t)$



DSGE Model Implications

- Formulas for autocovariance functions, spectra, and impulse response functions for a linearized DSGE model can be derived analytically.
- Analytical expressions can then be numerically evaluated for different vectors of parameter values θ.
- For general nonlinear general nonlinear DSGE model, the implied moments have to be computed using Monte Carlo simulation.
- Let $Y^*_{1:T}$ denote a sequence of observations simulated from the state-space representation of the DSGE model by drawing an initial state vector s_0 and innovations ϵ_t from their model-implied distributions, then

$$\frac{1}{T}\sum_{t=1}^{T} y_t^* \xrightarrow{a.s.} \mathbb{E}[y_t].$$

- The downside of Monte Carlo approximations is that they are associated with a simulation error.

Empirical Analogues

- Do implications of DSGE model match what we observe in the data?
- We can use the comparison of DSGE model implications and empirical analogues to
    - determine the parameterization of DSGE model (so far we made up a θ);
    - assess the fit of the model.

Empirical Analogues: Data from FRED

- **Real aggregate output:** we use quarterly, seasonally adjusted GDP at the annual rate that has been pegged to 2009 dollars (GDPC96). We turn GDP into growth rates by taking logs and then differencing.
- **Labor share:** Compensation of Employees (COE) divided by nominal GDP (GDP). Both series are quarterly and seasonally adjusted at the annual rate. We use the log labor share as the observable.
- **Inflation:** computed from the implicit price deflator (GDPDEF) by taking log differences.
- **Interest rates:** we use the Effective Federal Funds Rate (FEDFUNDS), which is monthly, and not seasonally adjusted. Quarterly interest rates are obtained by taking averages of the monthly rates.
- **Sample period:** post-Great Moderation and pre-Great Recession – from 1984:Q1 to 2007:Q4.

Empirical Analogues: Autocovariances

- The sample analog of the population autocovariance $\Gamma_{yy}(h)$ is defined as

$$\hat{\Gamma}_{yy}(h) = \frac{1}{T}\sum_{t=h}^{T}(y_t - \hat{\mu}_y)(y_{t-h} - \hat{\mu}_y)', \quad \text{where} \quad \hat{\mu}_y = \frac{1}{T}\sum_{t=1}^{T} y_t.$$

- Could be directly calculated from data.

Empirical Analogues: Autocovariances

- If the object of interest is a sequence of autocovariance matrices, then it might be more efficient to:
  1. estimate an auxiliary model;
  2. convert parameter estimates of the auxiliary model into estimates of the autocovariance sequence.

- E.g., estimate VAR(1):

$$y_t = \Phi_1 y_{t-1} + \Phi_0 + u_t, \quad u_t \sim iid(0, \Sigma).$$

- OLS Estimator

$$\hat{\Phi}_1 \approx \hat{\Gamma}_{yy}(1)\hat{\Gamma}_{yy}^{-1}(0), \quad \hat{\Sigma} \approx \hat{\Gamma}_{yy}(0) - \hat{\Gamma}_{yy}(1)\hat{\Gamma}_{yy}^{-1}(0)\hat{\Gamma}'_{yy}(1)$$

Empirical Analogues: Autocovariances

- Now convert $\hat{\Phi}_1$ and $\hat{\Sigma}$ into autocovariances using the same formulas as for DSGE model

$$\hat{\Gamma}^V_{yy}(0) \approx \hat{\Gamma}_{yy}(0), \quad \hat{\Gamma}^V_{yy}(h) \approx \left(\hat{\Gamma}_{yy}(1)\hat{\Gamma}^{-1}_{yy}(0)\right)^h \hat{\Gamma}_{yy}(0)$$

- For $h = 0, 1$ we obtain $\hat{\Gamma}^V_{yy}(1) = \hat{\Gamma}_{yy}(1) + O_p(T^{-1})$.
- For $h > 1$ the VAR(1) plug-in estimate of the autocovariance matrix differs from the sample autocovariance matrix.
- If the actual time series are well approximated by a VAR(1), then the plug-in autocovariance estimate tends to be more efficient than the sample autocovariance estimate $\hat{\Gamma}_{yy}(h)$.

Empirical Analogues: Autocovariances

Extension to VAR(p):

$$y_t = \Phi_1 y_{t-1} + \ldots + \Phi_p y_{t-p} + \Phi_0 + u_t, \quad u_t \sim iid(0, \Sigma).$$

- Determine lag length with model selection criterion.
- Write in companion form and use VAR(1) formulas:

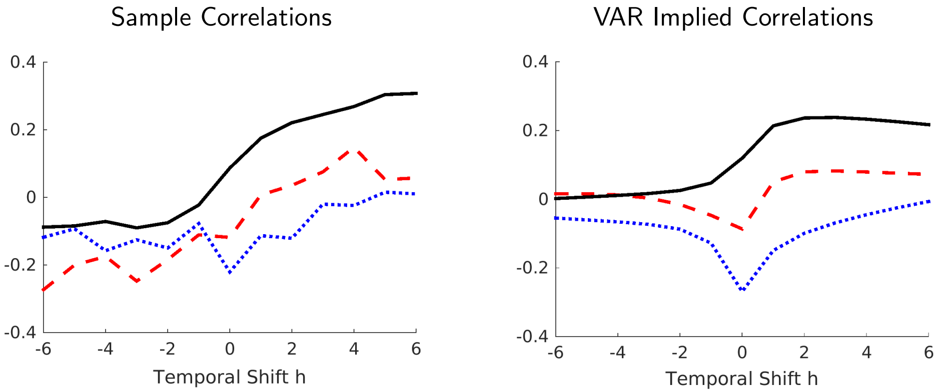
$$\tilde{y}_t = \tilde{\Phi}_1 \tilde{y}_{t-1} + \tilde{\Phi}_0 + \tilde{u}_t, \quad \tilde{u}_t \sim iid(0, \tilde{\Sigma}),$$

where

$$\tilde{\Phi}_1 = \begin{bmatrix} \Phi_1 & \ldots & \Phi_{p-1} & \Phi_p \\ I_{n\times n} & \ldots & 0_{n\times n} & 0_{n\times n} \\ \vdots & \ddots & \vdots & \vdots \\ 0_{n\times n} & \ldots & I_{n\times n} & 0_{n\times n} \end{bmatrix}, \quad \tilde{\Phi}_0 = \begin{bmatrix} \Phi_0 \\ 0_{n(p-1)\times 1} \end{bmatrix},$$

$$\tilde{\epsilon}_t = \begin{bmatrix} \epsilon_t \\ 0_{n(p-1)\times 1} \end{bmatrix}, \quad \tilde{\Sigma} = \begin{bmatrix} \Sigma & 0_{n\times n(p-1)} \\ 0_{n(p-1)\times n} & 0_{n(p-1)\times n(p-1)} \end{bmatrix}.$$

Empirical Cross-Correlations Corr$\left(\log(X_t/X_{t-1}), \log Z_{t-h}\right)$



Notes: Each plot shows the correlation of output growth $\log(X_t/X_{t-1})$ with interest rates (solid), inflation (dashed), and the labor share (dotted), respectively. Left panel: correlation functions are computed from sample autocovariance matrices $\hat{\Gamma}_{yy}(h)$. Right panel: correlation functions are computed from estimated VAR(1).

Empirical Analogues: Impulse Responses

- So far, we considered reduced-form VARs, say,

$$y_t = \Phi_1 y_{t-1} + u_t, \quad \mathbb{E}[u_t u_t'] = \Sigma, \quad y_t = \sum_{h=0}^{\infty} C_h u_{t-h} = \sum_{h=0}^{\infty} \Phi_1^h u_{t-h}.$$

- Error terms u_t have the interpretation of one-step ahead forecast errors.
- According to our DSGE model, one-step ahead forecast errors are functions of innovations to fundamental shocks.
- Need to link one-step ahead forecast errors u_t to structural innovations ϵ_t.

Empirical Analogues: Impulse Response Functions

- How are the two types of shocks related?
- We will assume that the one-step-ahead forecast errors are linear functions of the structural shocks:

$$
\begin{aligned}
u_{1,t} &= \phi_{\epsilon,11}\epsilon_{1,t} + \phi_{\epsilon,12}\epsilon_{2,t} \\
u_{2,t} &= \phi_{\epsilon,21}\epsilon_{1,t} + \phi_{\epsilon,22}\epsilon_{2,t}
\end{aligned}
$$

- Our, in more compact notation:

$$
u_t = \Phi_\epsilon \epsilon_t \tag{1}
$$

- How can we determine Φ_ϵ coefficients?
- Let's assume that the structural shocks are uncorrelated with each other and have unit variance:

$$
\begin{bmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \end{bmatrix} \sim iidN\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right).
$$

Empirical Analogues: Impulse Response Functions

- Ultimately, the Φ_ϵ matrix affects the variance of the one-step-ahead forecast errors u_t.
- Using the fact that the structural shocks $\epsilon_{1,t}$ and $\epsilon_{2,t}$ are *iid* standard normals, we obtain the restrictions:

$$\begin{aligned}
\Sigma_{11} &= \phi^2_{\epsilon,11} + \phi^2_{\epsilon,12} \\
\Sigma_{22} &= \phi^2_{\epsilon,21} + \phi^2_{\epsilon,22} \\
\Sigma_{12} &= \Sigma_{\epsilon,21} = \phi_{\epsilon,11}\phi_{\epsilon,21} + \phi_{\epsilon,12}\phi_{\epsilon,22}
\end{aligned}$$

- Since we know already how to estimate Σ_u, we could simply try to solve for the b's.
- There is a problem: we have four unknowns and only three equations!!!

Empirical Analogues: Impulse Response Functions

- Φ_ϵ has to satisfy the restriction

$$\Phi_\epsilon \Phi_\epsilon' = \Sigma$$

Notice that Φ_ϵ has n^2 elements but Σ only $n(n+1)/2$.

- Take a "square root," e.g. Cholesky, decomposition of

$$\Sigma = \Sigma_{tr}\Sigma_{tr}',$$

Σ_{tr} is lower triangular. If Σ is non-singular the decomp. is unique.

- Let Ω be an orthogonal matrix, meaning that $\Omega\Omega' = \Omega'\Omega = I$.
- Then

$$u_t = \Phi_\epsilon \epsilon_t = \Sigma_{tr}\Omega\epsilon_t,$$

where Σ_{tr} is identifiable and Ω is not, because:

$$\begin{aligned} \mathbb{E}[u_t u_t'] &= \mathbb{E}[\Sigma_{tr}\Omega\epsilon_t\epsilon_t'\Omega'\Sigma_{tr}'] = \Sigma_{tr}\Omega\mathbb{E}[\epsilon_t\epsilon_t']\Omega'\Sigma_{tr}' \\ &= \Sigma_{tr}\Omega\Omega'\Sigma_{tr}' = \Sigma_{tr}\Sigma_{tr}' = \Sigma. \end{aligned}$$

Empirical Analogues: Impulse Response Functions

- Literature on structural VARs is about the mechanics and the economics of imposing restrictions on Ω.
- Conditional on estimates $\hat{\Phi}$ and $\hat{\Sigma}$ and an identification scheme for one or more columns of Ω, IRFs are:

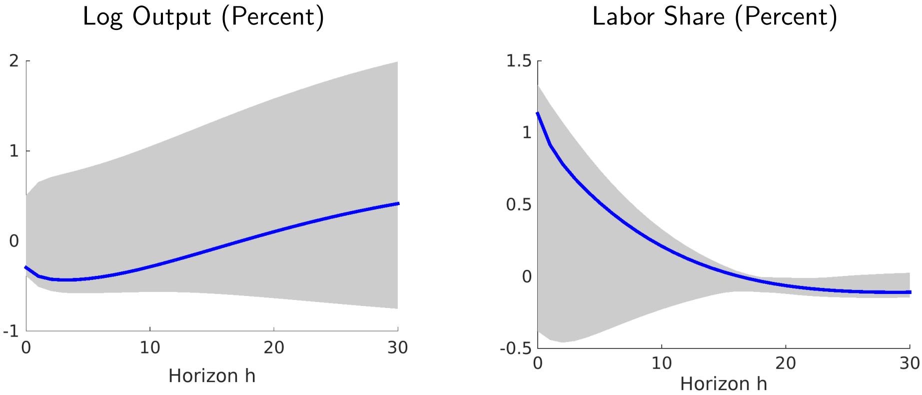
$$\widehat{IRF}^{V}(.,j,h) = C_h(\hat{\Phi})\hat{\Sigma}_{tr}[\Omega]_{.j}$$

- $C_h(\hat{\Phi})$ are moving average coefficient matrices. For VAR(1) $C_h(\Phi) = \Phi^h$.

Empirical Analogues: Impulse Response Functions

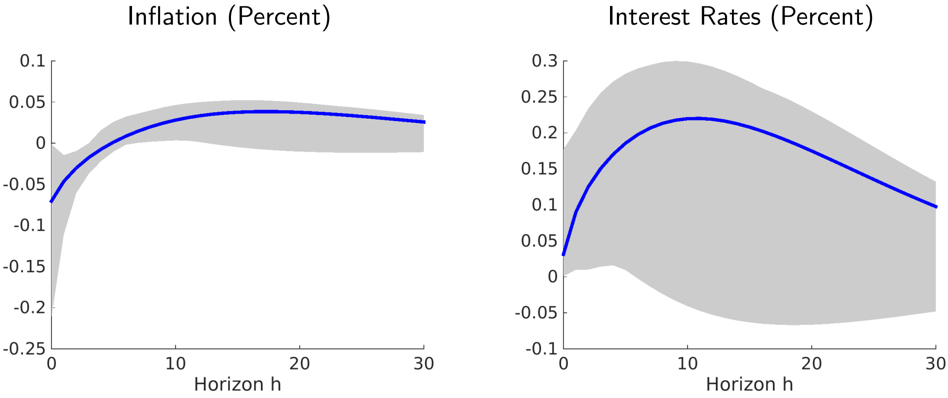
- Problem: it is difficult to find restrictions consistent with DSGE model.
- We follow literature on VARs identified with sign restrictions.
- Restrict Ω to a set $\mathcal{O}(\Phi, \Sigma)$ such that the implied impulse response functions satisfy certain sign restrictions. $\Longrightarrow$ set identification.
- We assume: in response to a contractionary monetary policy shock
  - interest rates increase
  - inflation is negative for four quarters.
- Without loss of generality, assume that MP shock is $\epsilon_{1,t}$. Then first column of Ω, denoted by q, captures the effect of MP shock.

Impulse Responses to a Monetary Policy Shock



Notes: Impulse responses to a one-standard-deviation monetary policy shock. Inflation and interest rate responses are not annualized. The bands indicate pointwise estimates of identified sets for the impulse responses based on the assumption that a contractionary monetary policy shock raises interest rates and lowers inflation for 4 quarters. The solid line represents a particular impulse response function contained in the identified set.

Impulse Responses to a Monetary Policy Shock

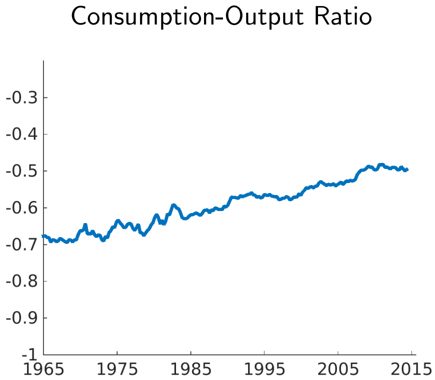


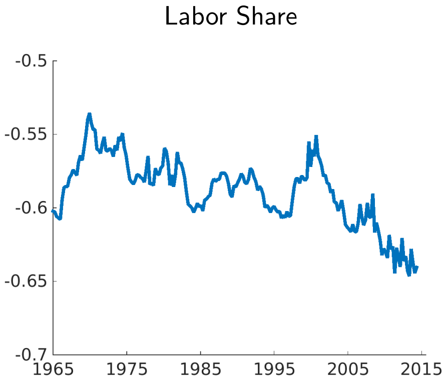
Notes: Impulse responses to a one-standard-deviation monetary policy shock. Inflation and interest rate responses are not annualized. The bands indicate pointwise estimates of identified sets for the impulse responses based on the assumption that a contractionary monetary policy shock raises interest rates and lowers inflation for 4 quarters. The solid line represents a particular impulse response function contained in the identified set.

Dealing with Trends in the Data

- Trends are a salient feature of macroeconomic time series.
- Our DSGE model features a stochastic trend generated by the productivity process $\log Z_t$, which evolves according to a random walk with drift:
  - common trend in consumption, output, and real wages;
  - log consumption-output ratio and the log labor share are stationary.

Consumption-Output Ratio and Labor Share (in Logs)





Dealing with Trends in the Data

- Remedies to address the mismatch between model and data.
  (i) Detrend each time series separately and fit DSGE model to detrended data.
  (ii) Apply an appropriate trend filter to both actual and model-implied data when confronting the DSGE model with data.
  (iii) Create a hybrid model: flexible nonstructural trend + DSGE for fluctuations around trend.
  (iv) Incorporate realistic trend directly into DSGE model.

- From a modeling perspective, option (i) is the least desirable and option (iv) is the most desirable choice.