Computing Sunspot Equilibria in Linear Rational Expectations Models

Technical Appendix

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Solving the New Keynesian Model

The DSGE model presented in Sections 2 and 5 of the paper can be expressed in terms of $\xi_t = [\xi_t^y, \xi_t^\pi]'$ as

$$\underbrace{\begin{bmatrix} 1 & \sigma \\ 0 & \beta \end{bmatrix}}_{\Gamma_0} \xi_t = \underbrace{\begin{bmatrix} 1 & \sigma\psi \\ -\kappa & 1 \end{bmatrix}}_{\Gamma_1} \xi_{t-1} + \underbrace{\begin{bmatrix} \sigma \\ 0 \end{bmatrix}}_{\Psi} \epsilon_t + \underbrace{\begin{bmatrix} 1 & \sigma\psi \\ -\kappa & 1 \end{bmatrix}}_{\Pi} \eta_t. \tag{1}$$

Premultiply the system by

$$\Gamma_0^{-1} = \begin{bmatrix} 1 & -\sigma/\beta \\ 0 & 1/\beta \end{bmatrix} \tag{2}$$

to obtain

$$\xi_t = \underbrace{\begin{bmatrix} 1+\frac{\kappa\sigma}{\beta} & \sigma(\psi - \frac{1}{\beta}) \\ -\frac{\kappa}{\beta} & \frac{1}{\beta} \end{bmatrix}}_{\Gamma_1^*} \xi_{t-1} + \underbrace{\begin{bmatrix} \sigma \\ 0 \end{bmatrix}}_{\Psi^*} \epsilon_t + \underbrace{\begin{bmatrix} 1+\frac{\kappa\sigma}{\beta} & \sigma(\psi - \frac{1}{\beta}) \\ -\frac{\kappa}{\beta} & \frac{1}{\beta} \end{bmatrix}}_{\Pi^*} \eta_t. \tag{3}$$

The eigenvalues of Γ_1^* are a solution to the equation

$$0 = \det \begin{bmatrix} (1+\kappa\sigma/\beta) - \lambda & \sigma(\psi - 1/\beta) \\ -\kappa/\beta & 1/\beta - \lambda \end{bmatrix} \tag{4}$$

which can be rewritten as

$$0 = \lambda^2 - \lambda \left[1 + \frac{1}{\beta}(1+\kappa\sigma) \right] + \frac{1}{\beta}(1+\kappa\sigma\psi). \tag{5}$$

The solution of this quadratic equation is

$$\lambda_1, \lambda_2 = \underbrace{\frac{1}{2}\left(1 + \frac{\kappa\sigma+1}{\beta}\right)}_{l_1} \mp \underbrace{\frac{1}{2}\sqrt{\left(\frac{\kappa\sigma+1}{\beta} - 1\right)^2 + \frac{4\kappa\sigma}{\beta}(1-\psi)}}_{l_2}. \tag{6}$$

The eigenvectors have to satisfy the relationship

$$\begin{bmatrix} 1+\kappa\sigma/\beta & \sigma(\psi - 1/\beta) \\ -\kappa/\beta & 1/\beta \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} = l_1 \begin{bmatrix} x \\ 1 \end{bmatrix} \mp l_2 \begin{bmatrix} x \\ 1 \end{bmatrix}. \tag{7}$$

Thus, x solves

$$-(\kappa/\beta)x + 1/\beta = l_1 \overset{-}{+} l_2, \tag{8}$$

which implies that

$$x = \frac{1}{\kappa}(1 - \beta l_1 \overset{+}{-} \beta l_2). \tag{9}$$

Thus, we obtain the following Jordan decomposition of Γ_1^*

$$\Gamma_1^* = J\Lambda J^{-1}, \tag{10}$$

where

$$J = \begin{bmatrix} \frac{1}{\kappa}(1 - \beta l_1 + \beta l_2) & \frac{1}{\kappa}(1 - \beta l_1 - \beta l_2) \\ 1 & 1 \end{bmatrix} \tag{11}$$

$$\Lambda = \begin{bmatrix} l_1 - l_2 & 0 \\ 0 & l_1 + l_2 \end{bmatrix} \tag{12}$$

$$J^{-1} = \frac{1}{2\beta l_2} \begin{bmatrix} \kappa & -1 + \beta l_1 + \beta l_2 \\ -\kappa & 1 - \beta l_1 + \beta l_2 \end{bmatrix}. \tag{13}$$

Let $w_t = J^{-1}\xi_t$. We can now write the LRE system in terms of the transformed variables

$$w_t = \Lambda w_{t-1} + J^{-1}\Psi^*\epsilon_t + J^{-1}\Pi^*\eta_t. \tag{14}$$

Determinacy

If both λ_1 and λ_2 are unstable then the (unique) solution is

$$\begin{aligned} \eta_t &= -\Pi_*^{-1}\Psi^*\epsilon_t \\ &= -\begin{bmatrix} 1 + \frac{\kappa\sigma}{\beta} & \sigma(\psi - \frac{1}{\beta}) \\ -\frac{\kappa}{\beta} & \frac{1}{\beta} \end{bmatrix}^{-1} \begin{bmatrix} \sigma \\ 0 \end{bmatrix} \\ &= -\frac{\beta}{1 + \kappa\sigma/\beta - \kappa\sigma/\beta + \kappa\sigma\psi} \begin{bmatrix} 1/\beta & \sigma(1/\beta - \psi) \\ \kappa/\beta & 1 + \kappa\sigma/\beta \end{bmatrix} \begin{bmatrix} \sigma \\ 0 \end{bmatrix} \\ &= -\frac{\sigma}{1 + \kappa\sigma\psi} \begin{bmatrix} 1 \\ \kappa \end{bmatrix} \epsilon_t. \end{aligned} \tag{15}$$

Indeterminacy

In order to solve the system under indeterminacy we have to calculate

$$J^{-1}\Psi^* = \frac{1}{2\beta l_2}\begin{bmatrix} \kappa & -1+\beta l_1+\beta l_2 \\ -\kappa & 1-\beta l_1+\beta l_2 \end{bmatrix}\begin{bmatrix} \sigma \\ 0 \end{bmatrix}. \tag{16}$$

The second row of this vector is

$$[J^{-1}\Psi]_{2.} = -\frac{\kappa\sigma}{2\beta l_2}. \tag{17}$$

Moreover,

$$J^{-1}\Pi^* = \frac{1}{2\beta l_2}\begin{bmatrix} \kappa & -1+\beta l_1+\beta l_2 \\ -\kappa & 1-\beta l_1+\beta l_2 \end{bmatrix}\begin{bmatrix} 1+\frac{\kappa\sigma}{\beta} & \sigma(\psi-\frac{1}{\beta}) \\ -\frac{\kappa}{\beta} & \frac{1}{\beta} \end{bmatrix} \tag{18}$$

and the second row of this matrix is

$$\begin{aligned} [J^{-1}\Pi^*]_{.2} &= \frac{1}{2\beta l_2}\begin{bmatrix} -\kappa-\kappa^2\sigma/\beta-\kappa/\beta+\kappa l_1-\kappa l_2 \\ -\kappa\sigma(\psi-1/\beta)+1/\beta-l_1+l_2 \end{bmatrix}' \\ &= \frac{1}{2\beta l_2}\begin{bmatrix} -\kappa\lambda_2 \\ \lambda_2-1-\kappa\sigma\psi \end{bmatrix}', \end{aligned} \tag{19}$$

since $1+\sigma\kappa/\beta+1/\beta = 2l_1$ and $\lambda_2 = l_1+l_2$. Thus, the stability condition can be expressed as

$$-\kappa\sigma\varepsilon_t - \kappa\lambda_2\eta_t^y + [\lambda_2-1-\kappa\sigma\psi]\,\eta_t^\pi = 0. \tag{20}$$

Note that in the paper we are using

$$[J^{-1}\Pi^*]_{2.} = [-\kappa\lambda_2 \quad (\lambda_2-1-\kappa\sigma\psi)]$$

and

$$[J^{-1}\Psi^*]_{2.} = -\kappa\sigma,$$

which corresponds to a slightly different normalization of the matrix of eigenvectors J, than the one used in above derivations.

The singular value decomposition of $[J^{-1}\Pi^*]_{2.}$ yields

$$\begin{aligned} U_{.1} &= 1 \\ D_{11} &= d = \sqrt{(\kappa\lambda_2)^2 + (\lambda_2 - 1 - \kappa\sigma\psi)^2} \\ V'_{.1} &= [-\kappa\lambda_2 \quad (\lambda_2 - 1 - \kappa\sigma\psi)]/d \\ V'_{.2} &= [(\lambda_2 - 1 - \kappa\sigma\psi) \quad \kappa\lambda_2]/d \end{aligned}$$

and $Q_{2.}\Psi$ in Propositions 2 and 3 corresponds to $-\kappa\sigma$. The application of the formulae in Propositions 1 to 4 now yields the expressions for η_t that are reported in the paper.