Sequential Monte Carlo Methods (for DSGE Models)

Frank Schorfheide*

*University of Pennsylvania, PIER, CEPR, and NBER

October 23, 2017

Some References

These lectures use material from our joint work:

- "Tempered Particle Filtering," 2016, *PIER Working Paper*, 16-017
- *Bayesian Estimation of DSGE Models*, 2015, Princeton University Press
- "Sequential Monte Carlo Sampling for DSGE Models," 2014, *Journal of Econometrics*

Sequential Monte Carlo (SMC) Methods

SMC can help to

Lecture 1

- approximate the posterior of θ: Chopin (2002) ... Durham and Geweke (2013) ... Creal (2007), Herbst and Schorfheide (2014)

Lecture 2

- approximate the likelihood function (particle filtering): Gordon, Salmond, and Smith (1993) ... Fernandez-Villaverde and Rubio-Ramirez (2007)
- or both: SMC^2: Chopin, Jacob, and Papaspiliopoulos (2012) ... Herbst and Schorfheide (2015)

Lecture 2

Filtering - General Idea

- State-space representation of nonlinear DSGE model

$$\begin{aligned} \text{Measurement Eq.} &: \quad y_t = \Psi(s_t, t; \theta) + u_t, \quad u_t \sim F_u(\cdot; \theta) \\ \text{State Transition} &: \quad s_t = \Phi(s_{t-1}, \epsilon_t; \theta), \quad \epsilon_t \sim F_\epsilon(\cdot; \theta). \end{aligned}$$

- Likelihood function:

$$p(Y_{1:T}|\theta) = \prod_{t=1}^{T} p(y_t|Y_{1:t-1}, \theta)$$

- A filter generates a sequence of conditional distributions $s_t|Y_{1:t}$.
- Iterations:
  - Initialization at time $t-1$: $p(s_{t-1}|Y_{1:t-1}, \theta)$
  - Forecasting t given $t-1$:
    1. Transition equation:
       $p(s_t|Y_{1:t-1}, \theta) = \int p(s_t|s_{t-1}, Y_{1:t-1}, \theta) p(s_{t-1}|Y_{1:t-1}, \theta) ds_{t-1}$
    2. Measurement equation:
       $p(y_t|Y_{1:t-1}, \theta) = \int p(y_t|s_t, Y_{1:t-1}, \theta) p(s_t|Y_{1:t-1}, \theta) ds_t$
  - Updating with Bayes theorem. Once y_t becomes available:

$$p(s_t|Y_{1:t}, \theta) = p(s_t|y_t, Y_{1:t-1}, \theta) = \frac{p(y_t|s_t, Y_{1:t-1}, \theta) p(s_t|Y_{1:t-1}, \theta)}{p(y_t|Y_{1:t-1}, \theta)}$$

Conditional Distributions for Kalman Filter (Linear Gaussian State-Space Model)

	Distribution	Mean and Variance
$s_{t-1}\|(Y_{1:t-1},\theta)$	$N\big(\bar{s}_{t-1\|t-1}, P_{t-1\|t-1}\big)$	Given from Iteration $t-1$
$s_t\|(Y_{1:t-1},\theta)$	$N\big(\bar{s}_{t\|t-1}, P_{t\|t-1}\big)$	$\bar{s}_{t\|t-1} = \Phi_1 \bar{s}_{t-1\|t-1}$ $P_{t\|t-1} = \Phi_1 P_{t-1\|t-1}\Phi_1' + \Phi_\epsilon \Sigma_\epsilon \Phi_\epsilon'$
$y_t\|(Y_{1:t-1},\theta)$	$N\big(\bar{y}_{t\|t-1}, F_{t\|t-1}\big)$	$\bar{y}_{t\|t-1} = \Psi_0 + \Psi_1 t + \Psi_2 \bar{s}_{t\|t-1}$ $F_{t\|t-1} = \Psi_2 P_{t\|t-1}\Psi_2' + \Sigma_u$
$s_t\|(Y_{1:t},\theta)$	$N\big(\bar{s}_{t\|t}, P_{t\|t}\big)$	$\bar{s}_{t\|t} = \bar{s}_{t\|t-1} + P_{t\|t-1}\Psi_2' F_{t\|t-1}^{-1}(y_t - \bar{y}_{t\|t-1})$ $P_{t\|t} = P_{t\|t-1} - P_{t\|t-1}\Psi_2' F_{t\|t-1}^{-1}\Psi_2 P_{t\|t-1}$

Approximating the Likelihood Function

- DSGE models are inherently nonlinear.
- Sometimes linear approximations are sufficiently accurate...
- but in other applications nonlinearities may be important:
  - asset pricing;
  - borrowing constraints;
  - zero lower bound on nominal interest rates;
  - ...
- Nonlinear state-space representation requires nonlinear filter:

$$
\begin{aligned}
y_t &= \Psi(s_t, t; \theta) + u_t, \quad u_t \sim F_u(\cdot; \theta) \\
s_t &= \Phi(s_{t-1}, \epsilon_t; \theta), \quad \epsilon_t \sim F_\epsilon(\cdot; \theta).
\end{aligned}
$$

Particle Filters

- There are many particle filters...
- We will focus on three types:
  - Bootstrap PF
  - A generic PF
  - A conditionally-optimal PF

Bootstrap Particle Filter

1. **Initialization.** Draw the initial particles from the distribution $s_0^j \stackrel{iid}{\sim} p(s_0)$ and set $W_0^j = 1$, $j = 1, \ldots, M$.
2. **Recursion.** For $t = 1, \ldots, T$:
   1. **Forecasting s_t.** Propagate the period $t-1$ particles $\{s_{t-1}^j, W_{t-1}^j\}$ by iterating the state-transition equation forward:

      $$\tilde{s}_t^j = \Phi(s_{t-1}^j, \epsilon_t^j; \theta), \quad \epsilon_t^j \sim F_\epsilon(\cdot; \theta). \tag{1}$$

      An approximation of $\mathbb{E}[h(s_t)|Y_{1:t-1}, \theta]$ is given by

      $$\hat{h}_{t,M} = \frac{1}{M} \sum_{j=1}^{M} h(\tilde{s}_t^j) W_{t-1}^j. \tag{2}$$

Bootstrap Particle Filter

1. **Initialization.**
2. **Recursion.** For $t = 1, \ldots, T$:
   1. **Forecasting s_t.**
   2. **Forecasting y_t.** Define the incremental weights

$$\tilde{w}_t^j = p(y_t|\tilde{s}_t^j, \theta). \tag{3}$$

The predictive density $p(y_t|Y_{1:t-1}, \theta)$ can be approximated by

$$\hat{p}(y_t|Y_{1:t-1}, \theta) = \frac{1}{M} \sum_{j=1}^{M} \tilde{w}_t^j W_{t-1}^j. \tag{4}$$

If the measurement errors are $N(0, \Sigma_u)$ then the incremental weights take the form

$$\tilde{w}_t^j = (2\pi)^{-n/2} |\Sigma_u|^{-1/2} \exp\left\{ -\frac{1}{2} \big(y_t - \Psi(\tilde{s}_t^j, t; \theta)\big)' \Sigma_u^{-1} \big(y_t - \Psi(\tilde{s}_t^j, t; \theta)\big) \right\}, \tag{5}$$

where n here denotes the dimension of y_t.

Bootstrap Particle Filter

1. **Initialization.**
2. **Recursion.** For $t = 1, \ldots, T$:
   1. **Forecasting s_t.**
   2. **Forecasting y_t.** Define the incremental weights

$$\tilde{w}_t^j = p(y_t | \tilde{s}_t^j, \theta). \tag{6}$$

   3. **Updating.** Define the normalized weights

$$\tilde{W}_t^j = \frac{\tilde{w}_t^j W_{t-1}^j}{\frac{1}{M} \sum_{j=1}^M \tilde{w}_t^j W_{t-1}^j}. \tag{7}$$

   An approximation of $\mathbb{E}[h(s_t)|Y_{1:t}, \theta]$ is given by

$$\tilde{h}_{t,M} = \frac{1}{M} \sum_{j=1}^M h(\tilde{s}_t^j) \tilde{W}_t^j. \tag{8}$$

Bootstrap Particle Filter

1. **Initialization.**
2. **Recursion.** For $t = 1, \ldots, T$:
   1. **Forecasting s_t.**
   2. **Forecasting y_t.**
   3. **Updating.**
   4. **Selection (Optional).** Resample the particles via multinomial resampling. Let $\{s_t^j\}_{j=1}^M$ denote M iid draws from a multinomial distribution characterized by support points and weights $\{\tilde{s}_t^j, \tilde{W}_t^j\}$ and set $W_t^j = 1$ for $j =, 1 \ldots, M$.
   An approximation of $\mathbb{E}[h(s_t)|Y_{1:t}, \theta]$ is given by

$$\bar{h}_{t,M} = \frac{1}{M} \sum_{j=1}^{M} h(s_t^j) W_t^j. \tag{9}$$

Likelihood Approximation

- The approximation of the log likelihood function is given by

$$\ln \hat{p}(Y_{1:T}|\theta) = \sum_{t=1}^{T} \ln \left(\frac{1}{M} \sum_{j=1}^{M} \tilde{w}_t^j W_{t-1}^j \right). \qquad (10)$$

- One can show that the approximation of the likelihood function is unbiased.

- This implies that the approximation of the log likelihood function is downward biased.

The Role of Measurement Errors

- Measurement errors may not be intrinsic to DSGE model.
- Bootstrap filter needs non-degenerate $p(y_t|s_t,\theta)$ for incremental weights to be well defined.
- Decreasing the measurement error variance Σ_u, holding everything else fixed, increases the variance of the particle weights, and reduces the accuracy of Monte Carlo approximation.

Generic Particle Filter

1. **Initialization.** Same as BS PF
2. **Recursion.** For $t = 1, \ldots, T$:
   1. **Forecasting s_t.** Draw $\tilde{s}_t^j$ from density $g_t(\tilde{s}_t|s_{t-1}^j, \theta)$ and define

      $$\omega_t^j = \frac{p(\tilde{s}_t^j|s_{t-1}^j, \theta)}{g_t(\tilde{s}_t^j|s_{t-1}^j, \theta)}. \tag{11}$$

      An approximation of $\mathbb{E}[h(s_t)|Y_{1:t-1}, \theta]$ is given by

      $$\hat{h}_{t,M} = \frac{1}{M}\sum_{j=1}^{M} h(\tilde{s}_t^j)\omega_t^j W_{t-1}^j. \tag{12}$$

   2. **Forecasting y_t.** Define the incremental weights

      $$\tilde{w}_t^j = p(y_t|\tilde{s}_t^j, \theta)\omega_t^j. \tag{13}$$

      The predictive density $p(y_t|Y_{1:t-1}, \theta)$ can be approximated by

      $$\hat{p}(y_t|Y_{1:t-1}, \theta) = \frac{1}{M}\sum_{j=1}^{M} \tilde{w}_t^j W_{t-1}^j. \tag{14}$$

   3. **Updating.** Same as BS PF
   4. **Selection.** Same as BS PF
3. **Likelihood Approximation.** Same as BS PF

Asymptotics

- The convergence results can be established recursively, starting from the assumption

$$\begin{aligned} \bar{h}_{t-1,M} &\stackrel{a.s.}{\longrightarrow} \mathbb{E}[h(s_{t-1})|Y_{1:t-1}], \\ \sqrt{M}\big(\bar{h}_{t-1,M} - \mathbb{E}[h(s_{t-1})|Y_{1:t-1}]\big) &\Longrightarrow N\big(0, \Omega_{t-1}(h)\big). \end{aligned}$$

- Forward iteration: draw s_t from $g_t(s_t|s^j_{t-1}) = p(s_t|s^j_{t-1})$.
- Decompose

$$\begin{aligned} & \hat{h}_{t,M} - \mathbb{E}[h(s_t)|Y_{1:t-1}] \qquad (15) \\ &= \frac{1}{M}\sum_{j=1}^{M}\Big(h(\tilde{s}^j_t) - \mathbb{E}_{p(\cdot|s^j_{t-1})}[h]\Big) W^j_{t-1} \\ &\quad + \frac{1}{M}\sum_{j=1}^{M}\Big(\mathbb{E}_{p(\cdot|s^j_{t-1})}[h] W^j_{t-1} - \mathbb{E}[h(s_t)|Y_{1:t-1}]\Big) \\ &= I + II, \end{aligned}$$

- Both I and II converge to zero (and potentially satisfy CLT).

Asymptotics

- Updating step approximates

$$\mathbb{E}[h(s_t)|Y_{1:t}] = \frac{\int h(s_t)p(y_t|s_t)p(s_t|Y_{1:t-1})ds_t}{\int p(y_t|s_t)p(s_t|Y_{1:t-1})ds_t} \approx \frac{\frac{1}{M}\sum_{j=1}^{M} h(\tilde{s}_t^j)\tilde{w}_t^j W_{t-1}^j}{\frac{1}{M}\sum_{j=1}^{M} \tilde{w}_t^j W_{t-1}^j} \tag{16}$$

- Define the normalized incremental weights as

$$v_t(s_t) = \frac{p(y_t|s_t)}{\int p(y_t|s_t)p(s_t|Y_{1:t-1})ds_t}. \tag{17}$$

- Under suitable regularity conditions, the Monte Carlo approximation satisfies a CLT of the form

$$\sqrt{M}\big(\tilde{h}_{t,M} - \mathbb{E}[h(s_t)|Y_{1:t}]\big) \tag{18}$$
$$\Longrightarrow \quad N\big(0, \tilde{\Omega}_t(h)\big), \quad \tilde{\Omega}_t(h) = \hat{\Omega}_t\big(v_t(s_t)(h(s_t) - \mathbb{E}[h(s_t)|Y_{1:t}])\big).$$

- Distribution of particle weights matters for accuracy! $\Longrightarrow$ Resampling!

Adapting the Generic PF

- Conditionally-optimal importance distribution:

$$g_t(\tilde{s}_t|s^j_{t-1}) = p(\tilde{s}_t|y_t, s^j_{t-1}).$$

  This is the posterior of s_t given s^j_{t-1}. Typically infeasible, but a good benchmark.

- Approximately conditionally-optimal distributions: from linearize version of DSGE model or approximate nonlinear filters.

- Conditionally-linear models: do Kalman filter updating on a subvector of s_t. Example:

$$\begin{aligned} y_t &= \Psi_0(m_t) + \Psi_1(m_t)t + \Psi_2(m_t)s_t + u_t, \quad u_t \sim N(0, \Sigma_u), \\ s_t &= \Phi_0(m_t) + \Phi_1(m_t)s_{t-1} + \Phi_\epsilon(m_t)\epsilon_t, \quad \epsilon_t \sim N(0, \Sigma_\epsilon), \end{aligned}$$

  where m_t follows a discrete Markov-switching process.

Nonlinear and Partially Deterministic State Transitions

- Example:

$$s_{1,t} = \Phi_1(s_{t-1}, \epsilon_t), \quad s_{2,t} = \Phi_2(s_{t-1}), \quad \epsilon_t \sim N(0,1).$$

- Generic filter requires evaluation of $p(s_t|s_{t-1})$.
- Define $\varsigma_t = [s_t', \epsilon_t']'$ and add identity $\epsilon_t = \epsilon_t$ to state transition.
- Factorize the density $p(\varsigma_t|\varsigma_{t-1})$ as

$$p(\varsigma_t|\varsigma_{t-1}) = p^{\epsilon}(\epsilon_t)p(s_{1,t}|s_{t-1}, \epsilon_t)p(s_{2,t}|s_{t-1}).$$

where $p(s_{1,t}|s_{t-1}, \epsilon_t)$ and $p(s_{2,t}|s_{t-1})$ are pointmasses.

- Sample innovation ϵ_t from $g_t^{\epsilon}(\epsilon_t|s_{t-1})$.
- Then

$$\omega_t^j = \frac{p(\tilde{\varsigma}_t^j|\varsigma_{t-1}^j)}{g_t(\tilde{\varsigma}_t^j|\varsigma_{t-1}^j)} = \frac{p^{\epsilon}(\tilde{\epsilon}_t^j)p(\tilde{s}_{1,t}^j|s_{t-1}^j, \tilde{\epsilon}_t^j)p(\tilde{s}_{2,t}^j|s_{t-1}^j)}{g_t^{\epsilon}(\tilde{\epsilon}_t^j|s_{t-1}^j)p(\tilde{s}_{1,t}^j|s_{t-1}^j, \tilde{\epsilon}_t^j)p(\tilde{s}_{2,t}^j|s_{t-1}^j)} = \frac{p^{\epsilon}(\tilde{\epsilon}_t^j)}{g_t^{\epsilon}(\tilde{\epsilon}_t^j|s_{t-1}^j)}.$$

Degenerate Measurement Error Distributions

- Our discussion of the conditionally-optimal importance distribution suggests that in the absence of measurement errors, one has to solve the system of equations

$$y_t = \Psi\big(\Phi(s^j_{t-1}, \tilde{\epsilon}^j_t)\big),$$

to determine $\tilde{\epsilon}^j_t$ as a function of s^j_{t-1} and the current observation y_t.

- Then define

$$\omega^j_t = p^\epsilon(\tilde{\epsilon}^j_t) \quad \text{and} \quad \tilde{s}^j_t = \Phi(s^j_{t-1}, \tilde{\epsilon}^j_t).$$

- Difficulty: one has to find all solutions to a nonlinear system of equations.

- While resampling duplicates particles, the duplicated particles do not mutate, which can lead to a degeneracy.

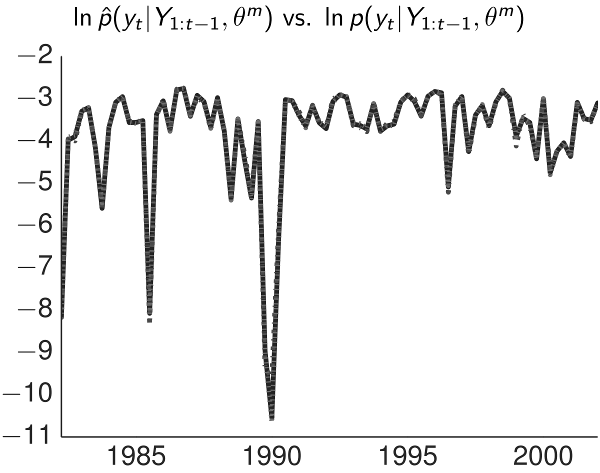
Next Steps

- We will now apply PFs to linearized DSGE models.
- This allows us to compare the Monte Carlo approximation to the "truth."
- Small-scale New Keynesian DSGE model
- Smets-Wouters model

Parameter Values For Likelihood Evaluation

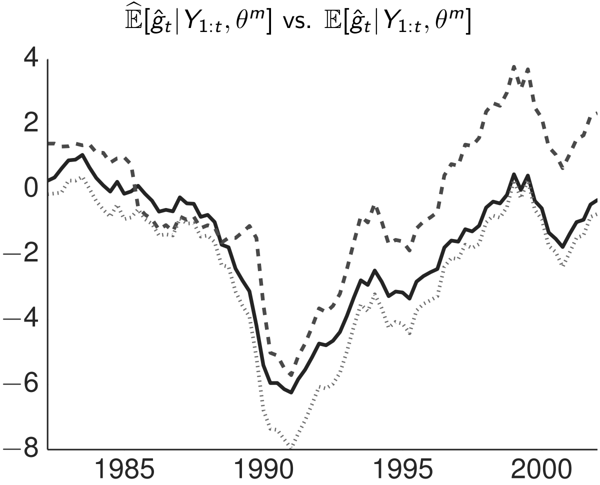
Parameter	θ^m	θ^l	Parameter	θ^m	θ^l
τ	2.09	3.26	κ	0.98	0.89
ψ_1	2.25	1.88	ψ_2	0.65	0.53
ρ_r	0.81	0.76	ρ_g	0.98	0.98
ρ_z	0.93	0.89	$r^{(A)}$	0.34	0.19
$\pi^{(A)}$	3.16	3.29	$\gamma^{(Q)}$	0.51	0.73
σ_r	0.19	0.20	σ_g	0.65	0.58
σ_z	0.24	0.29	$\ln p(Y\|\theta)$	-306.5	-313.4

Likelihood Approximation



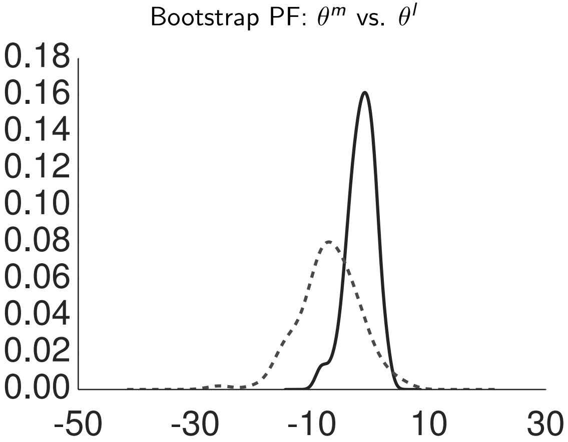
Notes: The results depicted in the figure are based on a single run of the bootstrap PF (dashed, $M = 40,000$), the conditionally-optimal PF (dotted, $M = 400$), and the Kalman filter (solid).

Filtered State



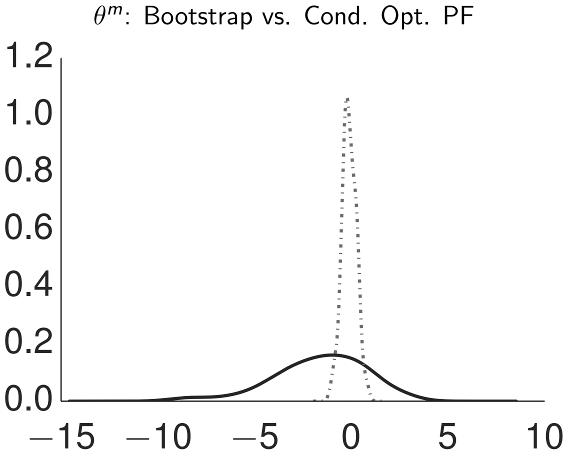
Notes: The results depicted in the figure are based on a single run of the bootstrap PF (dashed, $M = 40,000$), the conditionally-optimal PF (dotted, $M = 400$), and the Kalman filter (solid).

Distribution of Log-Likelihood Approximation Errors



Notes: Density estimate of $\hat{\Delta}_1 = \ln \hat{p}(Y_{1:T}|\theta) - \ln p(Y_{1:T}|\theta)$ based on $N_{run} = 100$ runs of the PF. Solid line is $\theta = \theta^m$; dashed line is $\theta = \theta^l$ $(M = 40,000)$.

Distribution of Log-Likelihood Approximation Errors



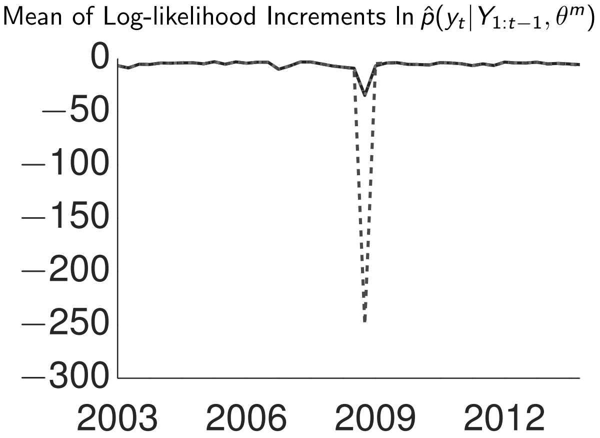
Notes: Density estimate of $\hat{\Delta}_1 = \ln \hat{p}(Y_{1:T}|\theta) - \ln p(Y_{1:T}|\theta)$ based on $N_{run} = 100$ runs of the PF. Solid line is bootstrap particle filter ($M = 40,000$); dotted line is conditionally optimal particle filter ($M = 400$).

Summary Statistics for Particle Filters

	Bootstrap	Cond. Opt.	Auxiliary
Number of Particles M	40,000	400	40,000
Number of Repetitions	100	100	100
High Posterior Density: $\theta = \theta^m$			
Bias $\hat{\Delta}_1$	-1.39	-0.10	-2.83
StdD $\hat{\Delta}_1$	2.03	0.37	1.87
Bias $\hat{\Delta}_2$	0.32	-0.03	-0.74
Low Posterior Density: $\theta = \theta^l$			
Bias $\hat{\Delta}_1$	-7.01	-0.11	-6.44
StdD $\hat{\Delta}_1$	4.68	0.44	4.19
Bias $\hat{\Delta}_2$	-0.70	-0.02	-0.50

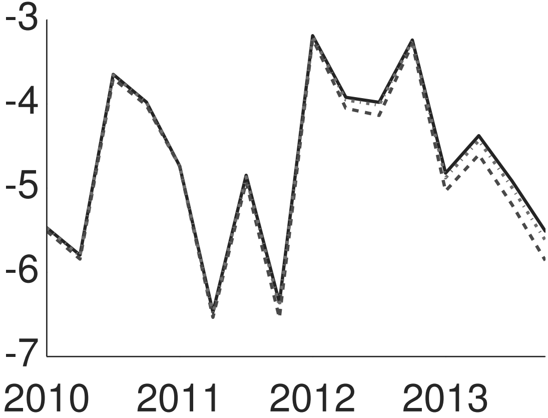
Notes: $\hat{\Delta}_1 = \ln \hat{p}(Y_{1:T}|\theta) - \ln p(Y_{1:T}|\theta)$ and $\hat{\Delta}_2 = \exp[\ln \hat{p}(Y_{1:T}|\theta) - \ln p(Y_{1:T}|\theta)] - 1$. Results are based on $N_{run} = 100$ runs of the particle filters.

Great Recession and Beyond

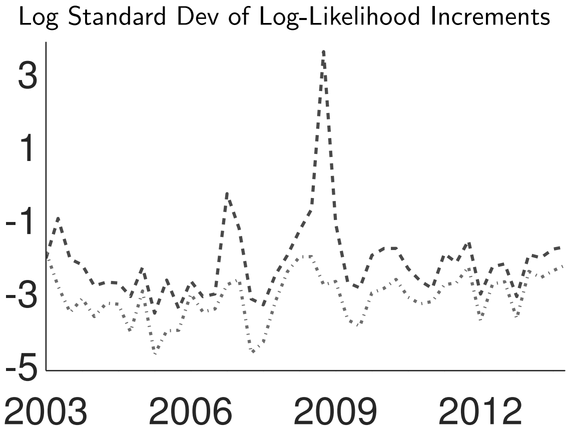


Notes: Solid lines represent results from Kalman filter. Dashed lines correspond to bootstrap particle filter ($M = 40,000$) and dotted lines correspond to conditionally-optimal particle filter ($M = 400$). Results are based on $N_{run} = 100$ runs of the filters.

Mean of Log-likelihood Increments $\ln \hat{p}(y_t|Y_{1:t-1}, \theta^m)$

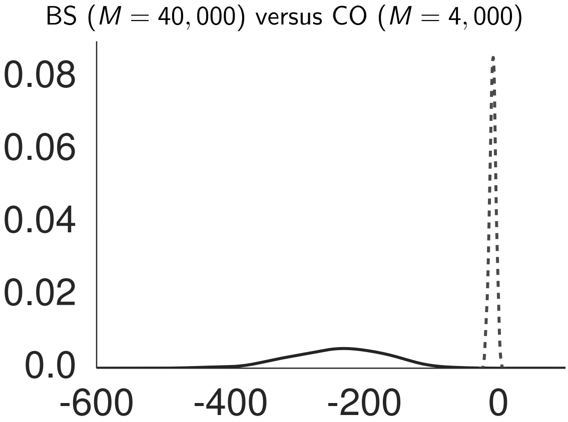


Notes: Solid lines represent results from Kalman filter. Dashed lines correspond to bootstrap particle filter ($M = 40,000$) and dotted lines correspond to conditionally-optimal particle filter ($M = 400$). Results are based on $N_{run} = 100$ runs of the filters.



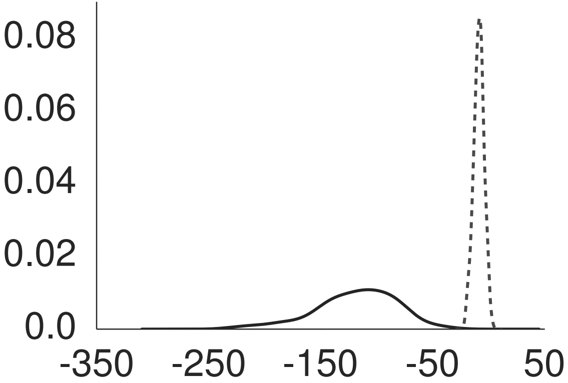
Notes: Solid lines represent results from Kalman filter. Dashed lines correspond to bootstrap particle filter ($M = 40,000$) and dotted lines correspond to conditionally-optimal particle filter ($M = 400$). Results are based on $N_{run} = 100$ runs of the filters.

SW Model: Distr. of Log-Likelihood Approximation Errors



Notes: Density estimates of $\hat{\Delta}_1 = \ln \hat{p}(Y|\theta) - \ln p(Y|\theta)$ based on $N_{run} = 100$. Solid densities summarize results for the bootstrap (BS) particle filter; dashed densities summarize results for the conditionally-optimal (CO) particle filter.

BS ($M = 400,000$) versus CO ($M = 4,000$)



Notes: Density estimates of $\hat{\Delta}_1 = \ln \hat{p}(Y|\theta) - \ln p(Y|\theta)$ based on $N_{run} = 100$. Solid densities summarize results for the bootstrap (BS) particle filter; dashed densities summarize results for the conditionally-optimal (CO) particle filter.

SW Model: Summary Statistics for Particle Filters

	Bootstrap		Cond. Opt.	
Number of Particles M	40,000	400,000	4,000	40,000
Number of Repetitions	100	100	100	100
High Posterior Density: $\theta = \theta^m$				
Bias $\hat{\Delta}_1$	-238.49	-118.20	-8.55	-2.88
StdD $\hat{\Delta}_1$	68.28	35.69	4.43	2.49
Bias $\hat{\Delta}_2$	-1.00	-1.00	-0.87	-0.41
Low Posterior Density: $\theta = \theta^l$				
Bias $\hat{\Delta}_1$	-253.89	-128.13	-11.48	-4.91
StdD $\hat{\Delta}_1$	65.57	41.25	4.98	2.75
Bias $\hat{\Delta}_2$	-1.00	-1.00	-0.97	-0.64

Notes: $\hat{\Delta}_1 = \ln \hat{p}(Y_{1:T}|\theta) - \ln p(Y_{1:T}|\theta)$ and $\hat{\Delta}_2 = \exp[\ln \hat{p}(Y_{1:T}|\theta) - \ln p(Y_{1:T}|\theta)] - 1$. Results are based on $N_{run} = 100$.

Embedding PF Likelihoods into Posterior Samplers

- Likelihood functions for nonlinear DSGE models can be approximated by the PF.
- We will now embed the likelihood approximation into a posterior sampler: PFMH Algorithm (a special case of PMCMC).
- The book also discusses SMC^2.

Embedding PF Likelihoods into Posterior Samplers

- Distinguish between:
  - $\{p(Y|\theta), p(\theta|Y), p(Y)\}$, which are related according to:

$$p(\theta|Y) = \frac{p(Y|\theta)p(\theta)}{p(Y)}, \quad p(Y) = \int p(Y|\theta)p(\theta)d\theta$$

  - $\{\hat{p}(Y|\theta), \hat{p}(\theta|Y), \hat{p}(Y)\}$, which are related according to:

$$\hat{p}(\theta|Y) = \frac{\hat{p}(Y|\theta)p(\theta)}{\hat{p}(Y)}, \quad \hat{p}(Y) = \int \hat{p}(Y|\theta)p(\theta)d\theta.$$

- Surprising result (Andrieu, Docet, and Holenstein, 2010): under certain conditions we can replace $p(Y|\theta)$ by $\hat{p}(Y|\theta)$ and still obtain draws from $p(\theta|Y)$.

PFMH Algorithm

For $i = 1$ to N:

1. Draw ϑ from a density $q(\vartheta|\theta^{i-1})$.
2. Set $\theta^i = \vartheta$ with probability

$$\alpha(\vartheta|\theta^{i-1}) = \min\left\{1,\ \frac{\hat{p}(Y|\vartheta)p(\vartheta)/q(\vartheta|\theta^{i-1})}{\hat{p}(Y|\theta^{i-1})p(\theta^{i-1})/q(\theta^{i-1}|\vartheta)}\right\}$$

   and $\theta^i = \theta^{i-1}$ otherwise. The likelihood approximation $\hat{p}(Y|\vartheta)$ is computed using a particle filter.

Why Does the PFMH Work?

- At each iteration the filter generates draws $\tilde{s}_t^j$ from the proposal distribution $g_t(\cdot|s_{t-1}^j)$.
- Let $\tilde{S}_t = \left(\tilde{s}_t^1, \ldots, \tilde{s}_t^M\right)'$ and denote the entire sequence of draws by $\tilde{S}_{1:T}^{1:M}$.
- Selection step: define a random variable A_t^j that contains this ancestry information. For instance, suppose that during the resampling particle $j = 1$ was assigned the value $\tilde{s}_t^{10}$ then $A_t^1 = 10$. Let $A_t = \left(A_t^1, \ldots, A_t^N\right)$ and use $A_{1:T}$ to denote the sequence of A_t's.
- PFMH operates on an enlarged probability space: θ, $\tilde{S}_{1:T}$ and $A_{1:T}$.

Why Does the PFMH Work?

- Use $U_{1:T}$ to denote random vectors for $\tilde{S}_{1:T}$ and $A_{1:T}$. $U_{1:T}$ is an array of *iid* uniform random numbers.

- The transformation of $U_{1:T}$ into $(\tilde{S}_{1:T}, A_{1:T})$ typically depends on θ and $Y_{1:T}$, because the proposal distribution $g_t(\tilde{s}_t|s^j_{t-1})$ depends both on the current observation y_t as well as the parameter vector θ.

- E.g., implementation of conditionally-optimal PF requires sampling from a $N(\bar{s}^j_{t|t}, P_{t|t})$ distribution for each particle j. Can be done using a prob integral transform of uniform random variables.

- We can express the particle filter approximation of the likelihood function as

$$\hat{p}(Y_{1:T}|\theta) = g(Y_{1:T}|\theta, U_{1:T}).$$

where

$$U_{1:T} \sim p(U_{1:T}) = \prod_{t=1}^{T} p(U_t).$$

Why Does the PFMH Work?

- Define the joint distribution

$$p_g\big(Y_{1:T}, \theta, U_{1:T}\big) = g(Y_{1:T}|\theta, U_{1:T})p\big(U_{1:T}\big)p(\theta).$$

- The PFMH algorithm samples from the joint posterior

$$p_g\big(\theta, U_{1:T}\big|Y_{1:T}\big) \propto g(Y|\theta, U_{1:T})p\big(U_{1:T}\big)p(\theta)$$

  and discards the draws of $\big(U_{1:T}\big)$.

- For this procedure to be valid, it needs to be the case that PF approximation is unbiased:

$$\mathbb{E}[\hat{p}(Y_{1:T}|\theta)] = \int g(Y_{1:T}|\theta, U_{1:T})p\big(U_{1:T}\big)d\theta = p(Y_{1:T}|\theta).$$

Why Does the PFMH Work?

- We can express acceptance probability directly in terms of $\hat{p}(Y_{1:T}|\theta)$.

- Need to generate a proposed draw for both θ and $U_{1:T}$: ϑ and $U^*_{1:T}$.

- The proposal distribution for $(\vartheta, U^*_{1:T})$ in the MH algorithm is given by $q(\vartheta|\theta^{(i-1)})p(U^*_{1:T})$.

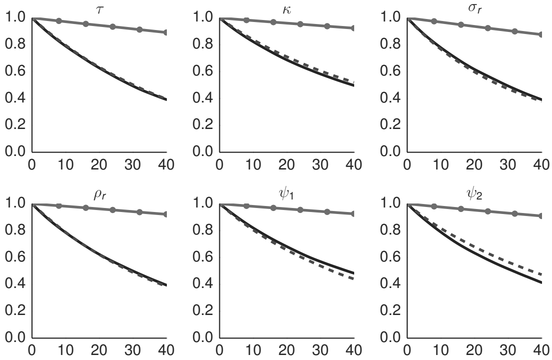
- No need to keep track of the draws $(U^*_{1:T})$.

- MH acceptance probability:

$$
\begin{aligned}
\alpha(\vartheta|\theta^{i-1}) &= \min \left\{ 1, \frac{\frac{g(Y|\vartheta, U^*)p(U^*)p(\vartheta)}{q(\vartheta|\theta^{(i-1)})p(U^*)}}{\frac{g(Y|\theta^{(i-1)}, U^{(i-1)})p(U^{(i-1)})p(\theta^{(i-1)})}{q(\theta^{(i-1)}|\theta^*)p(U^{(i-1)})}} \right\} \\
&= \min \left\{ 1, \frac{\hat{p}(Y|\vartheta)p(\vartheta)\big/q(\vartheta|\theta^{(i-1)})}{\hat{p}(Y|\theta^{(i-1)})p(\theta^{(i-1)})\big/q(\theta^{(i-1)}|\vartheta)} \right\}.
\end{aligned}
$$

Small-Scale DSGE: Accuracy of MH Approximations

- Results are based on $N_{run} = 20$ runs of the PF-RWMH-V algorithm.
- Each run of the algorithm generates $N = 100,000$ draws and the first $N_0 = 50,000$ are discarded.
- The likelihood function is computed with the Kalman filter (KF), bootstrap particle filter (BS-PF, $M = 40,000$) or conditionally-optimal particle filter (CO-PF, $M = 400$).
- "Pooled" means that we are pooling the draws from the $N_{run} = 20$ runs to compute posterior statistics.



Notes: The figure depicts autocorrelation functions computed from the output of the 1 Block RWMH-V algorithm based on the Kalman filter (solid), the conditionally-optimal particle filter (dashed) and the bootstrap particle filter (solid with dots).

Small-Scale DSGE: Accuracy of MH Approximations

	Posterior Mean (Pooled)			Inefficiency Factors			Std Dev of Means		
	KF	CO-PF	BS-PF	KF	CO-PF	BS-PF	KF	CO-PF	BS-PF
τ	2.63	2.62	2.64	66.17	126.76	1360.22	0.020	0.028	0.091
κ	0.82	0.81	0.82	128.00	97.11	1887.37	0.007	0.006	0.026
ψ_1	1.88	1.88	1.87	113.46	159.53	749.22	0.011	0.013	0.029
ψ_2	0.64	0.64	0.63	61.28	56.10	681.85	0.011	0.010	0.036
ρ_r	0.75	0.75	0.75	108.46	134.01	1535.34	0.002	0.002	0.007
ρ_g	0.98	0.98	0.98	94.10	88.48	1613.77	0.001	0.001	0.002
ρ_z	0.88	0.88	0.88	124.24	118.74	1518.66	0.001	0.001	0.005
$r^{(A)}$	0.44	0.44	0.44	148.46	151.81	1115.74	0.016	0.016	0.044
$\pi^{(A)}$	3.32	3.33	3.32	152.08	141.62	1057.90	0.017	0.016	0.045
$\gamma^{(Q)}$	0.59	0.59	0.59	106.68	142.37	899.34	0.006	0.007	0.018
σ_r	0.24	0.24	0.24	35.21	179.15	1105.99	0.001	0.002	0.004
σ_g	0.68	0.68	0.67	98.22	64.18	1490.81	0.003	0.002	0.011
σ_z	0.32	0.32	0.32	84.77	61.55	575.90	0.001	0.001	0.003
$\ln \hat{p}(Y)$	-357.14	-357.17	-358.32				0.040	0.038	0.949

SW Model: Accuracy of MH Approximations

- Results are based on $N_{run} = 20$ runs of the PF-RWMH-V algorithm.
- Each run of the algorithm generates $N = 10,000$ draws.
- The likelihood function is computed with the Kalman filter (KF) or conditionally-optimal particle filter (CO-PF).
- "Pooled" means that we are pooling the draws from the $N_{run} = 20$ runs to compute posterior statistics. The CO-PF uses $M = 40,000$ particles to compute the likelihood.

SW Model: Accuracy of MH Approximations

	Post. Mean (Pooled)		Ineff. Factors		Std Dev of Means	
	KF	CO-PF	KF	CO-PF	KF	CO-PF
$(100\beta^{-1} - 1)$	0.14	0.14	172.58	3732.90	0.007	0.034
$\bar{\pi}$	0.73	0.74	185.99	4343.83	0.016	0.079
$\bar{l}$	0.51	0.37	174.39	3133.89	0.130	0.552
α	0.19	0.20	149.77	5244.47	0.003	0.015
σ_c	1.49	1.45	86.27	3557.81	0.013	0.086
Φ	1.47	1.45	134.34	4930.55	0.009	0.056
φ	5.34	5.35	138.54	3210.16	0.131	0.628
h	0.70	0.72	277.64	3058.26	0.008	0.027
ξ_w	0.75	0.75	343.89	2594.43	0.012	0.034
σ_l	2.28	2.31	162.09	4426.89	0.091	0.477
ξ_p	0.72	0.72	182.47	6777.88	0.008	0.051
ι_w	0.54	0.53	241.80	4984.35	0.016	0.073
ι_p	0.48	0.50	205.27	5487.34	0.015	0.078
ψ	0.45	0.44	248.15	3598.14	0.020	0.078
r_π	2.09	2.09	98.32	3302.07	0.020	0.116
ρ	0.80	0.80	241.63	4896.54	0.006	0.025
r_y	0.13	0.13	243.85	4755.65	0.005	0.023
$r_{\Delta y}$	0.21	0.21	101.94	5324.19	0.003	0.022

SW Model: Accuracy of MH Approximations

	Post. Mean (Pooled)		Ineff. Factors		Std Dev of Means	
	KF	CO-PF	KF	CO-PF	KF	CO-PF
ρ_a	0.96	0.96	153.46	1358.87	0.002	0.005
ρ_b	0.22	0.21	325.98	4468.10	0.018	0.068
ρ_g	0.97	0.97	57.08	2687.56	0.002	0.011
ρ_i	0.71	0.70	219.11	4735.33	0.009	0.044
ρ_r	0.54	0.54	194.73	4184.04	0.020	0.094
ρ_p	0.80	0.81	338.69	2527.79	0.022	0.061
ρ_w	0.94	0.94	135.83	4851.01	0.003	0.019
ρ_{ga}	0.41	0.37	196.38	5621.86	0.025	0.133
μ_p	0.66	0.66	300.29	3552.33	0.025	0.087
μ_w	0.82	0.81	218.43	5074.31	0.011	0.052
σ_a	0.34	0.34	128.00	5096.75	0.005	0.034
σ_b	0.24	0.24	186.13	3494.71	0.004	0.016
σ_g	0.51	0.49	208.14	2945.02	0.006	0.021
σ_i	0.43	0.44	115.42	6093.72	0.006	0.043
σ_r	0.14	0.14	193.37	3408.01	0.004	0.016
σ_p	0.13	0.13	194.22	4587.76	0.003	0.013
σ_w	0.22	0.22	211.80	2256.19	0.004	0.012
$\ln \hat{p}(Y)$	-964	-1018			0.298	9.139

Conclusion

- Versatile set of tools...
- SMC can be used to compute Laplace-type estimators.
- Particle filters can be used for other nonlinear state-space models.
- If you are interested in more: *Tempered Particle Filtering* presentation in *Econometrics Workshop* on Wednesday.