Bayesian Estimation of DSGE Models[1]
Chapter 2: Turning a DSGE Model into a Bayesian Model

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Bayesian Inference

- DSGE model parameters θ.
- Likelihood function $p(Y|\theta)$; requires that the DSGE model is solved numerically given θ.
- Bayes Theorem

$$p(\theta|Y) \propto p(Y|\theta)p(\theta).$$

Solving DSGE Models

- Derive nonlinear equilibrium conditions:
  - System of nonlinear expectational difference equations;
  - transversality conditions.
- Find solution(s) of system of expectational difference methods:
  - Global (nonlinear) approximation
  - Local approximation near steady state
- We will focus on log-linear approximations around the steady state.
- More detail in: Fernandez-Villaverde, Rubio-Ramirez, and Schorfheide (2016) "Solution and Estimation Methods for DSGE Models," prepared for forthcoming *Handbook of Macroeconomics*.

What is a Local Approximation?

- In a nutshell... consider the backward-looking model

$$y_t = f(y_{t-1}, \sigma\epsilon_t).$$

- Guess that the solution is of the form

$$y_t = y_t^{(0)} + \sigma y_t^{(1)} + o(\sigma).$$

- Steady state:

$$y_t^{(0)} = y^{(0)} = f(y^{(0)}, 0)$$

- Suppose $y^{(0)} = 0$. Expand $f(\cdot)$ around $\sigma = 0$:

$$f(y_{t-1}, \sigma\epsilon_t) = f_y y_{t-1} + f_\epsilon \sigma\epsilon_t + o(|y_{t-1}|) + o(\sigma)$$

- Now plug-in conjectured solution:

$$\sigma y_t^{(1)} = f_y \sigma y_{t-1}^{(1)} + f_\epsilon \sigma\epsilon_t + o(\sigma)$$

- Deduce that $y_t^{(1)} = f_y y_{t-1}^{(1)} + f_\epsilon \epsilon_t$

What is a Log-Linear Approximation?

- Consider a Cobb-Douglas production function: $Y_t = A_t K_t^\alpha N_t^{1-\alpha}$.

- Linearization around Y_*, A_*, K_*, N_*:

$$\begin{aligned} Y_t - Y_* &= K_*^\alpha N_*^{1-\alpha}(A_t - A_*) + \alpha A_* K_*^{\alpha-1} N_*^{1-\alpha}(K_t - K_*) \\ &\quad +(1-\alpha)A_* K_*^\alpha N_*^{-\alpha}(N_t - N_*) \end{aligned}$$

- Log-linearization: Let $f(x) = f(e^v)$ and linearize with respect to v:

$$f(e^v) \approx f(e^{v_*}) + e^{v_*} f'(e^{v_*})(v - v_*).$$

Thus:

$$f(x) \approx f(x_*) + x_* f'(x_*)(\ln x/x_*) = f(x_*) + f'(x_*)\tilde{x}$$

- Cobb-Douglas production function:

$$\tilde{Y}_t = \tilde{A}_t + \alpha \tilde{K}_t + (1-\alpha)\tilde{N}_t$$

Loglinearization of New Keynesian Model

- Consumption Euler equation:

$$\hat{y}_t = \mathbb{E}_t[\hat{y}_{t+1}] - \frac{1}{\tau}\left(\hat{R}_t - \mathbb{E}_t[\hat{\pi}_{t+1}] - \mathbb{E}_t[\hat{z}_{t+1}]\right) + \hat{g}_t - \mathbb{E}_t[\hat{g}_{t+1}]$$

- New Keynesian Phillips curve:

$$\hat{\pi}_t = \beta\mathbb{E}_t[\hat{\pi}_{t+1}] + \kappa(\hat{y}_t - \hat{g}_t),$$

where

$$\kappa = \tau\frac{1-\nu}{\nu\pi^2\phi}$$

- Monetary policy rule:

$$\hat{R}_t = \rho_R\hat{R}_{t-1} + (1-\rho_R)\psi_1\hat{\pi}_t + (1-\rho_R)\psi_2\left(\hat{y}_t - \hat{g}_t\right) + \epsilon_{R,t}$$

Canonical Linear Rational Expectatins System

- Define

$$x_t = [\hat{y}_t, \hat{\pi}_t, \hat{R}_t, \epsilon_{R,t}, \hat{g}_t, \hat{z}_t]'.$$

- Augment x_t by $\mathbb{E}_t[\hat{y}_{t+1}]$ and $\mathbb{E}_t[\hat{\pi}_{t+1}]$.
- Define

$$s_t = \left[x_t', \mathbb{E}_t[\hat{y}_{t+1}], \mathbb{E}_t[\hat{\pi}_{t+1}]\right]'.$$

- Define rational expectations forecast errors forecast errors for inflation and output. Let

$$\eta_{y,t} = y_t - \mathbb{E}_{t-1}[\hat{y}_t], \quad \eta_{\pi,t} = \pi_t - \mathbb{E}_{t-1}[\hat{\pi}_t].$$

- Write system in canonical form (Sims, 2002):

$$\Gamma_0 s_t = \Gamma_1 s_{t-1} + \Psi \epsilon_t + \Pi \eta_t.$$

How Can One Solve Linear Rational Expectations Systems? A Simple Example

- Consider

$$y_t = \frac{1}{\theta}\mathbb{E}_t[y_{t+1}] + \epsilon_t, \tag{1}$$

where $\epsilon_t \sim iid(0,1)$ and $\theta \in \Theta = [0,2]$.

- Introduce conditional expectation $\xi_t = \mathbb{E}_t[y_{t+1}]$ and forecast error $\eta_t = y_t - \xi_{t-1}$.
- Thus,

$$\xi_t = \theta\xi_{t-1} - \theta\epsilon_t + \theta\eta_t. \tag{2}$$

A Simple Example

- Determinacy: $\theta > 1$. Then only stable solution:

$$\xi_t = 0, \quad \eta_t = \epsilon_t, \quad y_t = \epsilon_t \tag{3}$$

- Indeterminacy: $\theta \leq 1$ the stability requirement imposes no restrictions on forecast error:

$$\eta_t = \widetilde{M}\epsilon_t + \zeta_t. \tag{4}$$

- For simplicity assume now $\zeta_t = 0$. Then

$$y_t - \theta y_{t-1} = \widetilde{M}\epsilon_t - \theta\epsilon_{t-1}. \tag{5}$$

Solving a More General System

- Canonical form:

$$\Gamma_0(\theta)s_t = \Gamma_1(\theta)s_{t-1} + \Psi(\theta)\epsilon_t + \Pi(\theta)\eta_t, \tag{6}$$

- The system can be rewritten as

$$s_t = \Gamma_1^*(\theta)s_{t-1} + \Psi^*(\theta)\epsilon_t + \Pi^*(\theta)\eta_t. \tag{7}$$

- Replace Γ_1^* by $J\Lambda J^{-1}$ and define $w_t = J^{-1}s_t$.
- To deal with repeated eigenvalues and non-singular Γ_0 we use Generalized Complex Schur Decomposition (QZ) in practice.
- Let the i'th element of w_t be $w_{i,t}$ and denote the i'th row of $J^{-1}\Pi^*$ and $J^{-1}\Psi^*$ by $[J^{-1}\Pi^*]_{i.}$ and $[J^{-1}\Psi^*]_{i.}$, respectively.

Solving a More General System

- Rewrite model:

$$w_{i,t} = \lambda_i w_{i,t-1} + [J^{-1}\Psi^*]_{i.}\epsilon_t + [J^{-1}\Pi^*]_{i.}\eta_t. \tag{8}$$

- Define the set of stable AR(1) processes as

$$I_s(\theta) = \left\{ i \in \{1, \ldots n\} \middle| |\lambda_i(\theta)| \leq 1 \right\} \tag{9}$$

- Let $I_x(\theta)$ be its complement. Let Ψ_x^J and Π_x^J be the matrices composed of the row vectors $[J^{-1}\Psi^*]_{i.}$ and $[J^{-1}\Pi^*]_{i.}$ that correspond to unstable eigenvalues, i.e., $i \in I_x(\theta)$.
- Stability condition:

$$\Psi_x^J \epsilon_t + \Pi_x^J \eta_t = 0 \tag{10}$$

for all t.

Solving a More General System

- Solving for η_t. Define

$$\begin{aligned}
\Pi_x^J &= \begin{bmatrix} U_{.1} & U_{.2} \end{bmatrix} \begin{bmatrix} D_{11} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_{.1}' \\ V_{.2}' \end{bmatrix} \\
&= \underbrace{U}_{m\times m} \underbrace{D}_{m\times k} \underbrace{V'}_{k\times k} \\
&= \underbrace{U_{.1}}_{m\times r} \underbrace{D_{11}}_{r\times r} \underbrace{V_{.1}'}_{r\times k}.
\end{aligned} \tag{11}$$

- If there exists a solution to Eq. (10) that expresses the forecast errors as function of the fundamental shocks ϵ_t and sunspot shocks ζ_t, it is of the form

$$\begin{aligned}
\eta_t &= \eta_1 \epsilon_t + \eta_2 \zeta_t \\
&= (-V_{.1} D_{11}^{-1} U_{.1}' \Psi_x^J + V_{.2} \widetilde{M}) \epsilon_t + V_{.2} M_\zeta \zeta_t,
\end{aligned} \tag{12}$$

where $\widetilde{M}$ is an $(k-r) \times l$ matrix, M_ζ is a $(k-r) \times p$ matrix, and the dimension of $V_{.2}$ is $k \times (k-r)$. The solution is unique if $k = r$ and $V_{.2}$ is zero.

Proposition

If there exists a solution to Eq. (10) that expresses the forecast errors as function of the fundamental shocks ϵ_t and sunspot shocks ζ_t, it is of the form

$$\begin{aligned} \eta_t &= \eta_1 \epsilon_t + \eta_2 \zeta_t \\ &= (-V_{.1} D_{11}^{-1} U_{.1}' \Psi_x^J + V_{.2} \widetilde{M}) \epsilon_t + V_{.2} M_\zeta \zeta_t, \end{aligned} \tag{13}$$

where $\widetilde{M}$ is an $(k-r) \times l$ matrix, M_ζ is a $(k-r) \times p$ matrix, and the dimension of $V_{.2}$ is $k \times (k-r)$. The solution is unique if $k = r$ and $V_{.2}$ is zero.

At the End of the Day...

- We obtain a transition equation for the vector s_t:

$$s_t = \Phi_1(\theta)s_{t-1} + \Phi_\epsilon(\theta)\epsilon_t.$$

- The coefficient matrices $\Phi_1(\theta)$ and $\Phi_\epsilon(\theta)$ are functions of the parameters of the DSGE model.

Likelihood Function

- Relate model variables s_t to observables y_t.

- In NK model:

$$\begin{aligned} YGR_t &= \gamma^{(Q)} + 100(\hat{y}_t - \hat{y}_{t-1} + \hat{z}_t) \\ INFL_t &= \pi^{(A)} + 400\hat{\pi}_t \\ INT_t &= \pi^{(A)} + r^{(A)} + 4\gamma^{(Q)} + 400\hat{R}_t. \end{aligned}$$

  where

$$\gamma = 1 + \frac{\gamma^{(Q)}}{100}, \quad \beta = \frac{1}{1 + r^{(A)}/400}, \quad \pi = 1 + \frac{\pi^{(A)}}{400}.$$

- More generically:

$$y_t = \Psi_0(\theta) + \Psi_1(\theta)t + \Psi_2(\theta)s_t + u_t.$$

State-Space Representation and Likelihood

- Measurement:

$$y_t = \Psi_0(\theta) + \Psi_1(\theta)t + \Psi_2(\theta)s_t + u_t.$$

- State transition:

$$s_t = \Phi_1(\theta)s_{t-1} + \Phi_\epsilon(\theta)\epsilon_t.$$

- Joint density for the observations and latent states:

$$\begin{aligned} p(Y_{1:T}, S_{1:T}|\theta) &= \prod_{t=1}^{T} p(y_t, s_t|Y_{1:t-1}, S_{1:t-1}, \theta) \\ &= \prod_{t=1}^{T} p(y_t|s_t, \theta)p(s_t|s_{t-1}, \theta). \end{aligned}$$

- Problem: we need the marginal $p(Y_{1:T}|\theta)$.

Filtering - General Idea

- State-space representation of linearized DSGE model

$$\begin{aligned} y_t &= \Psi_0(\theta) + \Psi_1(\theta)t + \Psi_2(\theta)s_t(+u_t) \quad \text{measurement} \\ s_t &= \Phi_1(\theta)s_t + \Phi_\epsilon(\theta)\epsilon_t \quad \text{state transition} \end{aligned}$$

- Likelihood function:

$$p(Y_{1:T}|\theta) = \prod_{t=1}^{T} p(y_t|Y_{1:t-1}, \theta)$$

- A filter generates a sequence of conditional distributions $s_t|Y_{1:t}$.
- Iterations:
  - Initialization at time $t-1$: $p(s_{t-1}|Y_{1:t-1}, \theta)$
  - Forecasting t given $t-1$:
    1. Transition equation: $p(s_t|Y_{1:t-1}, \theta) = \int p(s_t|s_{t-1}, Y_{1:t-1}, \theta)p(s_{t-1}|Y_{1:t-1}, \theta)ds_{t-1}$
    2. Measurement equation: $p(y_t|Y_{1:t-1}, \theta) = \int p(y_t|s_t, Y_{1:t-1}, \theta)p(s_t|Y_{1:t-1}, \theta)ds_t$
  - Updating with Bayes theorem. Once y_t becomes available:

$$p(s_t|Y_{1:t}, \theta) = p(s_t|y_t, Y_{1:t-1}, \theta) = \frac{p(y_t|s_t, Y_{1:t-1}, \theta)p(s_t|Y_{1:t-1}, \theta)}{p(y_t|Y_{1:t-1}, \theta)}$$

Conditional Distributions for Kalman Filter (Linear Gaussian State-Space Model)

	Distribution	Mean and Variance
$s_{t-1} \| (Y_{1:t-1}, \theta)$	$N\big(\bar{s}_{t-1\|t-1}, P_{t-1\|t-1}\big)$	Given from Iteration $t-1$
$s_t \| (Y_{1:t-1}, \theta)$	$N\big(\bar{s}_{t\|t-1}, P_{t\|t-1}\big)$	$\bar{s}_{t\|t-1} = \Phi_1 \bar{s}_{t-1\|t-1}$ $P_{t\|t-1} = \Phi_1 P_{t-1\|t-1} \Phi_1' + \Phi_\epsilon \Sigma_\epsilon \Phi_\epsilon'$
$y_t \| (Y_{1:t-1}, \theta)$	$N\big(\bar{y}_{t\|t-1}, F_{t\|t-1}\big)$	$\bar{y}_{t\|t-1} = \Psi_0 + \Psi_1 t + \Psi_2 \bar{s}_{t\|t-1}$ $F_{t\|t-1} = \Psi_2 P_{t\|t-1} \Psi_2' + \Sigma_u$
$s_t \| (Y_{1:t}, \theta)$	$N\big(\bar{s}_{t\|t}, P_{t\|t}\big)$	$\bar{s}_{t\|t} = \bar{s}_{t\|t-1} + P_{t\|t-1} \Psi_2' F_{t\|t-1}^{-1} (y_t - \bar{y}_{t\|t-1})$ $P_{t\|t} = P_{t\|t-1} - P_{t\|t-1} \Psi_2' F_{t\|t-1}^{-1} \Psi_2 P_{t\|t-1}$

Prior Elicitation

- Group parameters:
  - steady-state related parameters
  - parameters assoc with exogenous shocks
  - parameters assoc with internal propagation
- Non-sample information $p(\theta|\mathcal{X}^0)$:
  - pre-sample information
  - micro-level information
- To guide the prior for θ, you can ask: what are its implications for observables Y?

Name	Domain	Prior Density	Para (1)	Para (2)
Steady State Related Parameters $\theta_{(ss)}$				
$r^{(A)}$	$\mathbb{R}^+$	Gamma	0.50	0.50
$\pi^{(A)}$	$\mathbb{R}^+$	Gamma	7.00	2.00
$\gamma^{(Q)}$	$\mathbb{R}$	Normal	0.40	0.20
Endogenous Propagation Parameters $\theta_{(endo)}$				
τ	$\mathbb{R}^+$	Gamma	2.00	0.50
κ	$[0,1]$	Uniform	0.00	1.00
ψ_1	$\mathbb{R}^+$	Gamma	1.50	0.25
ψ_2	$\mathbb{R}^+$	Gamma	0.50	0.25
ρ_R	$[0,1)$	Uniform	0.00	1.00
Exogenous Shock Parameters $\theta_{(exo)}$				
ρ_G	$[0,1)$	Uniform	0.00	1.00
ρ_Z	$[0,1)$	Uniform	0.00	1.00
$100\sigma_R$	$\mathbb{R}^+$	InvGamma	0.40	4.00
$100\sigma_G$	$\mathbb{R}^+$	InvGamma	1.00	4.00
$100\sigma_Z$	$\mathbb{R}^+$	InvGamma	0.50	4.00