Introduction to (Bayesian) Inference

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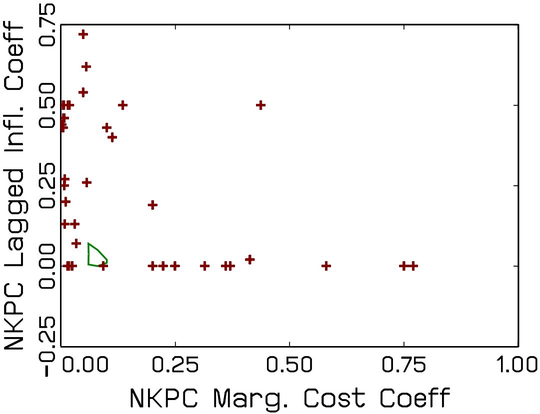
Statistical Inference

- **Econometric model:** collection of probability distributions $p(Y|\theta)$ indexed by parameter $\theta \in \Theta$. Examples: VAR, DSGE model, ...
- The "easy" part: pick values for parameter vector $\theta \Longrightarrow$ determine properties of model-simulated data $Y^{sim}(\theta)$.
- Statistical inference: observed data $Y^{obs} \Longrightarrow$ determine suitable values for parameter vector θ.
- **Basic Idea:** choose θ such that $Y^{sim}(\theta)$ look like Y^{obs}.
- Goals: estimates $\hat{\theta}$ as well as measures of uncertainty associated with these estimates.

Good Measures of Uncertainty are Important

NK Phillips Curve

$$\tilde{\pi}_t = \gamma_b \tilde{\pi}_{t-1} + \gamma_f \mathbb{E}_t[\tilde{\pi}_{t+1}] + \kappa \widetilde{MC}_t$$



Identification

- Econometric model generates a family of probability distributions $p(Y|\theta)$, $\theta \in \Theta$.
- Thought experiment: data are generated from the econometric model conditional on some "true" parameter θ_0.
- The parameter vector θ is globally identifiable at θ_0 if

$$p(Y|\theta) = p(Y|\theta_0) \quad \text{implies} \quad \theta = \theta_0.$$

- **Treatment of Y:**
  - Pre-experimental perspective: the sample is not yet observed and condition needs to hold with probability one under the distribution $p(Y|\theta_0)$.
  - Post-experimental perspective: sample has been observed, parameter θ may be identifiable for some trajectories Y, but not for others.
- **Example:**

$$y_{1,t}|(\theta, y_{2,t}) \sim iidN\big(\theta y_{2,t}, 1\big), \quad y_{2,t} = \begin{cases} 0 & \text{w.p. } 1/2 \\ \sim iidN(0,1) & \text{w.p. } 1/2 \end{cases}$$

With probability (w.p.) $1/2$, one observes a trajectory along which θ is not identifiable because $y_{2,t} = 0$ for all t.

Statistical Inference

- Frequentist:
  - pre-experimental perspective;
  - condition on "true" but unknown θ_0;
  - treat data Y as random;
  - study behavior of estimators and decision rules under repeated sampling.
- Bayesian:
  - post-experimental perspective;
  - condition on observed sample Y;
  - treat parameter θ as unknown and random;
  - derive estimators and decision rules that minimize expected loss (averaging over θ) conditional on observed Y.

Pre- vs. Post-Experimental Inference

- Suppose Y_1 and Y_2 are independently and identically distributed and

$$P_\theta^{Y_i}\{Y_i = \theta - 1\} = \frac{1}{2}, \quad P_\theta^{Y_i}\{Y_i = \theta + 1\} = \frac{1}{2}$$

- Consider the following coverage set

$$C(Y_1, Y_2) = \begin{cases} \frac{1}{2}(Y_1 + Y_2) & \text{if} \quad Y_1 \neq Y_2 \\ Y_1 - 1 & \text{if} \quad Y_1 = Y_2 \end{cases}$$

- Pre-experimental perspective: $C(Y_1, Y_2)$ is a 75% confidence interval. The probability (under repeated sampling, conditional on θ) that the confidence interval 75%.

- Post-experimental perspective: we are "100% confident" that $C(Y_1, Y_2)$ contains the "true" θ if $Y_1 \neq Y_2$, whereas we are only "50% percent" confident if $Y_1 = Y_2$.

Bayesian Inference - Overview

Model of interest (M_1) is assumed to be correctly specified, i.e. we believe the probabilistic structure is rich enough to assign high probability to the salient features of macroeconomic time series.

- Initial state of knowledge summarized in **prior** distribution $p(\theta)$.

- Update in view of data Y to obtain **posterior** distribution $p(\theta|Y)$:

$$p(\theta|Y,M_1) = \frac{p(Y|\theta,M_1)p(\theta|M_1)}{p(Y|M_1)}, \quad p(Y|M_1) = \int p(Y|\theta,M_1)p(\theta|M_1)d\theta.$$

- Make decisions that minimize posterior expected loss:

$$\delta_* = \text{argmin}_{\delta \in \mathcal{D}} \int L\big(h(\theta),\delta\big)p(\theta|Y,M_1)d\theta.$$

- Place probabilities on competing models and update:

$$\frac{\pi_{1,T}}{\pi_{2,T}} = \frac{\pi_{1,0}}{\pi_{2,0}} \frac{p(Y|M_1)}{p(Y|M_2)}.$$

Bayesian Inference – More Details

- Ingredients of Bayesian Analysis:
  - Likelihood function $p(Y|\theta)$
  - Prior density $p(\theta)$
  - Marginal data density $p(Y) = \int p(Y|\theta)p(\theta)d\phi$
- Bayes Theorem:

$$p(\theta|Y) = \frac{p(Y|\theta)p(\theta)}{p(Y)} \propto p(Y|\theta)p(\theta)$$

- Implementation: usually by generating a sequence of draws (not necessarily iid) from posterior

$$\theta^i \sim p(\theta|Y), \quad i = 1, \ldots, N$$

- Algorithms: direct sampling, accept/reject sampling, importance sampling, Markov chain Monte Carlo sampling, sequential Monte Carlo sampling...

Warm-Up: Linear Regression / AR Models

- Consider AR(1) model:

$$y_t = y_{t-1}\theta + u_t, \quad u_t \sim iidN(0,1).$$

- Let $x_t = y_{t-1}$. Write as

$$y_t = x_t'\theta + u_t, \quad u_t \sim iidN(0,1),$$

  or

$$Y = X\theta + U.$$

  We can easily allow for multiple regressors. Assume θ is $k \times 1$.

- Notice: we treat the variance of the errors as know. The generalization to unknown variance is straightforward but tedious.

- Likelihood function:

$$p(Y|\theta) = (2\pi)^{-T/2} \exp\left\{-\frac{1}{2}(Y - X\theta)'(Y - X\theta)\right\}.$$

A Convenient Prior

- Prior:

$$\theta \sim N\left(0_{k\times 1}, \tau^2 I_{k\times k}\right), \quad p(\theta) = (2\pi\tau^2)^{-k/2} \exp\left\{-\frac{1}{2\tau^2}\theta'\theta\right\}$$

- Large τ means diffuse prior.
- Small τ means tight prior.

Deriving the Posterior

- Bayes Theorem:
$$\begin{aligned} p(\theta|Y) &\propto p(Y|\theta)p(\theta) \\ &\propto \exp\left\{-\frac{1}{2}[(Y-X\theta)'(Y-X\theta)+\tau^{-2}\theta'\theta]\right\}. \end{aligned}$$
- Guess: what if $\theta|Y \sim N(\bar{\theta}_T, \bar{V}_T)$. Then
$$p(\theta|Y) \propto \exp\left\{-\frac{1}{2}(\theta-\bar{\theta}_T)'\bar{V}_T^{-1}(\theta-\bar{\theta}_T)\right\}.$$
- Rewrite exponential term
$$\begin{aligned} & Y'Y - \theta'X'Y - Y'X\theta + \theta'X'X\theta + \tau^{-2}\theta'\theta \\ &= Y'Y - \theta'X'Y - Y'X\theta + \theta'(X'X + \tau^{-2}\mathcal{I})\theta \\ &= \Big(\theta - (X'X+\tau^{-2}\mathcal{I})^{-1}X'Y\Big)'\Big(X'X+\tau^{-2}\mathcal{I}\Big) \\ &\quad \times\Big(\theta - (X'X+\tau^{-2}\mathcal{I})^{-1}X'Y\Big) \\ &\quad + Y'Y - Y'X(X'X+\tau^{-2}\mathcal{I})^{-1}X'Y. \end{aligned}$$

Deriving the Posterior

- Exponential term is a quadratic function of θ.
- Deduce: posterior distribution of θ must be a multivariate normal distribution

$$\theta|Y \sim N(\bar{\theta}_T, \bar{V}_T)$$

with

$$\begin{aligned} \bar{\theta}_T &= (X'X + \tau^{-2}I)^{-1}X'Y \\ \bar{V}_T &= (X'X + \tau^{-2}I)^{-1}. \end{aligned}$$

- $\tau \longrightarrow \infty$:

$$\theta|Y \overset{approx}{\sim} N\left(\hat{\theta}_{mle}, (X'X)^{-1}\right).$$

- $\tau \longrightarrow 0$:

$$\theta|Y \overset{approx}{\sim} \text{Pointmass at } 0$$

Marginal Data Density

- Plays an important role in Bayesian model selection and averaging.
- Write

$$
\begin{aligned}
p(Y) &= \frac{p(Y|\theta)p(\theta)}{p(\theta|Y)} \\
&= \exp\left\{ -\frac{1}{2}[Y'Y - Y'X(X'X + \tau^{-2}\mathcal{I})^{-1}X'Y] \right\} \\
&\quad \times (2\pi)^{-T/2}|\mathcal{I} + \tau^2 X'X|^{-1/2}.
\end{aligned}
$$

- The exponential term measures the goodness-of-fit.
- $|\mathcal{I} + \tau^2 X'X|$ is a penalty for model complexity.

Posterior

- We will often abbreviate posterior distributions $p(\theta|Y)$ by $\pi(\theta)$ and posterior expectations of $h(\theta)$ by

$$\mathbb{E}_\pi[h] = \mathbb{E}_\pi[h(\theta)] = \int h(\theta)\pi(\theta)d\theta = \int h(\theta)p(\theta|Y)d\theta.$$

- We will focus on algorithms that generate draws $\{\theta^i\}_{i=1}^N$ from posterior distributions of parameters in time series models.

- These draws can then be transformed into objects of interest, $h(\theta^i)$, and under suitable conditions a Monte Carlo average of the form

$$\bar{h}_N = \frac{1}{N}\sum_{i=1}^{N} h(\theta^i) \approx \mathbb{E}_\pi[h].$$

- Strong law of large numbers (SLLN), central limit theorem (CLT)...

Posterior Sampler 1: Direct Sampling

- In the simple linear regression model with Gaussian posterior it is possible to sample directly.
- For $i = 1$ to N, draw θ^i from $N\big(\bar{\theta}, \bar{V}_\theta\big)$.
- Provided that $\mathbb{V}_\pi[h(\theta)] < \infty$ we can deduce from Kolmogorov's SLLN and the Lindeberg-Levy CLT that

$$\begin{aligned} \bar{h}_N &\stackrel{a.s.}{\longrightarrow} \mathbb{E}_\pi[h] \\ \sqrt{N}\left(\bar{h}_N - \mathbb{E}_\pi[h]\right) &\Longrightarrow N\big(0, \mathbb{V}_\pi[h(\theta)]\big). \end{aligned}$$

Posterior Sampler 2: Importance Sampling

- Approximate $\pi(\cdot)$ by using a different, tractable density $g(\theta)$ that is easy to sample from.
- For more general problems, posterior density may be non-normalized. So we write

$$\pi(\theta) = \frac{p(Y|\theta)p(\theta)}{p(Y)} = \frac{f(\theta)}{\int f(\theta)d\theta}.$$

- Importance sampling is based on the identity

$$E_\pi[h(\theta)] = \int h(\theta)\pi(\theta)d\theta = \frac{\int_\Theta h(\theta)\frac{f(\theta)}{g(\theta)}g(\theta)d\theta}{\int_\Theta \frac{f(\theta)}{g(\theta)}g(\theta)d\theta}.$$

- The ratio

$$w(\theta) = \frac{f(\theta)}{g(\theta)}$$

is called the (unnormalized) importance weight.

Importance Sampling

1. For $i = 1$ to N, draw $\theta^i \overset{iid}{\sim} g(\theta)$ and compute the unnormalized importance weights

$$w^i = w(\theta^i) = \frac{f(\theta^i)}{g(\theta^i)}.$$

2. Compute the normalized importance weights

$$W^i = \frac{w^i}{\frac{1}{N}\sum_{i=1}^N w^i}.$$

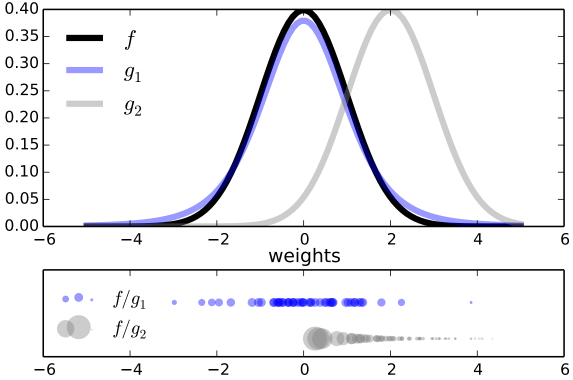
An approximation of $\mathbb{E}_\pi[h(\theta)]$ is given by

$$\bar{h}_N = \frac{\frac{1}{N}\sum_{i=1}^N w^i h(\theta^i)}{\frac{1}{N}\sum_{i=1}^N w^i} = \frac{1}{N}\sum_{i=1}^N W^i h(\theta^i).$$

Illustration

If θ^i's are draws from $g(\cdot)$ then

$$\mathbb{E}_\pi[h] \approx = \frac{\frac{1}{N}\sum_{i=1}^N h(\theta^i)w(\theta^i)}{\frac{1}{N}\sum_{i=1}^N w(\theta^i)}, \quad w(\theta) = \frac{f(\theta)}{g(\theta)}.$$



Accuracy

- Since we are generating *iid* draws from $g(\theta)$, it's fairly straightforward to derive a CLT:
- It can be shown that

$$\sqrt{N}(\bar{h}_N - \mathbb{E}_\pi[h]) \Longrightarrow N\big(0, \Omega(h)\big), \quad \text{where} \quad \Omega(h) = \mathbb{V}_g[(\pi/g)(h - \mathbb{E}_\pi[h])].$$

- Using a crude approximation (see, e.g., Liu (2008)), we can factorize $\Omega(h)$ as follows:

$$\Omega(h) \approx \mathbb{V}_\pi[h]\big(\mathbb{V}_g[\pi/g] + 1\big).$$

  The approximation highlights that the larger the variance of the importance weights, the less accurate the Monte Carlo approximation relative to the accuracy that could be achieved with an *iid* sample from the posterior.

- Users often monitor

$$ESS = N\frac{\mathbb{V}_\pi[h]}{\Omega(h)} \approx \frac{N}{1 + \mathbb{V}_g[\pi/g]}.$$

Decision Making

- The posterior expected loss associated with a decision $\delta(\cdot)$ is given by

$$\rho\big(\delta(\cdot)|Y\big) = \int_\Theta L\big(\theta, \delta(Y)\big) p(\theta|Y) d\theta.$$

- A Bayes decision is a decision that minimizes the posterior expected loss:

$$\delta^*(Y) = \text{argmin}_d \; \rho\big(\delta(\cdot)|Y\big).$$

- Since in most applications it is not feasible to derive the posterior expected risk analytically, we replace $\rho\big(\delta(\cdot)|Y\big)$ by a Monte Carlo approximation of the form

$$\bar{\rho}_N\big(\delta(\cdot)|Y\big) = \frac{1}{N}\sum_{i=1}^{N} L\big(\theta^i, \delta(\cdot)\big).$$

- A numerical approximation to the Bayes decision $\delta^*(\cdot)$ is then given by

$$\delta_N^*(Y) = \text{argmin}_d \; \bar{\rho}_N\big(\delta(\cdot)|Y\big).$$

Inference

- Point estimation:
  - Quadratic loss: posterior mean
  - Absolute error loss: posterior median
- Interval/Set estimation $\mathbb{P}_\pi\{\theta \in C(Y)\} = 1 - \alpha$:
  - highest posterior density sets
  - equal-tail-probability intervals

Point Estimation

- Interpret point estimation as decision problem.
- Consider quadratic loss:

$$L(\theta, \delta) = (\theta - \delta)^2$$

- Optimal decision rule is obtained by minimizing

$$\min_{\delta \in \mathcal{D}} \mathbb{E}_\pi[(\theta - \delta)^2]$$

- Solution: $\delta = \mathbb{E}_\pi[\theta]$, i.e., posterior mean.

Consistency of Posterior Mean

- Consistency: Suppose data are generated from the model $y_t = x_t'\theta_0 + u_t$. Asymptotically the Bayes estimator converges to the "true" parameter θ_0.
- Consider

$$
\begin{aligned}
\bar{\theta}_T &= (X'X + \tau^{-2}\mathcal{I})^{-1}X'Y \\
&= \theta_0 + \left[\left(\frac{1}{T}\sum x_t x_t' + \frac{1}{\tau^2 T}\mathcal{I} \right)^{-1} - \left(\frac{1}{T}\sum x_t x_t' \right)^{-1} \right] \\
&\quad \times \left(\frac{1}{T}\sum x_t x_t' \right)\theta_0 \\
&\quad + \left(\frac{1}{T}\sum x_t x_t' + \frac{1}{\tau^2 T}\mathcal{I} \right)^{-1} \left(\frac{1}{T}\sum x_t u_t \right) \\
&\stackrel{p}{\longrightarrow} \theta_0
\end{aligned}
$$

- Disagreement between two Bayesians who have different priors will asymptotically vanish.

Testing

- $H_0 : \theta \in \Theta_0$ versus $H_1 : \theta \in \Theta_1$.
- Decision space is 0 ("reject") and 1 ("accept").
- Loss function

$$L(\theta, \delta) = \begin{cases} 0 & \delta = \mathbb{I}\{\theta \in \Theta_0\} \quad \text{correct decision} \\ a_0 & \delta = 0, \ \theta \in \Theta_0 \quad \text{Type 1 error} \\ a_1 & \delta = 1, \ \theta \in \Theta_1 \quad \text{Type 2 error} \end{cases}$$

  Note that the parameters a_1 and a_2 are part of the econometricians preferences.
- Optimal decision:

$$\delta(Y) = \begin{cases} 1 & \mathbb{P}_\pi\{\theta \in \Theta_0\} \geq \frac{a_1}{a_0 + a_1} \\ 0 & \text{otherwise} \end{cases}$$

Testing

- Posterior odds:

$$\frac{\mathbb{P}_\pi\{\theta \in \Theta_0\}}{\mathbb{P}_\pi\{\theta \in \Theta_1\}}$$

- Often, hypotheses are evaluated according to Bayes factors:

$$B(Y) = \frac{\text{Posterior Odds}}{\text{Prior Odds}}$$

Credible Sets

- Set estimation is more difficult to cast into a decision problem $\mapsto$ two-player game.
- Bayesian credible set: $C_Y \subseteq \Theta$ is $1-\alpha$ credible if

$$\mathbb{P}^{\theta}_{Y}\{\underbrace{\theta}_{r.v.} \in C_Y\} \geq 1-\alpha$$

- A highest posterior density region (HPD) is of the form

$$C_Y = \{\theta : p(\theta|Y) \geq k_\alpha\} \quad \text{where } k_\alpha \text{ is chosen s.t. } \mathbb{P}^{\theta}_{Y}\{\theta \in C_Y\} = 1-\alpha.$$

  HPD regions have the smallest volume among all $1-\alpha$ credible regions.
- HPD regions are often difficult to compute. Thus, Bayesians often report equal-tail probability credible intervals.
- **Recall definition of frequentist confidence set:**

$$\mathbb{P}^{Y}_{\theta}\{\theta \in \underbrace{C_Y}_{r.v.}\} \geq 1-\alpha \quad \text{for all} \quad \theta \in \Theta.$$

Forecasting

- Example:

$$y_{T+h} = \theta^h y_T + \sum_{s=0}^{h-1} \theta^s u_{T+h-s}$$

- h-step ahead conditional distribution:

$$y_{T+h}|(Y_{1:T},\theta) \sim N\left(\theta^h y_T, \frac{1-\theta^h}{1-\theta}\right).$$

- Posterior predictive distribution:

$$p(y_{T+h}|Y_{1:T}) = \int p(y_{T+h}|y_T,\theta)p(\theta|Y_{1:T})d\theta.$$

- For each draw θ^i from the posterior distribution $p(\theta|Y_{1:T})$ sample a sequence of innovations $u^i_{T+1},\dots,u^i_{T+h}$ and compute y^i_{T+h} as a function of θ^i, $u^i_{T+1},\dots,u^i_{T+h}$, and $Y_{1:T}$.

Model Uncertainty

- Assign prior probabilities $\gamma_{j,0}$ to models M_j, $j = 1, \dots, J$.
- Posterior model probabilities are given by

$$\gamma_{j,T} = \frac{\gamma_{j,0} p(Y|M_j)}{\sum_{j=1}^J \gamma_{j,0} p(Y|M_j)},$$

where

$$p(Y|M_j) = \int p(Y|\theta_{(j)}, M_j) p(\theta_{(j)}|M_j) d\theta_{(j)}$$

- Log marginal data densities are one-step-ahead predictive scores:

$$\begin{aligned} &\ln p(Y|M_j) \\ &= \sum_{t=1}^T \ln \int p(y_t|\theta_{(j)}, Y_{1:t-1}, M_j) p(\theta_{(j)}|Y_{1:t-1}, M_j) d\theta_{(j)}. \end{aligned}$$

- Model averaging:

$$p(h|Y) = \sum_{j=1}^J \gamma_{j,T} p(h_j(\theta_{(j)})|Y, M_j).$$