Bayesian Estimation of DSGE Models[1]
Chapter 6: Three Applications

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Application 1: A New Keynesian Model with Correlated Shocks

- The assumption that exogenous shocks evolve according to independent AR(1) is to some extent arbitrary.
- Trying to generalize this assumption seems natural.
- However, the more elaborate the exogenous propagation mechanism, the more difficult it becomes to disentangle endogenous from exogenous propagation.
- This generates identification problems.

Application 1: A New Keynesian Model with Correlated Shocks

- Technology growth shock $\hat{z}_t$, government spending shock $\hat{g}_t$ evolve:

$$\left[\begin{array}{c} \hat{z}_t \\ \hat{g}_t \end{array}\right] = \left[\begin{array}{cc} \rho_z & \rho_{zg} \\ \rho_{gz} & \rho_g \end{array}\right] \left[\begin{array}{c} \hat{z}_{t-1} \\ \hat{g}_{t-1} \end{array}\right] + \left[\begin{array}{c} \epsilon_{z,t} \\ \epsilon_{g,t} \end{array}\right],$$

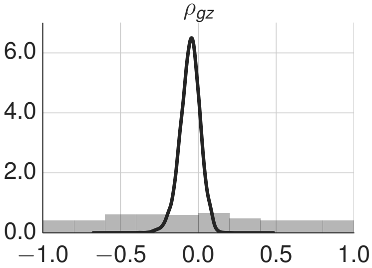
$$\left[\begin{array}{c} \epsilon_{z,t} \\ \epsilon_{g,t} \end{array}\right] \sim N\left(\left[\begin{array}{c} 0 \\ 0 \end{array}\right], \left[\begin{array}{cc} \sigma_z^2 & 0 \\ 0 & \sigma_g^2 \end{array}\right]\right).$$

- This VAR process is combined with:

$$\begin{array}{rcl} \hat{y}_t & = & \mathbb{E}_t[\hat{y}_{t+1}] - \dfrac{1}{\tau}\left(\hat{R}_t - \mathbb{E}_t[\hat{\pi}_{t+1}] - \mathbb{E}_t[\hat{z}_{t+1}]\right) \\ & & + \hat{g}_t - \mathbb{E}_t[\hat{g}_{t+1}], \\ \hat{\pi}_t & = & \beta \mathbb{E}_t[\hat{\pi}_{t+1}] + \kappa(\hat{y}_t - \hat{g}_t), \\ \hat{R}_t & = & \rho_R \hat{R}_{t-1} + (1-\rho_R)\psi_1 \hat{\pi}_t + (1-\rho_R)\psi_2\left(\hat{y}_t - \hat{g}_t\right) + \epsilon_{R,t}. \end{array}$$

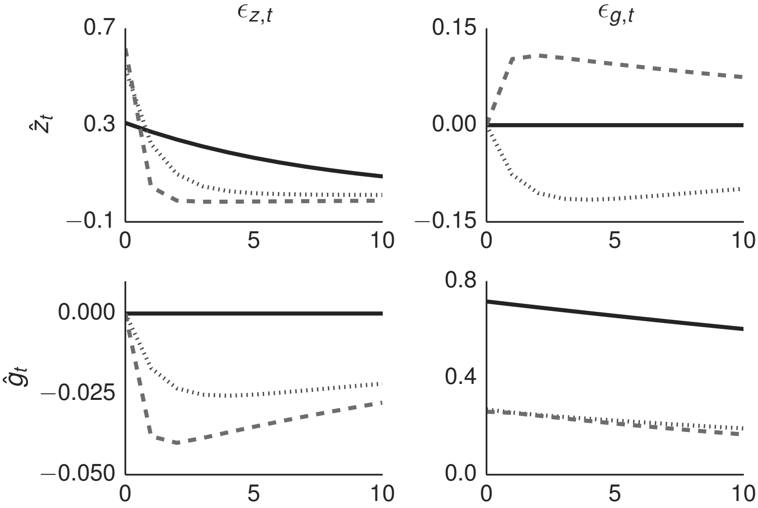
- We use agnostic priors: $\rho_g, \rho_z \sim U[0,1], \quad \rho_{gz}, \rho_{zg} \sim U[-1,1].$

Priors and Posteriors of ρ_{gz} and ρ_{zg}

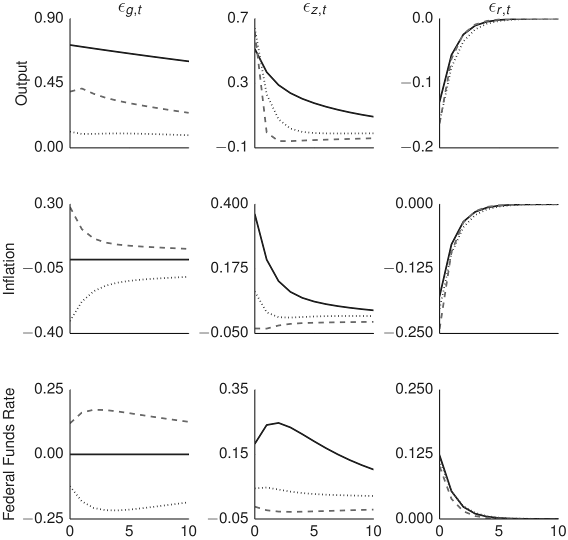


Notes: The two panels depict histograms of prior distributions (shaded area) and kernel density estimates of the posterior densities (solid lines).

Impulse Responses (Part 1)



Impulse Responses (Part 2)

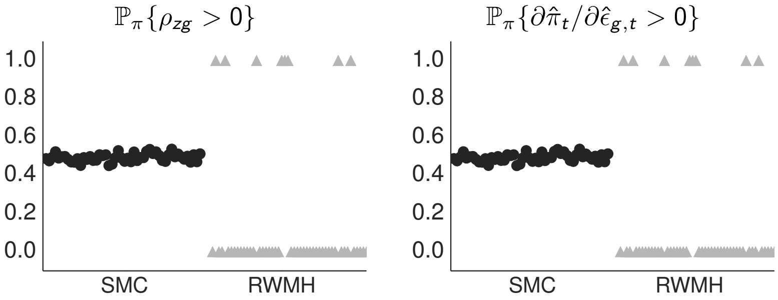


Algorithm Configuration

RWMH-V	SMC
$N = 100,000$	$N = 4,800$
$N_{burn} = 50,000$	$N_{\phi} = 500$
$N_{blocks} = 1$	$N_{blocks} = 6, \; N_{MH} = 1$
$c = 0.125$	$\lambda = 2$
Run Time: 00:28 (1 core)	Run Time: 05:52 (12 cores)

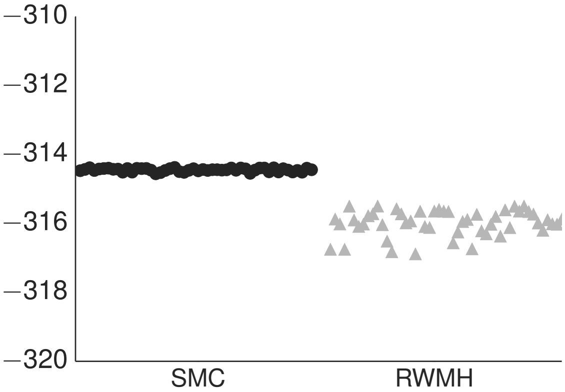
Note: We run each algorithm $N_{run} = 50$ times. Run time is reported as mm:ss.

Posterior Probability Approximations



Notes: Each symbol (50 in total) corresponds to one run of the SMC algorithm (dot) or the RWMH algorithm (triangle).

Marginal Data Density Approximations



Notes: Each symbol (50 in total) corresponds to one run of the SMC algorithm (dot) or the RWMH algorithm (triangle). The SMC algorithm automatically generates an estimate of the MDD; for the RWMH algorithm we use Geweke's modified harmonic mean estimator.

Marginal Data Density

Model	Mean($\ln \hat{p}(Y)$)	Std. Dev.($\ln \hat{p}(Y)$)
AR(1) Shocks	-346.16	(0.07)
VAR(1) Shocks	-314.45	(0.05)

Notes: Table shows mean and standard deviation of SMC-based estimate of the log marginal data density, computed over $N_{run} = 50$ runs of the SMC sampler.

Application 2: Estimation of Smets and Wouters (2007) Model

- Benchmark macro model, has been estimated many (many) times.
- "Core" of many larger-scale models.
- 36 estimated parameters.
- SW priors might be considered implausible because they seem to be informed by in-sample information.
- How does quality of posterior simulators change as one makes the priors more diffuse?
- Replace Beta by Uniform distributions; increase variances of parameters with Gamma and Normal prior by factor of 3.

Generating Quantile Estimates

- We will focus on the accuracy of the approximation of posterior quantiles.
- Quantile estimates can be computed in two different ways:
    - Sort the posterior draws $\{\theta_j^i\}_{i=1}^N$ and select the $\lfloor \tau N \rfloor$'th element.
    - Quantile regression (Koenker and Basset, 1978)

$$\begin{aligned}\hat{q}_\tau(\theta_j) &= \text{argmin}_q \left[(1-\tau)\frac{1}{N}\sum_{i:\,\theta_j^i<q}(\theta_j^i - q) \right. \\ & \left. +\tau\frac{1}{N}\sum_{i:\,\theta_j^i\geq q}(\theta_j^i - q)\right].\end{aligned}$$

Accuracy of Quantile Estimates

- Accuracy of the quantile estimates is given by the following CLT:

  $$\sqrt{N}(\hat{q}_\tau - q_\tau) \Longrightarrow N\left(0, \frac{\tau(1-\tau)}{\pi^2(q_\tau)}\right),$$

  where $\pi(\theta)$ is the posterior density.

- Finite sample inefficiency:

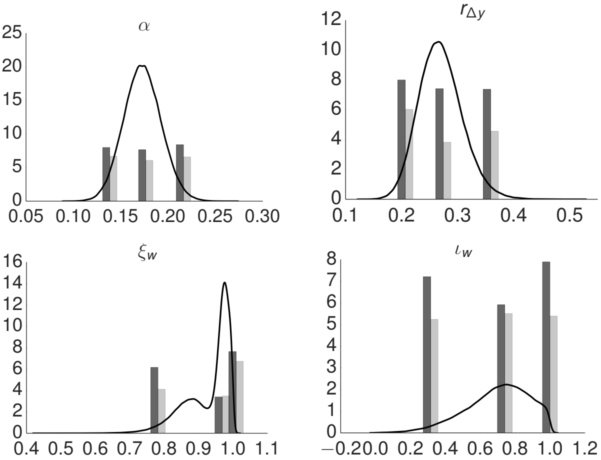
  $$\text{InEff}_N = \frac{\mathbb{V}[\hat{q}_\tau]}{\tau(1-\tau)/\big(N\pi^2(q_\tau)\big)}$$

Algorithm Configuration

RWMH-V	SMC
$N = 10,000,000$	$N = 12,000$
$N_{burn} = 5,000,000$	$N_{\phi} = 500$
$N_{blocks} = 1$	$N_{blocks} = 6, \; N_{MH} = 1$
$c = 0.08$	$\lambda = 2.1$
Run Time: 14:06 (1 core)	Run Time: 02:32 (24 cores)

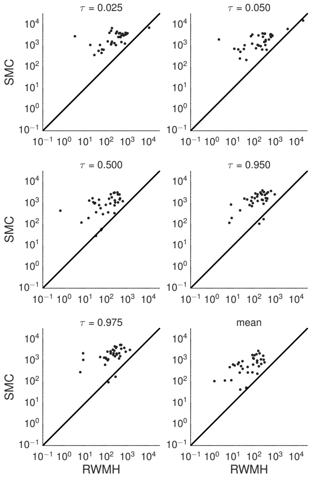
Note: We run each algorithm $N_{run} = 50$ times. Run time is reported as hh:mm.

Precision of Quantile Approximations (Part 1)



Notes: Each panel depicts a Kernel estimate of the posterior density (solid) and $\ln(N_{eff}) = \ln(N/\text{InEff}_N)$ (light gray hatched bars correspond to RWMH and solid bars correspond to SMC) for various choices of τ equal to 0.025, 0.05, 0.5, 0.95, and 0.975.

Precision of Quantile Approximations (Part 2)



Notes: N_{eff} for the RWMH-V and SMC quantile approximations. Each dot corresponds to one parameter. The 45-degree line appears in solid.

Application 3: A Fiscal Policy DSGE Model

- Based on Leeper, Plante, and Traum (2010)
- Incorporate elaborate fiscal policy rules (government spending, labor, capital, and consumption taxes) into DSGE model to study effects of tax and spending changes.
- Complex specification of fiscal policy creates identification problems.

Application 3: A Fiscal Policy DSGE Model

- The budget constraint of the households

$$\begin{aligned}(1+\tau_t^c)c_t + i_t + b_t \\ &= (1-\tau_t^l)w_t l_t + (1-\tau_t^k)R_t^k u_t k_{t-1} + R_{t-1}b_{t-1} + z_t.\end{aligned}$$

- The budget constraint for the government, using capital letters to denote aggregate quantities

$$B_t + \tau_t^k R_t^k u_t K_{t-1} + \tau_t^l w_t L_t + \tau_t^c C_t = R_{t-1}B_{t-1} + G_t + Z_t.$$

- The fiscal policy rules ($\hat{x}_t$: log deviation from steady state of x_t)

$$\begin{aligned}\hat{\tau}_t^k &= \varphi_k \hat{Y}_t + \gamma_k \hat{B}_{t-1} + \phi_{kl}\hat{u}_t^l + \phi_{kc}\hat{u}_t^c + \hat{u}_t^k, \\ \hat{\tau}_t^l &= \varphi_l \hat{Y}_t + \gamma_l \hat{B}_{t-1} + \phi_{lk}\hat{u}_t^k + \phi_{lc}\hat{u}_t^c + \hat{u}_t^l, \\ \hat{\tau}_t^c &= \phi_{ck}\hat{u}_t^k + \phi_{cl}\hat{u}_t^l + \hat{u}_t^c.\end{aligned}$$

Application 3: A Fiscal Policy DSGE Model

- The exogenous movements in taxes follow AR(1) processes

$$\begin{aligned}
\hat{u}^k_t &= \rho_k \hat{u}^k_{t-1} + \sigma_k \epsilon^k_t, \quad \epsilon^k_t \sim N(0,1),\\
\hat{u}^l_t &= \rho_l \hat{u}^l_{t-1} + \sigma_l \epsilon^l_t, \quad \epsilon^l_t \sim N(0,1),\\
\hat{u}^c_t &= \rho_c \hat{u}^c_{t-1} + \sigma_c \epsilon^c_t, \quad \epsilon^c_t \sim N(0,1).
\end{aligned}$$

- The government spending rule is given by

$$\begin{aligned}
\hat{G}_t &= -\varphi_g \hat{Y}_t - \gamma_g \hat{B}_{t-1} + \hat{u}^g_t,\\
\hat{u}^g_t &= \rho_g \hat{u}^g_{t-1} + \sigma_g \epsilon^g_t, \quad \epsilon^g_t \sim N(0,1).
\end{aligned}$$

- The transfer rule is given by

$$\begin{aligned}
\hat{Z}_t &= -\varphi_z \hat{Y}_t - \gamma_z \hat{B}_{t-1} + \hat{u}^z_t,\\
\hat{u}^z_t &= \rho_z \hat{u}^z_{t-1} + \sigma_z \epsilon^z_t, \quad \epsilon^z_t \sim N(0,1).
\end{aligned}$$

Prior Distributions for Fiscal Rule Parameters

	LPT Prior			Diffuse Prior		
	Type	Para (1)	Para (2)	Type	Para (1)	Para (2)
			Debt Response Parameters			
γ_g	G	0.4	0.2	U	0	5
γ_{tk}	G	0.4	0.2	U	0	5
γ_{tl}	G	0.4	0.2	U	0	5
γ_z	G	0.4	0.2	U	0	5
			Output Response Parameters			
φ_{tk}	G	1.0	0.3	N	1.0	1
φ_{tl}	G	0.5	0.25	N	0.5	1
φ_g	G	0.07	0.05	N	0.07	1
φ_z	G	0.2	0.1	N	0.2	1
			Exogenous Tax Comovement Parameters			
ϕ_{kl}	N	0.25	0.1	N	0.25	1
ϕ_{kc}	N	0.05	0.1	N	0.05	1
ϕ_{lc}	N	0.05	0.1	N	0.05	1

Notes: Para (1) and Para (2) correspond to the mean and standard deviation of the Beta (B), Gamma (G), and Normal (N) distributions and to the upper, lower bounds of the support for Uniform (U) distribution.

Common Prior Distributions

	Type	Para (1)	Para (2)		Type	Para (1)	Para (2)
			Endogenous Propagation Parameters				
γ	G	1.75	0.5	s''	G	5	0.5
κ	G	2.0	0.5	δ_2	G	0.7	0.5
h	B	0.5	0.2				
			Exogenous Process Parameters				
ρ_a	B	0.7	0.2	σ_a	IG	1	4
ρ_b	B	0.7	0.2	σ_b	IG	1	4
ρ_l	B	0.7	0.2	σ_l	IG	1	4
ρ_i	B	0.7	0.2	σ_i	IG	1	4
ρ_g	B	0.7	0.2	σ_g	IG	1	4
ρ_{tk}	B	0.7	0.2	σ_{tk}	IG	1	4
ρ_{tl}	B	0.7	0.2	σ_{tl}	IG	1	4
ρ_{tc}	B	0.7	0.2	σ_{tc}	IG	1	4
ρ_z	B	0.7	0.2	σ_z	IG	1	4

Notes: For the Inv. Gamma (IG) distribution, Para (1) and Para (2) refer to s and ν, where $p(\sigma|\nu, s) \propto \sigma^{-\nu-1} e^{-\nu s^2/2\sigma^2}$.

SMC Configuration

$N = 6,000$ $N_{\phi} = 500$

$N_{blocks} = 3$ $N_{MH} = 1$

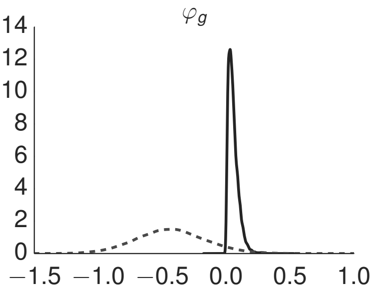
$\lambda = 4.0$

Run Time [mm:ss]: 48:00 (12 cores)

Posterior Moments

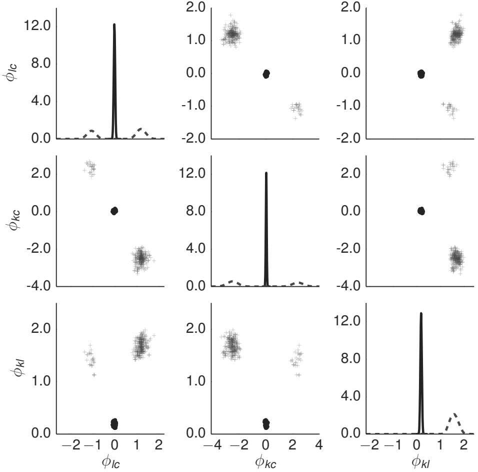
	Based on LPT Prior		Based on Diff. Prior	
	Mean	[5%, 95%] Int.	Mean	[5%, 95%] Int.
Debt Response Parameters				
γ_g	0.16	[0.07, 0.27]	0.10	[0.01, 0.23]
γ_{tk}	0.39	[0.22, 0.60]	0.38	[0.16, 0.62]
γ_{tl}	0.11	[0.04, 0.21]	0.04	[0.00, 0.11]
γ_z	0.32	[0.17, 0.47]	0.32	[0.14, 0.49]
Output Response Parameters				
φ_{tk}	1.67	[1.18, 2.18]	2.06	[1.44, 2.69]
φ_{tl}	0.29	[0.11, 0.53]	0.11	[-0.34, 0.58]
φ_g	0.06	[0.01, 0.13]	-0.43	[-0.87, 0.02]
φ_z	0.17	[0.06, 0.33]	-0.07	[-0.56, 0.41]
Exogenous Tax Comovement Parameters				
ϕ_{kl}	0.19	[0.14, 0.24]	1.57	[1.29, 1.87]
ϕ_{kc}	0.03	[-0.03, 0.08]	-0.33	[-2.84, 2.73]
ϕ_{lc}	-0.02	[-0.07, 0.04]	0.20	[-1.23, 1.40]
Innovations to Fiscal Rules				
σ_g	3.03	[2.79, 3.30]	2.91	[2.66, 3.19]
σ_{tk}	4.36	[4.01, 4.75]	1.26	[1.08, 1.46]
σ_{tl}	2.95	[2.71, 3.22]	2.00	[1.71, 2.33]
σ_{tc}	3.99	[3.67, 4.33]	1.14	[0.96, 1.35]
σ_z	3.34	[3.07, 3.63]	3.34	[3.07, 3.63]

Posterior of Output Response Parameters



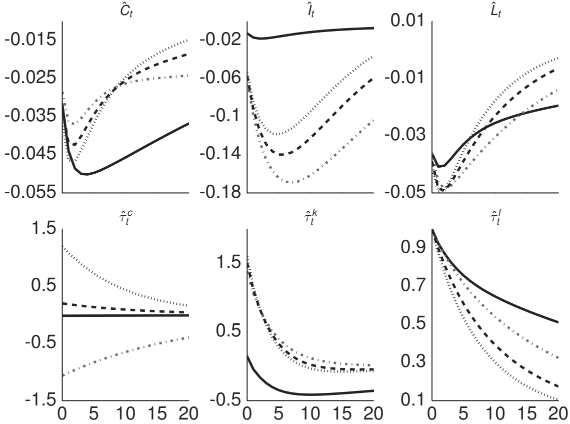
Notes: The figure depicts posterior densities under the LPT prior (solid) and the diffuse prior (dashed).

Posterior of Tax Comovement Parameters



Notes: The plots on the diagonal depict posterior densities under the LPT prior (solid) and the diffuse prior (dashed). The plots on the off-diagonals depict draws from the posterior distribution under the LPT prior (circles) and the diffuse prior (triangles).

Impulse Response to a Labor Tax Innovation



Notes: Figure depicts posterior mean impulse responses under LPT prior (solid); diffuse prior (dashed); diffuse prior with $\phi_{lc} > 0$, $\phi_{kl} < 0$ (dotted); and diffuse prior with $\phi_{lc} < 0$, $\phi_{kl} > 0$ (dots and short dashes). $\hat{C}_t$, $\hat{I}_t$ and $\hat{L}_t$ are consumption, investment, and hours worked in deviation from steady state.