Introduction to DSGE Modeling

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Introduction

- Estimated dynamic stochastic general equilibrium (DSGE) models are now widely used for
  - empirical research in macroeconomics;
  - quantitative policy analysis and prediction at central banks.
- We will consider a prototypical New Keynesian DSGE model...

Potential Empirical Questions

1. What is the optimal target inflation rate?
2. Was high inflation and output volatility in the 1970s due to loose monetary policy?
3. Effects of the zero lower bound on nominal interest rates on monetary policy.
4. How large are government spending multipliers?
5. Fiscal policy rules and the effect of a change in the labor tax rate.

A Small-Scale New Keynesian DSGE Model

- The model consists of
  - households;
  - final goods producing firms;
  - intermediate goods producing firms;
  - central bank and fiscal authority;
  - exogenous shock processes
- Let's take a look at the decision problems faced by economic agents...

Households

- Households maximize

$$\mathbb{E}_\tau \left[\sum_{t=\tau}^{\infty} \beta^{(t-\tau)} \left\{ \ln C_t - \frac{\phi_t}{1+\nu} L_t^{1+\nu} \right\} \right]$$

- subject to the constraints:

$$P_t C_t + B_{t+1} \leq P_t W_t L_t + \Pi_t + R_{t-1} B_t - T_t + \Omega_t.$$

- In a nutshell:
  - household cares about the future: intertemporal optimization
  - household likes consumption
  - household does not like to work...
  - there is a budget constraint: can't spend more than you earn and borrow; have to pay taxes;

Households

- Households maximize

$$\mathbb{E}_\tau \left[\sum_{t=\tau}^{\infty} \beta^{(t-\tau)} \left\{ \ln C_t - \frac{\phi_t}{1+\nu} L_t^{1+\nu} \right\} \right]$$

- subject to the constraints:

$$P_t C_t + B_{t+1} \leq P_t W_t L_t + \Pi_t + R_{t-1} B_t - T_t + \Omega_t.$$

- Possible modifications/generalizations:
  - let households on shares to the capital stock;
  - introduce money explicitly: cash-in-advance versus money in the utility function;
  - make taxes distortionary;
  - introduce differentiated labor.

Households: First-Order Conditions

- Households maximize

$$\mathbb{E}_\tau \left[\sum_{t=\tau}^{\infty} \beta^{(t-\tau)} \left\{ \ln C_t - \frac{\phi_t}{1+\nu} L_t^{1+\nu} \right\} \right]$$

- subject to the constraints:

$$P_t C_t + B_{t+1} \leq P_t W_t L_t + \Pi_t + R_{t-1} B_t - T_t + \Omega_t.$$

- Introduce Lagrange multiplier μ_t for budget constraint.
- Lagrangian

$$\begin{aligned} \mathcal{L} &= \mathbb{E}_\tau \Bigg[\sum_{t=\tau}^{\infty} \beta^{(t-\tau)} \Bigg\{ \ln C_t - \frac{\phi_t}{1+\nu} L_t^{1+\nu} \\ &\quad - \mu_t \Big(P_t C_t + B_{t+1} - \big[P_t W_t L_t + \Pi_t + R_{t-1} B_t - T_t + \Omega_t \big] \Big) \Bigg\} \Bigg] \end{aligned}$$

Households: First-Order Conditions

- Lagrangian

$$
\begin{aligned}
\mathcal{L} &= \mathbb{E}_\tau\Bigg[\sum_{t=\tau}^{\infty}\beta^{(t-\tau)}\Bigg\{\ln C_t - \frac{\phi_t}{1+\nu}L_t^{1+\nu} \\
&\quad -\mu_t\Big(P_tC_t + B_{t+1} - \big[P_tW_tL_t + \Pi_t + R_{t-1}B_t - T_t + \Omega_t\big]\Big)\Bigg\}\Bigg]
\end{aligned}
$$

- First-order condition for C_t:

$$
\frac{1}{C_t} = \mu_t P_t
$$

- First-order condition for B_{t+1}:

$$
\mu_t = \beta\mathbb{E}_t[\mu_{t+1}R_t]
$$

- Combine to consumption Euler equation (define $\pi_{t+1} = P_{t+1}/P_t$):

$$
\frac{1}{C_t} = \beta\mathbb{E}_t\left[\frac{1}{C_{t+1}}\frac{R_t}{\pi_{t+1}}\right]
$$

Households: First-Order Conditions

- Lagrangian

$$\mathcal{L} = \mathbb{E}_\tau \Bigg[\sum_{t=\tau}^{\infty} \beta^{(t-\tau)} \bigg\{ \ln C_t - \frac{\phi_t}{1+\nu} L_t^{1+\nu}$$
$$-\mu_t \Big(P_t C_t + B_{t+1} - \big[P_t W_t L_t + \Pi_t + R_{t-1} B_t - T_t + \Omega_t \big] \Big) \bigg\} \Bigg]$$

- Labor supply – first-order condition for L_t:

$$\phi_t L_t^\nu = \mu_t P_t W_t = \frac{W_t}{C_t}.$$

A Small-Scale New Keynesian DSGE Model

- households;
- **final goods producing firms;**
- intermediate goods producing firms;
- central bank and fiscal authority;
- exogenous shock processes

Final Goods Production

- Production: (these guys just buy and combine intermediate goods)

$$Y_t = \left[\int_0^1 Y_t(i)^{\frac{1}{1+\lambda}} di \right]^{1+\lambda_t}$$

- Profits

$$Y_t P_t - \int Y_t(i) P_t(i) di = \left[\int_0^1 Y_t(i)^{\frac{1}{1+\lambda_t}} di \right]^{1+\lambda_t} P_t - \int Y_t(i) P_t(i) di.$$

- Take prices as given and maximize profits by choosing optimal inputs $Y_t(i)$:

$$P_t(i) = P_t Y_t^{\lambda_t/(1+\lambda_t)} Y_t(i)^{-\lambda_t/(1+\lambda_t)} \quad \Longrightarrow \quad Y_t(i) = \left(\frac{P_t(i)}{P_t} \right)^{-\frac{1+\lambda_t}{\lambda_t}} Y_t$$

- Free entry leads to zero profits:

$$Y_t P_t = \int Y_t(i) P_t(i) di \quad \Longrightarrow \quad P_t = \left[\int_0^1 P_t(i)^{-\frac{1}{\lambda_t}} di \right]^{-\lambda_t}.$$

- Aggregate inflation is defined as $\pi_t = P_t / P_{t-1}$.

A Small-Scale New Keynesian DSGE Model

- households;
- final goods producing firms;
- **intermediate goods producing firms;**
- central bank and fiscal authority;
- exogenous shock processes

Intermediate Goods Production

- Production (these guys hire to produce something):

$$Y_t(i) = \max\left\{A_t L_t(i) - \mathcal{F}, 0\right\}.$$

- Firms are monopolistically competitive; face downward sloping demand curve:

$$Y_t(i) = \left(\frac{P_t(i)}{P_t}\right)^{-\frac{1+\lambda_t}{\lambda_t}} Y_t.$$

- Firms set prices to maximize profits, but there is a friction:
  - firms can only re-optimize their prices with probability $1 - \zeta_p$;
  - remaining $1 - \iota$ firms adjust their prices by $\bar{\pi}$
- Once prices are set, firms have to produce whatever quantity is demanded.

Intermediate Goods Production

- Define the real marginal costs of producing a unit Y_{it} as

$$MC_t = \frac{W_t}{A_t}$$

- Decision problem ($\beta^s \Xi_{t+s|t}$ is today's value of a future dollar)

$$\max_{\tilde{P}_t(i)} \quad \mathbb{E}_t \left\{ \sum_{s=0}^{\infty} \zeta_p^s \beta^s \Xi_{t+s|t} Y_{t+s}(i) \left[\tilde{P}_t(i) \bar{\pi}^s - P_{t+s} MC_{t+s} \right] \right\}$$

$$\text{s.t.} \quad Y_{t+s}(i) = \left(\frac{\tilde{P}_t(i) \bar{\pi}^s}{P_{t+s}} \right)^{-\frac{1+\lambda_t}{\lambda_t}} Y_{t+s}$$

- Differentiate with respect to $\tilde{P}_t(i)$ to obtain first-order condition for optimal price.

Intermediate Goods Production

- First-order condition to determine $\tilde{P}_t(i)$:

$$\mathbb{E}_t \left\{ \sum_{s=0}^{\infty} \zeta_p^s \beta^s \Xi_{t+s|t} \left(\frac{\partial Y_{t+s}(i)}{\partial \tilde{P}_t(i)} \left(\tilde{P}_t(i) \bar{\pi}^s - P_{t+s} MC_{t+s} \right) + Y_{t+s}(i) \bar{\pi}^s \right) \right\} = 0,$$

- where

$$\frac{\partial Y_{t+s}(i)}{\partial \tilde{P}_t(i)} = -\frac{1+\lambda_t}{\lambda_t} \frac{\bar{\pi}^s}{P_{t+s}} \left(\frac{\tilde{P}_t(i) \bar{\pi}^s}{P_{t+s}} \right)^{-\frac{1+\lambda_t}{\lambda_t}-1} Y_{t+s} = -\frac{1+\lambda_t}{\lambda_t} \frac{1}{\tilde{P}_t(i)} Y_{t+s}(i)$$

- Assume all optimizing firms choose the same price: $\tilde{P}_t(i) = \tilde{P}_t$.

Intermediate Goods Production

- Divide FOC by P_t and impose symmetry. Let $\tilde{p}_t = \tilde{P}_t / P_t$.
- First-order condition to determine $\tilde{p}_t$:

$$\mathbb{E}_t \left\{ \sum_{s=0}^{\infty} \zeta_p^s \beta^s \frac{\Xi_{t+s|t}}{\lambda_t \tilde{p}_t} \left(\frac{\tilde{p}_t \bar{\pi}^s}{\prod_{j=1}^{s} \pi_{t+j}} \right)^{-\frac{1+\lambda_t}{\lambda_t}} Y_{t+s} \left[\tilde{p}_t \bar{\pi}^s - (1+\lambda_t) \left(\prod_{j=1}^{s} \pi_{t+j} \right) MC_{t+s} \right] \right\} = 0,$$

- New Keynesian Phillips curve: relationship between $\tilde{p}_t$, inflation π_t, and real marginal costs MC_t.

Intermediate Goods Production

- Recall from final goods producers:

$$P_t = \left[\int_0^1 P_t(i)^{-\frac{1}{\lambda_t}} di \right]^{-\lambda_t}.$$

- Fraction ζ_p will index previous price $P_{t-1}(i)$ by inflation, whereas fraction $(1-\zeta_p)$ will charge $\tilde{P}_t$:

$$\begin{aligned} P_t &= \left[(1-\zeta_p)\tilde{P}_t^{-\frac{1}{\lambda_t}} + \zeta_p \bar{\pi}^{-\frac{1}{\lambda_t}} \int_0^1 P_{t-1}(i)^{-\frac{1}{\lambda_t}} di \right]^{-\lambda_t} \\ &= \left[(1-\zeta_p)\tilde{P}_t^{-\frac{1}{\lambda_t}} + \zeta_p \bar{\pi}^{-\frac{1}{\lambda_t}} P_{t-1}^{-\frac{1}{\lambda_t}} \right]^{-\lambda_t} \end{aligned}$$

- Inflation satisfies (let $\tilde{p}_t = \tilde{P}_t / P_t$):

$$\pi_t = \left[(1-\zeta_p)(\pi_t \tilde{p}_t)^{-\frac{1}{\lambda_t}} + \zeta_p \bar{\pi}^{-\frac{1}{\lambda_t}} \right]^{-\lambda_t}$$

Intermediate Goods Production

- Most complicated part of the model...
- generates a relationship between real marginal costs and inflation.
- So, it connects nominal and real side of the economy.
- **Exercise:** if $\zeta_p = 0$ prices are flexible. Simplify the formulas!

- households;
- final goods producing firms;
- intermediate goods producing firms;
- **central bank and fiscal authority;**
- exogenous shock processes

Monetary Policy

- We did not specify a money demand equation, but we could. It would depend on the nominal interest rate. The higher R_t, the lower the demand for money.
- Central bank prints enough money so that demand is satisfied at interest rate implied by monetary policy rule:

$$R_t = R_{*,t}^{1-\rho_R} R_{t-1}^{\rho_R} \exp\{\sigma_R \epsilon_{R,t}\}, \quad R_{*,t} = (r\pi_*) \left(\frac{\pi_t}{\pi_*}\right)^{\psi_1} \left(\frac{Y_t}{\gamma Y_{t-1}}\right)^{\psi_2}$$

- r is equilibrium real rate.
- π_* is target inflation rate.
- $\epsilon_{R,t}$ is exogenous monetary policy shock. Interpretation?

Fiscal Policy

- For now, it's passive and not very interesting.
- Budget constraint:

$$P_t G_t + R_{t-1} B_t + M_t = T_t + B_t + M_{t+1}$$

- Lump-sum taxes/transfer balance the budget in every period. Seigniorage does not matter.
- Government spending is exogenous. Re-scale:

$$G_t = \left(1 - \frac{1}{g_t}\right) Y_t.$$

A Small-Scale New Keynesian DSGE Model

- households;
- final goods producing firms;
- intermediate goods producing firms;
- central bank and fiscal authority;
- **exogenous shock processes.**

Exogenous shock processes

- Total factor productivity A_t.
- Preference / labor demand shifter ϕ_t.
- Mark-up shock λ_t.
- Monetary policy shock $\epsilon_{R,t}$.
- Government spending shock g_t.
- We will specify exogenous laws of motions for these processes, e.g.,

$$\ln g_t = (1-\rho_g)\ln g^* + \rho_g \ln g_{t-1} + \sigma_g \epsilon_{g,t}, \quad \epsilon_{g,t} \sim N(0,1).$$

Aggregate Resource Constraint

- Combine household and government budget constraints:

$$P_t C_t + P_t G_t = P_t W_t \int L_t(i)di + \int \Pi_t(i)di$$

- Final goods producers make zero profits, which implies:

$$P_t Y_t = \int P_t(i) Y_t(i) di.$$

- Profits of intermediate goods producers:

$$\begin{aligned}\int \Pi_t(i)di &= \int Y_t(i)P_t(i)di - P_t W_t \int L_t(i)di - \mathcal{F} \\ &= P_t Y_t - P_t W_t L_t - \mathcal{F}.\end{aligned}$$

- Thus, assuming $\mathcal{F} = 0$:

$$C_t + G_t = Y_t.$$

Aggregate Resource Constraint

- Production:

$$Y_t(i) = A_t L_t(i)$$

- Using the demand function for $Y_t(i)$ we can write

$$Y_t \left(\frac{P_t(i)}{P_t} \right)^{-\frac{1+\lambda_t}{\lambda_t}} = A_t L_t(i).$$

- Integrating over the firms i yields:

$$Y_t = \frac{1}{D_t} A_t L_t, \quad D_t = \int \left(\frac{P_t(i)}{P_t} \right)^{-\frac{1+\lambda_t}{\lambda_t}} di \geq 1$$

- Price dispersion creates a loss of output!

Evolution of Price Dispersion

- Recall

$$D_t = \int \left(\frac{P_t(i)}{P_t} \right)^{-\frac{1+\lambda_t}{\lambda_t}} di$$

- A fraction of ζ_p firms changes its price in each period. Thus,

$$\begin{aligned} D_t &= (1-\zeta_p) \sum_{j=0}^{\infty} \zeta^j \left(\frac{\bar{\pi}^j \tilde{P}_{t-j}}{\pi_t \pi_{t-1} \cdots \pi_{t-j+1} P_{t-j}} \right)^{-\frac{1+\lambda_t}{\lambda_t}} \\ &= (1-\zeta_p) \sum_{j=0}^{\infty} \zeta^j \left(\frac{\bar{\pi}^j}{\pi_t \pi_{t-1} \cdots \pi_{t-j+1}} \tilde{p}_t \right)^{-\frac{1+\lambda_t}{\lambda_t}} \end{aligned}$$

Complete Markets

- Firms discount future profits using the households stochastic discount factor:

$$\Xi_{t+s|t} = \frac{C_t}{C_{t+1}}$$

So far

- We now have a small-scale New Keynesian DSGE model! What are the policy trade-offs? What policies can we study?

- **Monetary policy:**
  - systematic part (react to inflation and output growth): what happens if we change inflation target π^*? What happens if CB reacts more aggressively to inflation deviations?
  - discretionary component: what happens if CB raises interest rates in an unanticipated fashion, i.e., $\epsilon_{R,t} > 0$?

- **Fiscal policy:**
  - systematic part: what happens if g^* increases?
  - unanticipated: reaction to $\epsilon_{g,t}$.

- To answer other questions, we need to enrich the model:
  - ZLB constraint;
  - role for unconventional monetary policy;
  - distortionary taxes;
  - more interesting debt dynamics.

Model Solution

- After deriving the equilibrium conditions of the model, we now need to solve for the dynamics of the endogenous variables.
- System of nonlinear expectational difference equations;
- Find solution(s) of system of expectational difference equations:
  - global (nonlinear) approximation methods;
  - local approximation near steady state.
- We will focus on log-linear approximations around the steady state.
- Many more details in FVRRS.

Our Goal: State-space Representation of DSGE Model

- $n_y \times 1$ vector of observables:

$$y_t = M_y'[\log(X_t/X_{t-1}), \log lsh_t, \log \pi_t, \log R_t]'.$$

- $n_s \times 1$ vector of econometric state variables s_t

$$s_t = [\phi_t, \lambda_t, z_t, \epsilon_{R,t}, \widehat{x}_{t-1}]'$$

- DSGE model parameters:

$$\theta = [\beta, \gamma, \lambda, \pi^*, \zeta_p, \nu, \rho_\phi, \rho_\lambda, \rho_z, \sigma_\phi, \sigma_\lambda, \sigma_z, \sigma_R]'.$$

- Measurement equation:

$$y_t = \Psi_0(\theta) + \Psi_1(\theta) s_t.$$

- State-transition equation:

$$s_t = \Phi_1(\theta) s_{t-1} + \Phi_\epsilon(\theta)\epsilon_t, \quad \epsilon_t = [\epsilon_{\phi,t}, \epsilon_{\lambda,t}, \epsilon_{z,t}, \epsilon_{R,t}]'$$

Our Goal: State-Space Representation of DSGE Model

State-space representation:

$$
\begin{aligned}
y_t &= \Psi_0(\theta) + \Psi_1(\theta)s_t \\
s_t &= \Phi_1(\theta)s_{t-1} + \Phi_\epsilon(\theta)\epsilon_t
\end{aligned}
$$

System matrices:

$$
\Psi_0(\theta) = M_y' \begin{bmatrix} \log\gamma \\ \log(lsh) \\ \log\pi^* \\ \log(\pi^*\gamma/\beta) \end{bmatrix}, \quad x_\phi = -\frac{\kappa_p\psi_p/\beta}{1-\psi_p\rho_\phi}, \quad x_\lambda = -\frac{\kappa_p\psi_p/\beta}{1-\psi_p\rho_\lambda}, \quad x_z = \frac{\rho_z\psi_p}{1-\psi_p\rho_z}, \quad x_{\epsilon_R} = -\psi_p\sigma_R
$$

$$
\Psi_1(\theta) = M_y' \begin{bmatrix} x_\phi & x_\lambda & x_z + 1 & x_{\epsilon_R} & -1 \\ 1+(1+\nu)x_\phi & (1+\nu)x_\lambda & (1+\nu)x_z & (1+\nu)x_{\epsilon_R} & 0 \\ \frac{\kappa_p}{1-\beta\rho_\phi}(1+(1+\nu)x_\phi) & \frac{\kappa_p}{1-\beta\rho_\lambda}(1+(1+\nu)x_\lambda) & \frac{\kappa_p}{1-\beta\rho_z}(1+\nu)x_z & +\kappa_p(1+\nu)x_{\epsilon_R} & 0 \\ \frac{\kappa_p/\beta}{1-\beta\rho_\phi}(1+(1+\nu)x_\phi) & \frac{\kappa_p/\beta}{1-\beta\rho_\lambda}(1+(1+\nu)x_\lambda) & \frac{\kappa_p/\beta}{1-\beta\rho_z}(1+\nu)x_z & (\kappa_p(1+\nu)x_{\epsilon_R}/\beta + \sigma_R) & 0 \end{bmatrix}
$$

$$
\Phi_1(\theta) = \begin{bmatrix} \rho_\phi & 0 & 0 & 0 & 0 \\ 0 & \rho_\lambda & 0 & 0 & 0 \\ 0 & 0 & \rho_z & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ x_\phi & x_\lambda & x_z & x_{\epsilon_R} & 0 \end{bmatrix}, \quad \Phi_\epsilon(\theta) = \begin{bmatrix} \sigma_\phi & 0 & 0 & 0 \\ 0 & \sigma_\lambda & 0 & 0 \\ 0 & 0 & \sigma_z & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}
$$

M_y' is an $n_y \times 4$ selection matrix that selects rows of Ψ_0 and Ψ_1.

Steady State

- Shut down aggregate uncertainty: set all shock standard deviations $\sigma_{\cdot} = 0$.
- **Technology:**

$$\ln A_t = \ln\gamma + \ln A_{t-1} + z_t, \quad z_t = \rho_z z_{t-1} + \sigma_z \epsilon_{z,t}.$$

  Set $\sigma_z = 0$: $\ln A_t^* = \gamma t$.
- **Preferences:**

$$\ln\phi_t = (1-\rho_\phi)\ln\phi + \rho_\phi \ln\phi_{t-1} + \sigma_\phi \epsilon_{\phi,t}.$$

- **Mark-up:**

$$\ln\lambda_t = (1-\rho_\lambda)\ln\lambda + \rho_\lambda \ln\lambda_{t-1} + \sigma_\lambda \epsilon_{\lambda,t}.$$

- **Government Spending:**

$$\ln g_t = (1-\rho_g)\ln g^* + \rho_g \ln g_{t-1} + \sigma_g \epsilon_{g,t}$$

Steady State

- Problem: this economy grows... which does not lead to a **steady** state.
- Solution: detrend model variables by A_t.
- Model has steady state in terms of detrended variables.

Households' Euler Equation

- Recall:

$$\frac{1}{C_t} = \beta \mathbb{E}_t \left[\frac{1}{C_{t+1}} \frac{R_t}{\pi_{t+1}} \right]$$

- Rewrite:

$$\frac{A_t}{C_t} = \beta \mathbb{E}_t \left[\frac{A_{t+1}}{C_{t+1}} \frac{A_t}{A_{t+1}} \frac{R_t}{\pi_{t+1}} \right] \quad \Longrightarrow \quad \frac{1}{c_t} = \beta \mathbb{E}_t \left[\frac{1}{c_{t+1}} \frac{1}{\gamma e^{z_{t+1}}} \frac{R_t}{\pi_{t+1}} \right]$$

- Steady state:

$$R = \pi \frac{\gamma}{\beta} = \pi r.$$

Households' Labor Supply

- Recall:

$$\phi_t L_t^{\nu} = \frac{W_t}{C_t}$$

- Rewrite:

$$\phi_t L_t^{\nu} = \frac{W_t/A_t}{C_t/A_t} \quad \Longrightarrow \quad \phi_t L_t^{\nu} = \frac{w_t}{c_t}$$

- Steady state:

$$\phi L^{\nu} = \frac{w}{c}.$$

Intermediate Goods Production

- Recall:

$$MC_t = \frac{W_t}{A_t}.$$

- Steady state:

$$mc = w.$$

- Recall:

$$\pi_t = \left[(1-\zeta_p)(\pi_t \tilde{p}_t)^{-\frac{1}{\lambda_t}} + \zeta_p \bar{\pi}^{-\frac{1}{\lambda_t}}\right]^{-\lambda_t}.$$

- Steady state:

$$\pi = \left[(1-\zeta_p)(\pi \tilde{p})^{-\frac{1}{\lambda}} + \zeta_p \bar{\pi}^{-\frac{1}{\lambda}}\right]^{-\lambda}.$$

Intermediate Goods Production

- Recall:

$$C_t \mathbb{E}_t \left\{ \sum_{s=0}^{\infty} \zeta_p^s \beta^s \frac{Y_{t+s}/C_{t+s}}{\lambda_t \tilde{p}_t} \left(\frac{\tilde{p}_t \bar{\pi}^s}{\prod_{j=1}^{s} \pi_{t+j}} \right)^{-\frac{1+\lambda_t}{\lambda_t}} \left[\tilde{p}_t \bar{\pi}^s - (1+\lambda_t) \left(\prod_{j=1}^{s} \pi_{t+j} \right) MC_{t+s} \right] \right\} = 0,$$

- Steady state:

$$\frac{c/y}{\lambda \tilde{p}} \tilde{p}^{-\frac{1+\lambda}{\lambda}} \left\{ \sum_{s=0}^{\infty} \zeta_p^s \beta^s \left(\frac{\bar{\pi}^s}{\pi^s} \right)^{-\frac{1+\lambda}{\lambda}} [\tilde{p}_t \bar{\pi}^s - (1+\lambda) \pi^s mc] \right\} = 0,$$

Monetary and Fiscal Policy - Steady States

- **Monetary policy rule:**

$$R = r\pi_* \left(\frac{\pi}{\pi_*} \right)^{\psi_1}$$

- **Government spending:**

$$g = \left(1 - \frac{1}{g_*} \right) y$$

Aggregate Resource Constraint and Price Dispersion

- **Market clearing:**

$$c + \left(1 - \frac{1}{g_*}\right) y = y \quad \Longrightarrow \quad c = \frac{1}{g_*} y.$$

- **Aggregate production:**

$$y = \frac{1}{D} L.$$

- **Price dispersion:**

$$D = (1 - \zeta_p) \sum_{j=0}^{\infty} \zeta_p^j \left(\frac{\bar{\pi}^j}{\pi^j} \tilde{p} \right)^{-\frac{1+\lambda}{\lambda}} = \tilde{p}^{-\frac{1+\lambda}{\lambda}} \frac{1 - \zeta_p}{1 - \zeta_p \left(\bar{\pi} / \tilde{\pi} \right)^{-\frac{1+\lambda}{\lambda}}}.$$

Combining Bits and Pieces

- Steady state equations are quite complicated.
- Special case: $\bar{\pi} = \pi_*$, i.e., price setters index prices by target inflation rate.
- Verify that $\pi = \pi_* = \bar{\pi}$ is an equilibrium:
  - Policy rule and Euler equation imply $R = \pi r$, where $r = \gamma/\beta$.
  - For $\bar{\pi} = \pi$ the condition

$$\pi = \left[(1-\zeta_p)(\pi\tilde{p})^{-\frac{1}{\lambda}} + \zeta_p \bar{\pi}^{-\frac{1}{\lambda}}\right]^{-\lambda}.$$

    implies $\tilde{p} = 1$.
  - Thus, there is no steady state price dispersion: $D = 1$.
  - The firms' FOC imply that

$$mc = w = \frac{1}{1+\lambda} \quad \Longrightarrow \quad \tilde{p} = (1+\lambda)mc.$$

  - Using $c = y/g_*$ and $y = l$, the households' labor supply condition implies

$$\phi y^{\nu} = \frac{w}{c} = \frac{1}{1+\lambda}\frac{g_*}{y} \quad \Longrightarrow \quad y = \left(\frac{g_*}{\phi(1+\lambda)}\right)^{1/(1+\nu)}.$$

Monetary and Fiscal Policy Experiments - Thus Far...

- Change the target inflation rate π_*, assuming that indexation to $\bar{\pi}$ does not change. Crucial parameter: ζ_p.
- Change the amount of government spending through g_* and compute long-run multipliers. Crucial parameter ν.
- Estimate model to obtain policy-effect relevant parameters.
- Parameter uncertainty translates into policy uncertainty.

(Log) Linearization Around Steady State

- We will now approximate the equilibrium dynamics of the model.
- Taylor series expansion around around the steady state.
- Linear rational exectations system:

$$
\begin{aligned}
\widehat{c}_t &= \mathbb{E}_{t+1}[\widehat{c}_{t+1}] - \left(\widehat{R}_t - \mathbb{E}[\widehat{\pi}_{t+1}]\right) + \mathbb{E}_t[z_{t+1}] \\
\widehat{\pi}_t &= \beta \mathbb{E}_t[\widehat{\pi}_{t+1}] + \kappa_p\left(\widehat{lsh}_t + \lambda_t\right) \\
\widehat{R}_t &= \psi_1 \widehat{\pi}_t + \psi_2(\widehat{y}_t - \widehat{y}_{t-1} + z_t) + \sigma_R \epsilon_{R,t} \\
\widehat{lsh}_t &= (1+\nu)\widehat{c}_t + \nu \widehat{g}_t + \phi_t \\
\widehat{y}_t &= \widehat{c}_t + \widehat{g}_t
\end{aligned}
$$

State-space Representation of DSGE Model

- $n_y \times 1$ vector of observables:

$$y_t = M_y'[\log(X_t/X_{t-1}), \log lsh_t, \log \pi_t, \log R_t]'.$$

- $n_s \times 1$ vector of econometric state variables s_t

$$s_t = [\phi_t, \lambda_t, z_t, \epsilon_{R,t}, \widehat{x}_{t-1}]'$$

- DSGE model parameters:

$$\theta = [\beta, \gamma, \lambda, \pi^*, \zeta_p, \nu, \rho_\phi, \rho_\lambda, \rho_z, \sigma_\phi, \sigma_\lambda, \sigma_z, \sigma_R]'.$$

- Measurement equation:

$$y_t = \Psi_0(\theta) + \Psi_1(\theta) s_t.$$

- State-transition equation:

$$s_t = \Phi_1(\theta) s_{t-1} + \Phi_\epsilon(\theta)\epsilon_t, \quad \epsilon_t = [\epsilon_{\phi,t}, \epsilon_{\lambda,t}, \epsilon_{z,t}, \epsilon_{R,t}]'$$

State-Space Representation of DSGE Model

State-space representation:

$$
\begin{aligned}
y_t &= \Psi_0(\theta) + \Psi_1(\theta) s_t \\
s_t &= \Phi_1(\theta) s_{t-1} + \Phi_\epsilon(\theta)\epsilon_t
\end{aligned}
$$

System matrices:

$$
\Psi_0(\theta) = M_y' \begin{bmatrix} \log\gamma \\ \log(lsh) \\ \log\pi^* \\ \log(\pi^*\gamma/\beta) \end{bmatrix}, \quad x_\phi = -\frac{\kappa_p\psi_p/\beta}{1-\psi_p\rho_\phi}, \quad x_\lambda = -\frac{\kappa_p\psi_p/\beta}{1-\psi_p\rho_\lambda}, \quad x_z = \frac{\rho_z\psi_p}{1-\psi_p\rho_z}, \quad x_{\epsilon_R} = -\psi_p\sigma_R
$$

$$
\Psi_1(\theta) = M_y' \begin{bmatrix} x_\phi & x_\lambda & x_z+1 & x_{\epsilon_R} & -1 \\ 1+(1+\nu)x_\phi & (1+\nu)x_\lambda & (1+\nu)x_z & (1+\nu)x_{\epsilon_R} & 0 \\ \frac{\kappa_p}{1-\beta\rho_\phi}(1+(1+\nu)x_\phi) & \frac{\kappa_p}{1-\beta\rho_\lambda}(1+(1+\nu)x_\lambda) & \frac{\kappa_p}{1-\beta\rho_z}(1+\nu)x_z & +\kappa_p(1+\nu)x_{\epsilon_R} & 0 \\ \frac{\kappa_p/\beta}{1-\beta\rho_\phi}(1+(1+\nu)x_\phi) & \frac{\kappa_p/\beta}{1-\beta\rho_\lambda}(1+(1+\nu)x_\lambda) & \frac{\kappa_p/\beta}{1-\beta\rho_z}(1+\nu)x_z & (\kappa_p(1+\nu)x_{\epsilon_R}/\beta+\sigma_R) & 0 \end{bmatrix}
$$

$$
\Phi_1(\theta) = \begin{bmatrix} \rho_\phi & 0 & 0 & 0 & 0 \\ 0 & \rho_\lambda & 0 & 0 & 0 \\ 0 & 0 & \rho_z & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ x_\phi & x_\lambda & x_z & x_{\epsilon_R} & 0 \end{bmatrix}, \quad \Phi_\epsilon(\theta) = \begin{bmatrix} \sigma_\phi & 0 & 0 & 0 \\ 0 & \sigma_\lambda & 0 & 0 \\ 0 & 0 & \sigma_z & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}
$$

M_y' is an $n_y \times 4$ selection matrix that selects rows of Ψ_0 and Ψ_1.

What is a Local Approximation?

- In a nutshell... consider the backward-looking model

$$y_t = f(y_{t-1}, \sigma\epsilon_t). \tag{1}$$

- Suppose there is a steady state y^* satisfies $y^* = f(y^*, 0)$.
- Guess that the solution to (1) is of the form

$$y_t = y^* + \sigma y_t^{(1)} + o(\sigma). \tag{2}$$

- Taylor series expansion of $f(\cdot)$ around steady state:

$$f(y_{t-1}, \sigma\epsilon_t) = y^* + f_y y_{t-1} + f_\epsilon \sigma\epsilon_t + o(|y_{t-1}|) + o(\sigma)$$

- Now plug-in conjectured solution (2) into (1) using approx of $f(\cdot)$:

$$y^* + \sigma y_t^{(1)} + o(\sigma) = y^* + f_y \sigma y_{t-1}^{(1)} + f_\epsilon \sigma\epsilon_t + o(\sigma)$$

- Deduce that $y_t^{(1)} = f_y y_{t-1}^{(1)} + f_\epsilon \epsilon_t$.

What is a Log-Linear Approximation?

- Consider Cobb-Douglas production function: $Y_t = Z_t K_t^\alpha H_t^{1-\alpha}$.
- Linearization around Y_*, Z_*, K_*, H_*:

$$\begin{aligned} Y_t - Y_* &= K_*^\alpha H_*^{1-\alpha}(Z_t - Z_*) + \alpha Z_* K_*^{\alpha-1} H_*^{1-\alpha}(K_t - K_*) \\ &\quad +(1-\alpha) Z_* K_*^\alpha H_*^{-\alpha}(H_t - K_*) \end{aligned}$$

- Log-linearization: Let $f(x) = f(e^v)$ and linearize with respect to v:

$$f(e^v) \approx f(e^{v_*}) + e^{v_*} f'(e^{v_*})(v - v_*).$$

Thus:

$$f(x) \approx f(x_*) + x_* f'(x_*)(\ln x/x_*) = f(x_*) + f'(x_*)\hat{x}$$

- Cobb-Douglas production function:

$$\tilde{Y}_t = \hat{Z}_t + \alpha \hat{K}_t + (1-\alpha)\hat{H}_t$$

Let's Try the Log-linearizations

- **Euler Equation:**

$$\frac{1}{c_t} = \beta \mathbb{E}_t \left[\frac{1}{c_{t+1}} \frac{1}{\gamma e^{z_{t+1}}} \frac{R_t}{\pi_{t+1}} \right].$$

- **Log-linearized:**

$$-\widehat{c}_t = \mathbb{E}_t \left[-\widehat{c}_{t+1} - z_{t+1} + \widehat{R}_t - \widehat{\pi}_{t+1} \right] \quad \Longrightarrow \quad \widehat{c}_t = \mathbb{E}_t[\widehat{c}_{t+1}] - (\widehat{R}_t - \mathbb{E}[\widehat{\pi}_{t+1}]) + \mathbb{E}_t[z_{t+1}].$$

- **Labor Supply:**

$$\phi_t L_t^{\nu} = \frac{w_t}{c_t}.$$

- **Log-linearized:**

$$\widehat{\phi}_t + \nu \widehat{L}_t = \widehat{w}_t - \widehat{c}_t$$

Let's Try the Log-linearizations

- **Aggregate Resource Constraint:**

$$y_t = \frac{L_t}{D_t}, \quad c_t + \left(1 - \frac{1}{g_t}\right) y_t = y_t \quad \Longrightarrow c_t g_t = y_t.$$

- **Log-linearized:**

$$\widehat{y}_t = \widehat{L}_t - \widehat{D}_t, \quad \widehat{c}_t + \widehat{g}_t = \widehat{y}_t.$$

- **Monetary Policy Rule:**

$$R_t = R_{*,t}^{1-\rho_R} R_{t-1}^{\rho_R} \exp\{\sigma_R \epsilon_{R,t}\}, \quad R_{*,t} = (r\pi_*) \left(\frac{\pi_t}{\pi_*}\right)^{\psi_1} \left(\frac{Y_t}{\gamma Y_{t-1}}\right)^{\psi_2}.$$

- **Log-linearized**

$$\widehat{R}_t = (1 - \rho_R)\widehat{R}_{*,t} + \rho_R \widehat{R}_{t-1} + \sigma_R \epsilon_{R,t}, \quad \widehat{R}_{*,t} = \psi_1 \widehat{\pi}_t + \psi_2 \left[\widehat{y}_t - \widehat{y}_{t-1} + z_t\right].$$

New Keynesian Phillips Curve

- This is fairly complicated... let's focus on the result.
- Assume: $\pi = \bar{\pi} = \pi_*$
- Note that
$$\widehat{mc}_t = \widehat{w}_t = \widehat{lsh}_t.$$
- **Log-linearized:**
$$\widehat{\pi}_t = \beta \mathbb{E}_t[\widehat{\pi}_{t+1}] + \kappa_p(\widehat{lsh}_t + \lambda_t), \quad \kappa_p = \frac{(1-\zeta_p\beta)(1-\zeta_p)}{\zeta_p}.$$
- We also get $\widehat{D}_t = 0$.

Combining Bits and Pieces

- Notation: write x_t instead of y_t for output.
- Assume: $\pi = \bar{\pi} = \pi_*$, $\psi_1 = 1/\beta$, $\psi_2 = 0$, $\rho_R = 0$.
- **Linear rational expectations (LRE) system:**

$$
\begin{aligned}
\widehat{c}_t &= \mathbb{E}_{t+1}[\widehat{c}_{t+1}] - \left(\widehat{R}_t - \mathbb{E}[\widehat{\pi}_{t+1}]\right) + \mathbb{E}_t[z_{t+1}] \\
\widehat{\pi}_t &= \beta \mathbb{E}_t[\widehat{\pi}_{t+1}] + \kappa_p \left(\widehat{lsh}_t + \lambda_t\right) \\
\widehat{R}_t &= \frac{1}{\beta}\widehat{\pi}_t + \sigma_R \epsilon_{R,t} \\
\widehat{lsh}_t &= (1+\nu)\widehat{c}_t + \nu \widehat{g}_t + \phi_t \\
\widehat{x}_t &= \widehat{c}_t + \widehat{g}_t \\
\widehat{g}_t &= \rho_g \widehat{g}_{t-1} + \sigma_g \epsilon_{g,t} \\
\phi_t &= \rho_\phi \phi_{t-1} + \sigma_\phi \epsilon_{\phi,t} \\
\lambda_t &= \rho_\lambda \lambda_{t-1} + \sigma_\lambda \epsilon_{\lambda,t} \\
z_t &= \rho_z z_{t-1} + \sigma_z \epsilon_{z,t}
\end{aligned}
$$

How Can One Solve LRE Systems? A Simple Example

Simple model:

$$y_t = \frac{1}{\theta}\mathbb{E}_t[y_{t+1}] + \epsilon_t, \quad \epsilon_t \sim iid(0,1), \quad \theta \in \Theta = [0,2].$$

- **Method 1:** Introduce conditional expectation $\xi_t = \mathbb{E}_t[y_{t+1}]$ and forecast error $\eta_t = y_t - \xi_{t-1}$:

  $$\xi_t = \theta\xi_{t-1} - \theta\epsilon_t + \theta\eta_t.$$

  Nonexplosive solutions:

  - Determinacy: $\theta > 1$. The only stable solution:

    $$\xi_t = 0, \quad \eta_t = \epsilon_t \quad \Longrightarrow \quad y_t = \epsilon_t$$

  - Indeterminacy: $\theta \leq 1$ the stability requirement imposes no restrictions on forecast error:

    $$\eta_t = \widetilde{M}\epsilon_t + \zeta_t \quad \Longrightarrow y_t = \theta y_{t-1} + \widetilde{M}\epsilon_t + \zeta_t - \theta\epsilon_{t-1}$$

How Can One Solve LRE Systems? A Simple Example

Simple model:

$$y_t = \frac{1}{\theta}\mathbb{E}_t[y_{t+1}] + \epsilon_t, \quad \epsilon_t \sim iid(0,1), \quad \theta \in \Theta = [0,2].$$

- **Method 2:** Construct nonexplosive solutions as follows:
  - Determinacy: $\theta > 1$. Solve equation forward:

$$y_t = \epsilon_t + \frac{1}{\theta}\mathbb{E}_t\left[\frac{1}{\theta}\mathbb{E}_{t+1}[y_{t+2}] + \epsilon_{t+1}\right] = \sum_{s=0}^{\infty}\mathbb{E}_t\left[\left(\frac{1}{\theta}\right)^s \epsilon_{t+s}\right] = \epsilon_t.$$

  - Indeterminacy: $\theta \leq 1$. Express model in terms of $\xi_t = \mathbb{E}_t[y_{t+1}]$ and solve backward (as in previous slide).

How Can One Solve LRE Systems? A Simple Example

Simple model:

$$y_t = \frac{1}{\theta}\mathbb{E}_t[y_{t+1}] + \epsilon_t, \quad \epsilon_t \sim iid(0,1), \quad \theta \in \Theta = [0,2].$$

- **Method 3:** Undetermined coefficients. Guess that $y_t = \gamma_1 y_{t-1} + \gamma_2 \epsilon_t + \gamma_3 \epsilon_{t-1}$. Thus,

$$y_t = \frac{1}{\theta}\mathbb{E}_t\big[\gamma_1 y_t + \gamma_2 \epsilon_{t+1} + \gamma_3 \epsilon_t\big] + \epsilon_t$$

Nonexplosive solutions:

  - Indeterminacy: $\theta \le 1$

$$\begin{aligned} y_t &: \quad \gamma_1 = \gamma_1^2/\theta \quad \Longrightarrow \quad \gamma_1 = 0 \text{ or } \gamma_1 = \theta \\ \epsilon_t &: \quad \gamma_2 \text{ is unrestricted} \\ \epsilon_{t-1} &: \quad 0 = \gamma_3/\theta + 1 \quad \Longrightarrow \quad \gamma_3 = 0 \text{ or } \gamma_3 = -\theta \end{aligned}$$

  - Determinacy: $\theta > 1$. We cannot set $\gamma_1 = \theta$. Thus,

$$\gamma_1 = 0, \quad \gamma_2 = 1, \quad \gamma_3 = 0.$$

More generally...

- Linearized DSGE leads to linear rational expectations (LRE) system.
- Sims (2002) provides solution algorithm for canonical form

$$\Gamma_0(\theta)s_t = \Gamma_1(\theta)s_{t-1} + \Psi\epsilon_t + \Pi\eta_t$$

  where
  - s_t is a vector of model variables, ϵ_t is a vector of exogenous shocks,
  - η_t is a vector of RE errors with elements $\eta_t^x = \hat{x}_t - \mathbb{E}_{t-1}[\hat{x}_t]$, and
  - s_t contains (among others) the conditional expectation terms $\mathbb{E}_t[\widetilde{x}_{t+1}]$.
- Overall the solution in terms of s_t is of the form

$$s_t = \Phi_1(\theta)s_{t-1} + \Phi_\epsilon(\theta)\epsilon_t.$$

- Other solution methods for LREs: Blanchard and Kahn (1980), King and Watson (1998), Uhlig (1999), Anderson (2000), Klein (2000), Christiano (2002).

Solving Our LRE Model

- Assumption: $\psi_2 = 1/\beta$, $\widehat{g}_t = 0$.
- Eliminate nominal interest rate from the consumption Euler equation using policy rule

$$\widehat{x}_t = \mathbb{E}_{t+1}[\widehat{x}_{t+1}] - \left(\frac{1}{\beta}\widehat{\pi}_t + \sigma_R \epsilon_{R,t} - \mathbb{E}[\widehat{\pi}_{t+1}]\right) + \mathbb{E}_t[z_{t+1}].$$

- Rewrite NKPC:

$$\frac{1}{\beta}\widehat{\pi}_t - \mathbb{E}_t[\widehat{\pi}_{t+1}] = \frac{\kappa_p}{\beta}\big((1+\nu)\widehat{x}_t + \phi_t + \lambda_t\big).$$

Solving our LRE Model

Substitute NKPC into consumption Euler equation:

$$\widehat{x}_t = \psi_p \mathbb{E}_t[\widehat{x}_{t+1}] - \frac{\kappa_p \psi_p}{\beta}(\phi_t + \lambda_t) + \psi_p \mathbb{E}_t[z_{t+1}] - \psi_p \sigma_R \epsilon_{R,t},$$

where $0 \leq \psi_p \leq 1$ is given by

$$\psi_p = \left(1 + \frac{\kappa_p}{\beta}(1+\nu)\right)^{-1}.$$

Solving our LRE Model – Output

- **Recall:**

$$\widehat{x}_t = \psi_p \mathbb{E}_t[\widehat{x}_{t+1}] - \frac{\kappa_p \psi_p}{\beta}(\phi_t + \lambda_t) + \psi_p \mathbb{E}_t[z_{t+1}] - \psi_p \sigma_R \epsilon_{R,t},$$

- We now need to find a law of motion for output (and, equivalently, consumption) of the form

$$\widehat{x}_t = \widehat{x}(\phi_t, \lambda_t, z_t, \epsilon_{R,t}) = x_\phi \phi_t + x_\lambda \lambda_t + x_z z_t + x_{\epsilon_R} \epsilon_{R,t}$$

- that solves the functional equation:

$$\begin{aligned} 0 \quad = \quad & \mathbb{E}_t \Bigg[\widehat{x}(\phi_t, \lambda_t, z_t, \epsilon_{R,t}) \\ & -\psi_p \widehat{x}\big(\rho_\phi \phi_t + \sigma_\phi \epsilon_{\phi,t+1}, \rho_\lambda \lambda_t + \sigma_\lambda \epsilon_{\lambda,t+1}, \rho z_t + \sigma_z \epsilon_{z,t+1}, \epsilon_{R,t+1}\big) \\ & + \frac{\kappa_p \psi_p}{\beta}(\phi_t + \lambda_t) - \psi_p z_{t+1} + \psi_p \sigma_R \epsilon_{R,t} \Bigg]. \end{aligned}$$

Solving our LRE Model – Output

- Decision rule for output:

$$\widehat{x}_t = \widehat{x}(\phi_t, \lambda_t, z_t, \epsilon_{R,t}) = x_\phi \phi_t + x_\lambda \lambda_t + x_z z_t + x_{\epsilon_R} \epsilon_{R,t}$$

- where

$$x_\phi = -\frac{\kappa_p \psi_p / \beta}{1 - \psi_p \rho_\phi}, \quad x_\lambda = -\frac{\kappa_p \psi_p / \beta}{1 - \psi_p \rho_\lambda}, \quad x_z = \frac{\rho_z \psi_p}{1 - \psi_p \rho_z} z_t, \quad x_{\epsilon_R} = -\psi_p \sigma_R.$$

Solving our LRE Model – Labor Share

- **Recall:** $\widehat{lsh}_t = (1+\nu)\widehat{x}_t + \phi_t$.
- Deduce

$$\widehat{lsh}_t = \left[1 + (1+\nu)x_\phi\right]\phi_t + (1+\nu)x_\lambda \lambda_t + (1+\nu)x_z z_t + (1+\nu)x_{\epsilon_R}\epsilon_{R,t}.$$

Solving our LRE Model – Inflation

The NKPC yields the following functional equation:

$$\begin{aligned} 0 &= \mathbb{E}_t\Big[\widehat{\pi}\big(\phi_t,\lambda_t,z_t,\epsilon_{R,t}\big) - \beta\widehat{\pi}\big(\rho_\phi\phi_t+\sigma_\phi\epsilon_{\phi,t+1},\rho_\lambda\lambda_t+\sigma_\lambda\epsilon_{\lambda,t+1},\rho z_t+\sigma_z\epsilon_{z,t+1},\epsilon_{R,t+1}\big) \\ &\quad -\kappa_p\widehat{lsh}\big(\phi_t,\lambda_t,z_t,\epsilon_{R,t}\big)-\kappa_p\lambda_t\Big], \end{aligned}$$

where $\widehat{lsh}(\cdot)$ was given on previous slide.

The solution takes the form

$$\begin{aligned} \widehat{\pi}_t &= \frac{\kappa_p}{1-\beta\rho_\phi}\big[1+(1+\nu)x_\phi\big]\phi_t + \frac{\kappa_p}{1-\beta\rho_\lambda}\big[1+(1+\nu)x_\lambda\big]\lambda_t \\ &\quad + \frac{\kappa_p}{(1-\beta\rho_z)}(1+\nu)x_z z_t + \kappa_p(1+\nu)x_{\epsilon_R}\epsilon_{R,t}. \end{aligned}$$

Solving our LRE Model – Interest Rates

Combining the decision rule for inflation with the monetary policy rule yields

$$\begin{aligned}\widehat{R}_t &= \frac{\kappa_p/\beta}{1-\beta\rho_\phi}\big[1+(1+\nu)x_\phi\big]\phi_t + \frac{\kappa_p/\beta}{1-\beta\rho_\lambda}\big[1+(1+\nu)x_\lambda\big]\lambda_t \\ &\quad + \frac{\kappa_p/\beta}{1-\beta\rho_z}(1+\nu)x_z z_t + \big[\kappa_p(1+\nu)x_{\epsilon_R}/\beta + \sigma_R\big]\epsilon_{R,t}.\end{aligned}$$

Linking Model Variables and Observables

- To confront the model with data, one has to account for the presence of the model-implied stochastic trend in aggregate output and to add the steady states to all model variables.
- Measurement equations:

$$
\begin{aligned}
\log(X_t/X_{t-1}) &= \widehat{x}_t - \widehat{x}_{t-1} + z_t + \log\gamma \\
\log(lsh_t) &= \widehat{lsh}_t + \log(lsh) \\
\log\pi_t &= \widehat{\pi}_t + \log\pi^* \\
\log R_t &= \widehat{R}_t + \log(\pi^*\gamma/\beta).
\end{aligned}
$$

State-space Representation of DSGE Model

- $n_y \times 1$ vector of observables:

$$y_t = M_y'[\log(X_t/X_{t-1}), \log lsh_t, \log \pi_t, \log R_t]'.$$

- $n_s \times 1$ vector of econometric state variables s_t

$$s_t = [\phi_t, \lambda_t, z_t, \epsilon_{R,t}, \widehat{x}_{t-1}]'$$

- DSGE model parameters:

$$\theta = [\beta, \gamma, \lambda, \pi^*, \zeta_p, \nu, \rho_\phi, \rho_\lambda, \rho_z, \sigma_\phi, \sigma_\lambda, \sigma_z, \sigma_R]'.$$

- Measurement equation:

$$y_t = \Psi_0(\theta) + \Psi_1(\theta)s_t.$$

- State-transition equation:

$$s_t = \Phi_1(\theta)s_{t-1} + \Phi_\epsilon(\theta)\epsilon_t, \quad \epsilon_t = [\epsilon_{\phi,t}, \epsilon_{\lambda,t}, \epsilon_{z,t}, \epsilon_{R,t}]'$$

State-Space Representation of DSGE Model

State-space representation:

$$
\begin{aligned}
y_t &= \Psi_0(\theta) + \Psi_1(\theta) s_t \\
s_t &= \Phi_1(\theta) s_{t-1} + \Phi_\epsilon(\theta)\epsilon_t
\end{aligned}
$$

System matrices:

$$
\Psi_0(\theta) = M_y' \begin{bmatrix} \log\gamma \\ \log(lsh) \\ \log\pi^* \\ \log(\pi^*\gamma/\beta) \end{bmatrix}, \quad x_\phi = -\frac{\kappa_p\psi_p/\beta}{1-\psi_p\rho_\phi}, \quad x_\lambda = -\frac{\kappa_p\psi_p/\beta}{1-\psi_p\rho_\lambda}, \quad x_z = \frac{\rho_z\psi_p}{1-\psi_p\rho_z}, \quad x_{\epsilon_R} = -\psi_p\sigma_R
$$

$$
\Psi_1(\theta) = M_y' \begin{bmatrix}
x_\phi & x_\lambda & x_z + 1 & x_{\epsilon_R} & -1 \\
1 + (1+\nu)x_\phi & (1+\nu)x_\lambda & (1+\nu)x_z & (1+\nu)x_{\epsilon_R} & 0 \\
\frac{\kappa_p}{1-\beta\rho_\phi}(1+(1+\nu)x_\phi) & \frac{\kappa_p}{1-\beta\rho_\lambda}(1+(1+\nu)x_\lambda) & \frac{\kappa_p}{1-\beta\rho_z}(1+\nu)x_z & +\kappa_p(1+\nu)x_{\epsilon_R} & 0 \\
\frac{\kappa_p/\beta}{1-\beta\rho_\phi}(1+(1+\nu)x_\phi) & \frac{\kappa_p/\beta}{1-\beta\rho_\lambda}(1+(1+\nu)x_\lambda) & \frac{\kappa_p/\beta}{1-\beta\rho_z}(1+\nu)x_z & (\kappa_p(1+\nu)x_{\epsilon_R}/\beta + \sigma_R) & 0
\end{bmatrix}
$$

$$
\Phi_1(\theta) = \begin{bmatrix}
\rho_\phi & 0 & 0 & 0 & 0 \\
0 & \rho_\lambda & 0 & 0 & 0 \\
0 & 0 & \rho_z & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
x_\phi & x_\lambda & x_z & x_{\epsilon_R} & 0
\end{bmatrix}, \quad \Phi_\epsilon(\theta) = \begin{bmatrix}
\sigma_\phi & 0 & 0 & 0 \\
0 & \sigma_\lambda & 0 & 0 \\
0 & 0 & \sigma_z & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix}
$$

M_y' is an $n_y \times 4$ selection matrix that selects rows of Ψ_0 and Ψ_1.

DSGE Model Implications

- We want to understand the implications of the DSGE model.
- We could simulate data from the state-space representation

$$\begin{aligned} y_t &= \Psi_0(\theta) + \Psi_1(\theta) s_t \\ s_t &= \Phi_1(\theta) s_{t-1} + \Phi_\epsilon(\theta)\epsilon_t \end{aligned}$$

- using:

$$\epsilon_t \sim iidN(0, I).$$

- But some calculations are better done analytically.

DSGE Model Implications: Questions

- What is the correlation between consumption growth this quarter and four quarters ago?
- What is the correlation between inflation and interest rates?
- Does the labor share predict consumption growth one-year ahead?

DSGE Model Implications: Fix Parameters

Parameter	Value	Parameter	Value
β	$1/1.01$	γ	$\exp(0.005)$
λ	0.15	π^*	$\exp(0.005)$
ζ_p	0.65	ν	0
ρ_ϕ	0.94	ρ_λ	0.88
ρ_z	0.13		
σ_ϕ	0.01	σ_λ	0.01
σ_z	0.01	σ_R	0.01

DSGE Model Implications: Autocovariances

- State-space representation:

$$\begin{aligned} y_t &= \Psi_0(\theta) + \Psi_1(\theta) s_t \\ s_t &= \Phi_1(\theta) s_{t-1} + \Phi_\epsilon(\theta)\epsilon_t \end{aligned}$$

- Notation:

$$\Gamma_{yy}(h) = \mathbb{E}[y_t y_{t-h}], \quad \Gamma_{ss}(h) = \mathbb{E}[s_t s_{t-h}], \quad \text{and} \quad \Gamma_{ys}(h) = \mathbb{E}[y_t s'_{t-h}]$$

- Covariance matrix of s_t is solution to Lyapunov equation:

$$\Gamma_{ss}(0) = \Phi_1 \Gamma_{ss}(0) \Phi_1' + \Phi_\epsilon \Phi_\epsilon'.$$

- Autocovariance matrices for $h \neq 0$:

$$\Gamma_{ss}(h) = \Phi_1^h \Gamma_{ss}(0).$$

- Using the measurement equation, we deduce that

$$\Gamma_{yy}(h) = \Psi_1 \Gamma_{ss}(h) \Psi_1', \quad \Gamma_{ys}(h) = \Psi_1 \Gamma_{ss}(h).$$

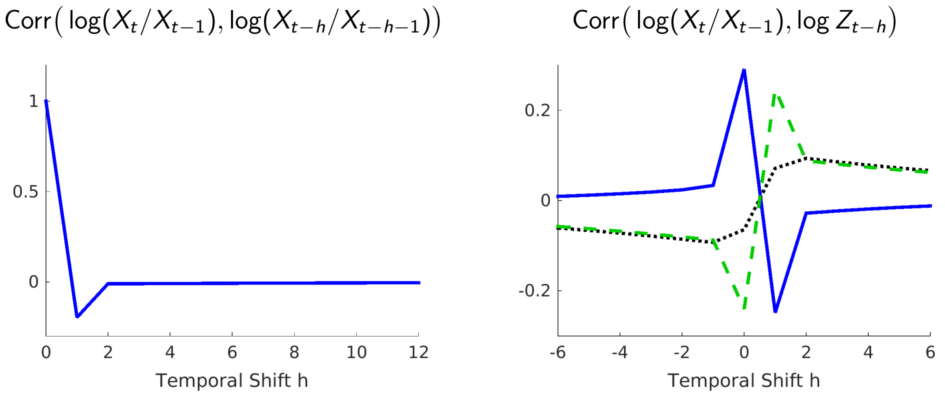
DSGE Model Implications: Autocovariances

1. Fix a set of DSGE model parameters.
2. Solve model and compute matrices $\Psi_0(\theta)$, $\Psi_1(\theta)$, $\Phi_1(\theta)$, $\Phi_\epsilon(\theta)$ in state-space representation:

$$\begin{aligned} y_t &= \Psi_0(\theta) + \Psi_1(\theta)s_t \\ s_t &= \Phi_1(\theta)s_{t-1} + \Phi_\epsilon(\theta)\epsilon_t \end{aligned}$$

3. Compute autocovariances based on $\Psi_0(\theta)$, $\Psi_1(\theta)$, $\Phi_1(\theta)$, $\Phi_\epsilon(\theta)$.

Compute Autocovariances and Plot



Notes: Right panel: correlations of output growth with labor share (solid), inflation (dotted), and interest rates (dashed).

DSGE Model Implications: Forecast Error Variance Decomposition

- Fluctuations in the model are driven by shocks:
  - technology growth z_t;
  - mark-up λ_t;
  - preference ϕ_t;
  - monetary policy $\epsilon_{R,t}$;
  - government spending $\widehat{g}_t$.
- Shocks generate uncertainty about future macroeconomic outcomes
- Question: what is the contribution of monetary policy shocks to forecast errors in inflation rates?

DSGE Model Implications: Forecast Error Variance Decomposition

- The law of motion for s_t can be expressed as the infinite-order vector moving average (MA)

$$y_t = \Psi_0 + \Psi_1 \sum_{s=0}^{\infty} \Phi_1^s \Phi_\epsilon \epsilon_{t-s}.$$

- h-step-ahead forecast error is

$$e_{t|t-h} = y_t - \mathbb{E}_{t-h}[y_t] = \Psi_1 \sum_{s=0}^{h-1} \Phi_1^s \Phi_\epsilon \epsilon_{t-s}.$$

- h-step-ahead forecast error covariance matrix is

$$\mathbb{E}[e_{t|t-h} e'_{t|t-h}] = \Psi_1 \left(\sum_{s=0}^{h-1} \Phi_1^s \Phi_\epsilon \Phi'_\epsilon \Phi_1^{s'} \right) \Psi'_1 \quad \text{with} \quad \lim_{h \longrightarrow \infty} \mathbb{E}[e_{t|t-h} e'_{t|t-h}] = \Gamma_{ss}(0).$$

DSGE Model Implications: Forecast Error Variance Decomposition

- Recall $\mathbb{E}[\epsilon_t \epsilon_t'] = I$. Let $I^{(j)}$ be defined by setting all but the j-th diagonal element of the identity matrix I to zero:

$$I = \sum_{j=1}^{n_\epsilon} I^{(j)}.$$

- Express the contribution of shock j to the forecast error for y_t as

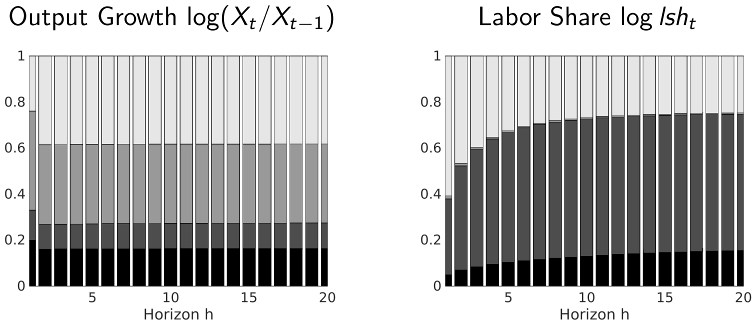
$$e_{t|t-h}^{(j)} = \Psi_1 \sum_{s=0}^{h-1} \Phi_1^s \Phi_\epsilon I^{(j)} \epsilon_{t-s}.$$

- The contribution of shock j to the forecast error variance of observation $y_{i,t}$ is

$$\mathrm{FEVD}(i,j,h) = \frac{\left[\Psi_1 \left(\sum_{s=0}^{h-1} \Phi_1^s \Phi_\epsilon I^{(j)} \Phi_\epsilon' \Phi_1^{s'}\right) \Psi_1'\right]_{ii}}{\left[\Psi_1 \left(\sum_{s=0}^{h-1} \Phi_1^s \Phi_\epsilon \Phi_\epsilon' \Phi_1^{s'}\right) \Psi_1'\right]_{ii}},$$

where $[A]_{ij}$ denotes element (i,j) of a matrix A.

Compute Forecast Error Variance Decomposition and Plot



Notes: The stacked bar plots represent the cumulative forecast error variance decomposition. The bars, from darkest to lightest, represent the contributions of ϕ_t, λ_t, z_t, and $\epsilon_{R,t}$.

DSGE Model Implications: Impulse Response Functions

- What is the dynamic effect of a 25 basis point unanticipated reduction in the interest rate?
- What is the effect of an unanticipated increase in technology growth?

DSGE Model Implications: Impulse Response Functions

- **Definition:**

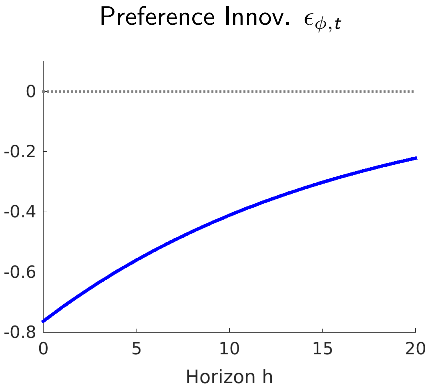
$$\text{IRF}(i,j,h|s_{t-1}) = \mathbb{E}\big[y_{i,t+h} \,\big|\, s_{t-1}, \epsilon_{j,t} = 1\big] - \mathbb{E}\big[y_{i,t+h} \,\big|\, s_{t-1}\big].$$

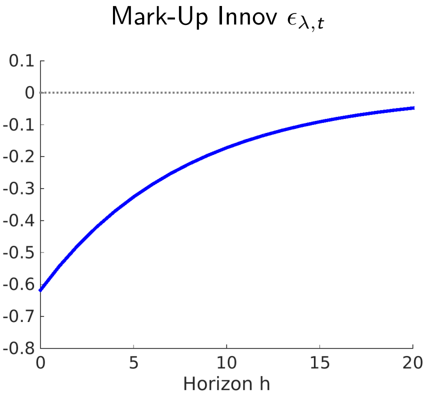
- Both expectations are conditional on the initial state s_{t-1} and integrate over current and future realizations of the shocks ϵ_t.
- First term also conditions on $\epsilon_{j,t} = 1$, whereas the second term averages of $\epsilon_{j,t}$.
- In a linearized DSGE model we have $\mathbb{E}[\epsilon_{t+h}|s_{t-1}] = 0$ for $h = 0, 1, \ldots$ and deduce

$$\text{IRF}(.,j,h) = \Psi_1 \frac{\partial}{\partial \epsilon_{j,t}} s_{t+h} = \Psi_1 \Phi_1^h [\Phi_\epsilon]_{.j},$$

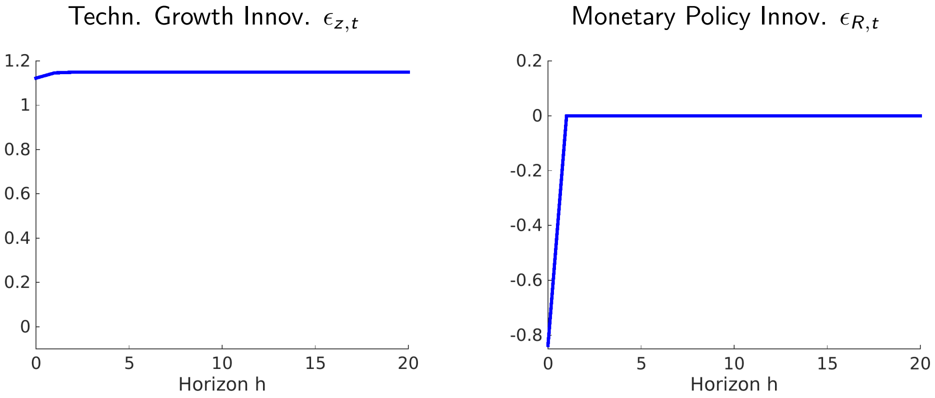
where $[A]_{.j}$ is the j-th column of a matrix A. We dropped s_{t-1} from the conditioning set to simplify the notation.

Compute and Plot Impulse Responses of Log Output $100 \log(X_{t+h}/X_t)$





Compute and Plot Impulse Responses of Log Output $100\log(X_{t+h}/X_t)$



DSGE Model Implications

- Formulas for autocovariance functions, spectra, and impulse response functions for a linearized DSGE model can be derived analytically.
- Analytical expressions can then be numerically evaluated for different vectors of parameter values θ.
- For general nonlinear general nonlinear DSGE model, the implied moments have to be computed using Monte Carlo simulation.
- Let $Y^*_{1:T}$ denote a sequence of observations simulated from the state-space representation of the DSGE model by drawing an initial state vector s_0 and innovations ϵ_t from their model-implied distributions, then

$$\frac{1}{T}\sum_{t=1}^{T} y_t^* \xrightarrow{a.s.} \mathbb{E}[y_t].$$

- The downside of Monte Carlo approximations is that they are associated with a simulation error.