Why Is an Empty Shampoo Bottle So Easy to Knock Over?

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It becomes dramatically and annoyingly easier to knock over a bottle as the last of its contents are removed. This is an easily-observed and curiosity-provoking phenomenon which yields insights into center of mass and impacts.

INTRODUCTION

An object placed on a tilted surface and released will tip over if the horizontal position of the center of mass lies outside the geometric outline of the base of the object. This means that an object with a low center of mass will be stable for larger incline angles. Students are often surprised to see that the altitude of the center of mass of a shampoo bottle is a non-linear function of the fraction occupied by the contents. This can, however, be seen in a straightforward manner by recognizing that for a plastic bottle, the mass of the liquid contents is usually far greater than the mass of the bottle. When the bottle is either completely full or completely empty, the center of mass must be approximately at the geometric center of the bottle, but the center of mass gets significantly lower in altitude for small liquid levels in between those two extremes. Pahwa et al. have recently published an extremely nice and comprehensive analysis of the center of mass of the system of bottle and liquid. In this paper, we simply address the subsequent question of sensitivity to impacts. Not only does the center of mass get dramatically higher as the last bit of
liquid is removed, but due to the decreasing mass, the kinetic energy required for an impact to tip the bottle becomes drastically smaller.

CALCULATIONS AND RESULTS

We start with a considerably simpler model than Pahwa et al. The empty shampoo bottle chosen had a mass of \( m_B = 42 \) g, height 21 cm, and center of mass altitude of \( y_B = 13 \) cm as determined by the balance point when resting on its side.\(^2\) (The center of mass of the empty bottle is actually above the geometric midpoint due to the higher mass of the complicated cap.) The shampoo liquid has a mass of \( m_L = 379 \) g and liquid level of 17 cm when the bottle is completely full, giving the liquid a center of mass altitude of \( y_L = 8.5 \) cm when 100% full. We make the simplifying assumptions that the bottle has a uniform cross section and that the liquid has a high enough viscosity so that it does not move appreciably during the brief time period of impact and tipping. We introduce the variable \( x \) ranging from zero to one to indicate the fraction of the device filled and then derive the equation for the altitude of the center of mass as

\[
y_{CM} = \frac{m_B y_B + (xm_L)(xy_L)}{m_B + xm_L}
\]

This relationship is plotted in Figure 1a, which shows the nonlinear features described above. The liquid level \( x \) that yields the minimum altitude of the center of mass for this bottle was 31.5% and will be referred to for convenience as “maximum stability” (for tipping on an incline). We introduce the symbol \( a \) to indicate half the width of the base, which we measure as \((4.2 \text{ cm})/2 = 2.1 \text{ cm}\). Then the angle of incline required to tip is given by \( \arctan(a/y_{CM}) \) and is plotted in Figure 1b.

Our work considers a bottle on a horizontal surface that receives an idealized instantaneous impact that gives the bottle a kinetic energy of rotation around the point on the
base of the bottle on the far side from the impact. The criteria for whether the bottle will return to the starting position or tip over is whether this kinetic energy is greater or smaller than the increase in potential energy required to raise the center of mass directly above that support point. This occurs at the tipping angle as displayed in Fig. 1b, requiring a potential energy increase of only $mg\left\{\sqrt{\left(y_{CM}^2 + a^2\right)} - y_{CM}\right\}$ as plotted in Fig. 1c.

As the liquid level in the bottle drops from the maximum stability level to completely empty, the tipping angle decreases to 43% of its value at maximum stability (from 21.8° to 9.4°). Over the same range, however, the potential energy required to tip has an even more dramatic decrease to only 11% of its value at maximum stability (from 6.6 mJ to 0.7 mJ). An empty bottle is predicted to have a potential energy required to tip of only 7% that of a full bottle (0.73 mJ compared to 10.5 mJ).

We constructed a demonstration to produce impacts from a pendulum bob. A tennis ball of mass 57.7 g striking the bottle at an altitude of 13.7 cm turned out to be a good match of mechanical impedance and center of percussion. A string was tied to the tennis ball so its center would travel in an arc of radius 37 cm, and a protractor was mounted to determine the release angle of the pendulum, with 0 degrees indicating the point of impact when the string was vertical. (Fig. 1d) The launch angles required for the impact to tip the bottle could be determined to ±0.5° and were approximately 20.5° for a full bottle, 12.0° for the maximum stability level, and only 3.0° for the completely empty bottle. The initial potential energy of the tennis ball at these angles is given by $mgl(1 - \cos(\theta))$ and yields corresponding values of 13.2 mJ, 4.6 mJ, and 0.3 mJ, illustrating the drastic decrease of energy required to tip the empty bottle. (The trend relative to the corresponding predicted values of 10.5 mJ, 6.6 mJ, and 0.7 mJ is
quite satisfying considering the large change of collision conditions as the mass of the bottle and
the altitude of its center of mass change with liquid level.)

A video of this demonstration is available.³
FIG. 1. a) Upper graph shows the non-linear behavior of the altitude of the center of mass. b) Center graph shows the angle required to tip. c) Lower graph shows drastic decrease in energy to tip. d) Impact demonstration apparatus.

2 L’Oréal Paris Elvive Color Vibrancy Protecting Shampoo, 375 ml bottle.

3 https://www.lehigh.edu/~jcl3/ShampooBottle