AN ALTERNATIVE APPROACH TO QUANTUM MECHANICS BASED ON FIVE DIMENSIONS

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Abstract: In the absence of a quantized version of four-dimensional general relativity, a non-compact fifth dimension is used which leads to a semi-classical version of relativistic wave mechanics. The extra coordinate oscillates around a hypersurface where the energy density of the vacuum diverges. Waves and particles in spacetime move on paths in five dimensions which are *null*. Wave-particle duality, quantization and uncertainty follow naturally. The wish to find a more logical explanation for these things motivates the present study. It should therefore be seen not as a replacement for the standard theory but as an alternative approach.

Key Words: Wave-Particle Duality, Quantization, Uncertainty, Vacuum, Higher Dimension.

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1. Introduction

Assuming it is in fact possible to unify gravitation with the interactions of particles, the preferred way has historically been to quantize general relativity. Another approach, which is not exclusive of the former, is to add dimensions to Einstein's theory in the hope of accounting for the symmetry groups of elementary particles. The prototype for this kind of theory was due to Kaluza, who in 1921 used a fifth dimension to unify Einstein's equations for gravity and Maxwell's equations for electromagnetism. That approach was later modified by Klein, who in 1926 took the extra dimension to be compact as a means of quantizing the electron charge. The opinion today appears to be that an acceptable quantum version of general relativity cannot be attained, and that Kaluza-Klein theory (at least in its original form) is too simplistic. In the present work an alternative approach is suggested, wherein the fifth dimension is used in a novel way to account for quantum phenomena.

Before proceeding to this analysis, it is instructive to recall a few relevant facts from previous studies. Thus in 1928, Robertson found that the curved metrics of certain standard solutions of four-dimensional general relativity could be embedded in *flat* five-dimensional manifolds [1]. Then in 1935, Dirac showed that it was possible to reformulate the properties of elementary particles in terms of an embedding of de Sitter space in 5D [2]. Finally, in 1938 Einstein and Bergmann drew attention to oscillating solutions and averred "we ascribe physical reality to the fifth dimension" [3]. These three facts were apparently overlooked by the authors of more modern works. For example, it was in ignorance that Wesson and coworkers in 1992 showed the utility of five dimensions in explaining things not covered by conventional theory, such as the origin of matter and the nature of the big bang [4]. In subsequent years, other workers established several significant results, recognizing that the 5D canonical metric provided an embedding for all vacuum solutions of Einstein's 4D field equations and recovering an old (and likewise forgotten) theorem of Campbell that ensured the embedding in 5D of all of general relativity [5]. The emergent formalism became known as Space-Time-Matter theory, though its mathematical structure and certain physical results were shared by Membrane theory [6-8]. Irrespective of name, it is now acknowledged that the modern form of Kaluza-Klein theory, albeit with a fifth dimension that is *not* compact, is based on the original ideas of Robertson, Dirac and Einstein.

Both STM theory and M theory are in agreement with observation, though certain topics such as inflationary cosmology and the nature of elementary particles remain controversial [9.10]. The application of 5D relativity to cosmology is by now fairly well understood, because it goes over to 4D general relativity in the appropriate limit. The same cannot be said of the implications of the fifth dimension for quantum mechanics, because the latter subject even in 4D is riddled with quandries [11-15]. Notwithstanding this, a consistent account of 5D quantum mechanics has appeared in the last few years. It cures many of the conundrums of the conventional theory, while offering some new tests. Axiomatically, the theory involves three things which are different from general relativity:

(A) Covariance in *five* dimensions. (The group of coordinate transformations in 5D is broader than in 4D, so a change which includes the extra coordinate may alter the form of 4D relations, a process sometimes called transformity.)

(B) Geodesics are null in *five* dimensions. (It is known that a path which is null in 5D can correspond to the path not only of a photon but also a massive particle in 4D.)

(C) The fifth coordinate can be spacelike *or* timelike. (Good solutions of the field equations exist with both signs for the extra metric coefficient, but the extra coordinate does not have the physical nature of a time, so there is no problem with closed timelike paths.)

The Weak Equivalence Principle is not taken as a postulate of the 5D theory. Rather, it is recovered as a symmetry, when the metric is independent of the extra coordinate and the acceleration of a test particle has no anomalous component and is due solely to the mass of the source. Gravitational mass is combined with Newton's constant G and the speed of light c in the potentials, usually as the Schwarzschild factor GM / c^2. Inertial mass is combined with Planck's constant h and c in the equations of motion, usually as the Compton wavelength h / mc. The use of the universal constants in this way produces lengths which allow both types of mass to be geometrized. Unlike other versions of 5D relativity, there is no direct link between the extra coordinate and the mass of a particle, and in non-cosmological situations m is a constant as in standard quantum theory.

The nomenclature used below is standard, though extended to 5D. Thus the coordinates are $x^A = x^0$ (time), x^{123} (space) and $x^4 = l$, where the last is chosen to avoid confusion with the Cartesian measure and the usage in *M* theory. The field equations in 5D are commonly taken in terms of the Ricci tensor to be $R_{AB} = 0$ (A,B = = 0 – 4). These by the aforementioned embedding theorem of Campbell actually contain Einstein's 4D field equations, with an induced or effective energy-momentum tensor that depends on the extra dimension, meaning that matter has a geometrical origin [4]. The source will also in general include a vacuum term, with magnitude measured by the cosmological constant Λ. The equation of state is the same as the Einstein vacuum, with pressure and energy density given by $p_v = -\rho_v c^2 = -\Lambda c^4 / 8\pi G$. However, it will be seen later that in application to particle physics the cosmological 'constant' can actually vary with the extra coordinate, $\Lambda = \Lambda(l)$. The precise form of this, and other results noted below, are derived in the literature [5]. It will transpire that the behavior of $\Lambda(l)$ is crucial for the stability of particles and their quantization.

2. A 5D Theory of Quantum Mechanics

In this section, the aim is to construct a new model for quantum mechanics by considering dynamics not in 4D but in 5D. The starting postulate is that particles exist in the (energetic) vacuum.

The most general 5D metric which contains all solutions of the 4D Einstein equations with vacuum energy has the so-called canonical form:

$$dS^2 = \left(\frac{l - l_o}{L}\right)^2 ds^2 + \varepsilon dl^2 \quad . \tag{1}$$

Here $x^4 = l$ is the extra coordinate, l_0 and L are constants with the physical dimensions of lengths, and $ds^2 = g_{\alpha\beta}\left(x^\gamma\right)dx^\alpha dx^\beta$ is the 4D interval for a vacuum solution of Einstein's equations. The indicator $\varepsilon = \pm 1$ denotes whether the extra coordinate is spacelike ($\varepsilon = -1$) or timelike ($\varepsilon = +1$). The metric (1) may be viewed as an embedding where 4D spacetime is a kind of spherical surface in 5D, but with the centre displaced from $l = 0$ to $l = l_0$ (this can be better visualized by replacing l by a radius measure). The displacement or shift $l \to \left(l - l_0\right)$ results in a striking form for the energy density of the vacuum as measured by the cosmological constant:

$$\Lambda = -\frac{3\varepsilon}{L^2}\left(\frac{l}{l - l_0}\right)^2 \quad . \tag{2}$$

This is the value of Λ measured by an observer intrinsic to the spacetime of (1). It is positive or negative depending on whether the extra coordinate is spacelike or timelike, and diverges at $l = l_0$. The behaviour of $\Lambda\left(l\right)$ is shown in Fig. 1.

Null geodesics of the metric (1) define paths in the 5D manifold and allow of the definition of a 4D wave function. Geodesics in general for 5D metrics may be found by the usual variational method, using $\delta\left[\int dS\right] = 0$ around the null path [5]. However, for the canonical form the null geodesics may be found directly by putting $dS^2 = 0$ into (1).

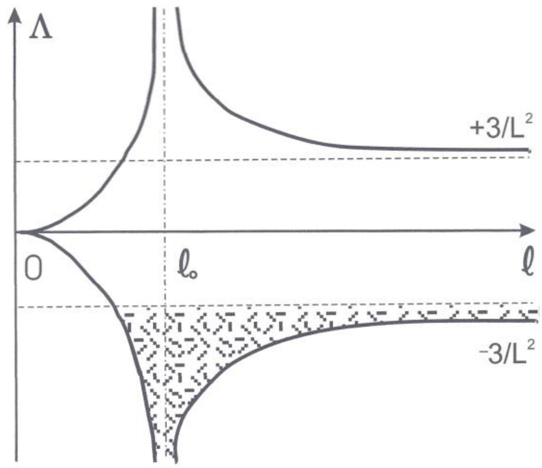

Fig. 1. The behaviour of the cosmological 'constant' as a function of the extra coordinate, according to equation (2).

Doing this, and integrating with the introduction of an arbitrary constant l_*, gives the relation $(l-l_0)/l_* = \exp[\pm(i)s/L]$. The latter is, of course, just the 4D wave function ψ. The choice of signs $(\pm)$ arises from taking the square root of terms in (1), and reflects the reversibility of the motion in the extra dimension. The (i) is shorthand for the fact that $i=\sqrt{-1}$ is or is not present, depending on whether ε =+1 or ε = - 1 in (1). These choices will later be dropped for ease of nomenclature, but for now they are kept to indicate that $l(s)$ plays the role of the wave function and is either monotonic or oscillatory. Thus the null geodesic $(dS^2 = 0)$ in (1) is equivalent to

$$l = l_0 + l_* \exp[\pm(i)s/L] \quad . \tag{3}$$

For the monotonic mode, a particle in the extra dimension either approaches or recedes from l_0 as a function of the 4D proper time s. For the oscillatory mode, the wave has locus l_0, amplitude l_* and wavelength L. The behaviour of $l(s)$ is shown in Fig. 2.

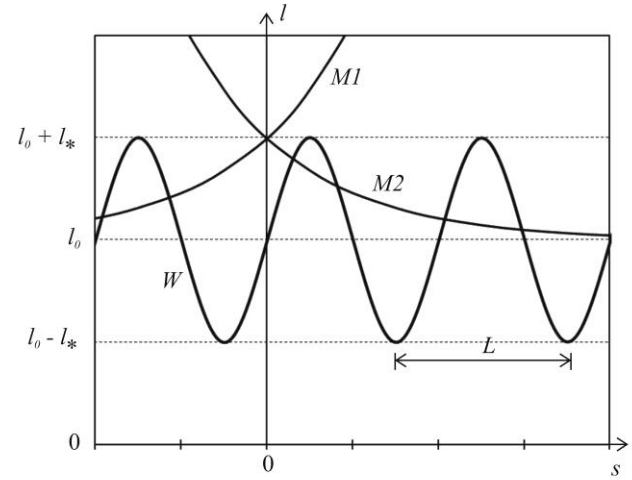

Fig. 2. The behaviour of the extra coordinate as a function of the 4D proper time, according to equation (3). The wave W is for a timelike extra dimension, while the monotonic lines M1, M2 illustrate the two allowed behaviours for a spacelike extra dimension. [W is given by $l = l_0 + l_* \exp(is/L)$, M1 by $l = l_0 + l_* \exp(s/L)$ and M2 by $l = l_0 + l_* \exp(-s/L)$.]

Wave-particle duality is implied in the above description. All that is needed is to identify the monotonic / spacelike mode $(\varepsilon = -1,\ \Lambda > 0)$ with the path of a particle, and the oscillatory / timelike mode ($\varepsilon = +1$, $\Lambda < 0$) with the associated wave. This interpretation is confirmed by setting $\psi = l(s)$ for the wave mode of (3), and substituting into the extra component of the 5D geodesic equation (derived from the variational principle mentioned above). The result is

$$\Box_4^2 \psi + (mc/h)^2 \psi = 0 \quad , \tag{4}$$

the Klein-Gordon equation. (Here $\Box_4^2 \psi \equiv g^{\alpha\beta} \psi_{,\alpha;\beta}$ where the comma denotes the partial derivative and the semicolon denotes the covariant or curved-space derivative). The length L in (3) has been identified as the Compton wavelength $L = h / mc$ in (4). The phase factor in (3) may as usual be split into time and space components via $s/L = (Et + p_i x^i)/h$ where E is the energy and p_i are the components of the linear momentum ($i = 1 - 3$). Alternatively, the 4-momenta can be defined via $p^\alpha \equiv mcu^\alpha$ where $u^\alpha \equiv dx^\alpha / ds$ are the 4-velocities $(\alpha = 0 - 3)$. Then the momenta are related to the wave function of (4) by $p_\alpha = (h/i\psi)(\partial\psi/\partial x^\alpha)$. This relation with (4) forms the basis of wave mechanics as developed in the 1920s by de Broglie. The latter subject lacks the sophistication of modern quantum field theory, but it is interesting to note that wave mechanics is equivalent to the wave-particle interpretation of (3) as based on the canonical metric (1). Indeed, when the 4D vacuum solution which is embedded in the 5D metric (1) is chosen to be of de Sitter type [5], the result is an oscillation in ordinary 3D space with properties identical to those of a de Broglie wave.

Local spacetime around a wave-particle pair is close to flat. To see this, consider a more general metric than (1) and the value of the Ricci or curvature scalar R [4]. It is instructive to split the 5D line element into a 4D part and an extra part involving a scalar field Φ, thus:

$$dS^2 = g_{\alpha\beta}\left(x^\gamma, l\right)dx^\alpha dx^\beta + \varepsilon\Phi^2\left(x^\gamma, l\right)dl^2 \tag{5.1}$$

$$R = \frac{\varepsilon}{4\Phi^2}\left[g^{\mu\nu}_{,4}g_{\mu\nu,4} + \left(g^{\mu\nu}g_{\mu\nu,4}\right)^2\right] \quad . \tag{5.2}$$

Clearly, the wave $(\varepsilon = +1)$ and the particle $(\varepsilon = -1)$ considered above have opposite curvatures, so the combination will not produce any significant departure from flatness. This agrees with observations, and helps resolve the cosmological-constant problem, wherein models of particles have unacceptably intense vacuum fields. There are several ways to model the internal structure of particles using 5D theory [5, 10]. It is possible that while $|g_{44}| \approx 1$, the scalar field is complex, with real and imaginary modes that lead to an effective change of signature $(\varepsilon = \pm 1)$. Generally, complex metric coefficients in 5D metrics are acceptable, provided the observable quantities calculated from the field equations are real. This applies to Λ of (2) and R of (5.2). For 4D metrics which are non-flat, it is often convenient to use the Gaussian curvature in place of the Ricci curvature, where the two are related by $K = -R/12$. Surfaces with $K < 0$ are open, while those with $K > 0$ are closed. The radius of curvature is just $|K|^{-1/2}$, and this should be moderate for any model that is to agree with observation.

The hypersurface $l = l_0$ has some interesting properties. By equation (2) or Fig. 1, the cosmological 'constant' diverges there for both the monotonic $(\varepsilon = -1)$ and oscillatory $(\varepsilon = +1)$ paths. But by (3) or Fig. 2, the monotonic paths are only in the neighbourhood of l_0 asymptotically $(s \to \pm\infty)$ and do not cross it. While contrariwise, the oscillatory paths have locus l_0 and cross it freely. This is different from the membrane of *M* theory, which is singular. Also, that membrane is symmetric, whereas in the present model $\Lambda(l)$ is asymmetric about $l = l_0$. Despite these differences, the surface at l_0 may for convenience be called a membrane. By (3), when the amplitude $l_* \to 0$ the wave's envelope shrinks to the line $l = l_0 =$constant. This suggests that the hypersurface which defines spacetime should be taken as $l = l_0$. However, as noted above, the particle-like paths are not confined to l_0 and the wave-like paths traverse the $l_0 \pm l_*$ band, so in a sense spacetime is fuzzy.

Quantization is connected with the motion of waves and particles near the surface $l \simeq l_0$ of spacetime. Both the monotonic and oscillatory modes follow from the axiom of null 5D geodesics $(dS^2 = 0)$. Both motions are reversible, entailing a sign choice; and mathematically they differ only by the symbol (i) as used in (3). As mentioned in the previous discussion, these choices can be dropped to streamline the nomenclature. Then the null condition applied to the canonical metric (1) gives

$$\frac{dl}{(l - l_0)} = \frac{ds}{L} = \frac{mc\,ds}{h} \quad , \tag{6}$$

where $L = h / mc$ has been used (see above). These relations are general. But on approaching the membrane, it is clear that $dl \rightarrow 0$ as $l \rightarrow l_0$ (see Fig. 2). Thus (6) implies

$$\frac{dl}{(l-l_0)} \rightarrow 1, \quad ds \rightarrow L, \quad mcds \rightarrow h \quad . \tag{7}$$

That is, the 4D proper time becomes discrete with unit L, and the action is quantized in units of Planck's constant. Due to its inevitable nature, this is a kind of automatic quantization.

3. Discussion and Conclusion

The theory outlined above is based on the three principles of 5D covariance, null 5D geodesics, and the admissibility of both spacelike and timelike extra dimensions (Section 1). Presumably, particles are located in the vacuum (Section 2). Then the appropriate 5D metric is the canonical one (1), which leads to an energy density for the vacuum that is variable with the extra coordinate $x^4 = l$ and is measured by the cosmological 'constant' (2). Null geodesics include both particle-like and wave-like paths (3), which provide an explanation for wave-particle duality. The extra coordinate, as a conformal prefactor on the 4D vacuum metric, plays the role of the wave function and is consistent with the Klein-Gordon equation (4). The spacetime curvatures for the particle and the wave have opposite signs (5), so there is no gross departure from local flatness provided both components are realized. There is , however, a hypersurface which is in some ways similar to the singular one of Membrane theory, though it is now traversable by the wave and asymmetric in the vacuum energy. The dynamics in the neighbourhood of this membrane are restricted as in (6), causing the 4D proper time to become discreet and the action to be quantized (7), the latter according to the standard law involving Planck's constant.

The theory as presented above develops logically from the 5D canonical metric to 4D quantization and follows a plan suggested by recent applications of five dimensions to cosmology [5, 6]. However, it is not intended to be a replacement for standard quantum mechanics, but an alternative approach. As with any new approach, it is necessary to inquire if it runs into problems with known physics and whether new tests of it can be devised.

One subject of concern is the Heisenberg uncertainty limit. This is normally understood to mean that there is a lower limit to the measurement of increments in the 4-momenta and the coordinates, of the symbolic form $\Delta p \Delta x = h$. The question arises of whether a relation of the type just quoted can result from a 5D approach. It should be appreciated that the noted relation is rather peculiar from the viewpoint of standard 4D mechanics, because it implies a force acting parallel to the motion. However such a force arises naturally in 5D mechanics, and has been isolated for both Space-Time-Matter theory and Membrane theory [5, 7]. A short calculation shows that for metrics of canonical type there is indeed an anomalous interaction of the required sort provided spacetime is affected by the extra coordinate [for details see 16]. In fact, the resulting interaction leads to a relation of the same form as the uncertainty limit.

Another subject of interest concerns virtual particles. These are inferred to exist from data on real particles, but are not directly observed because their interactions are below the uncertainty limit noted above. This behaviour, it may be shown from preceding relations, is possible for particles that are remote from the membrane. These have ε = -1 in (1), so $\Lambda > 0$ by (2). In ordinary 3D space, two such particles at distance r from each other feel the standard acceleration $\Lambda c^2 r/3$. Therefore, a population of these particles tends to disperse, forming a diffuse vacuum with a positive energy density.

This leads to the nature of the waves that make up the other part of the 5D model. These have $\varepsilon = +1$ in (1), so $\Lambda < 0$ by (2) and the waves are constrained in 3D as well as in the fifth dimension. They exist because of the restoring properties of the ($\Lambda < 0$) vacuum. It was mentioned above that when the 4D vacuum solution in (1) is taken to be of de Sitter type, the resulting oscillation resembles a de Broglie wave. Closer examination shows that the 5D theory has other consequences close to those of de Broglie wave mechanics. The underlying reason is that any 5D metric with signature $(+---+)$ allows for velocities in ordinary 3D space that exceed the speed of light. This is true even for 5D Minkowski space, as may easily be verified. Thus 5D dynamics matches the usual interpretation of wave mechanics, where a de Broglie wave has a phase velocity that is greater than c but a group velocity that is less than c. It should be noted that in the theory outlined above, the wave repeatedly pierces the surface of 4D spacetime, leaving a hyphenated track that resembles those seen in a bubble chamber. Recalling that geodesics in 5D are defined by $dS^2 = 0$ rather than $ds^2 \geq 0$, it is seen that the waves are causal in nature.

It is possible that the interpretation of 5D quantum mechanics would be clarified if it were formulated directly in 5D terms, rather than being reduced from 5D to 4D as above. For example, a 5D wave function can be defined, with the plane-wave form $\Psi = \exp\left[i\left(Et + px + mcl \right)/h \right]$. This satisfies the 5D analog of the 4D Klein-Gordon equation (4), namely $\Box_5^2 \Psi = 0$, which gives the usual energy-normalization condition $E^2 - p^2c^2 - m^2c^4 = 0$. Alternatively, the latter relation can be obtained by inverting the usual de Broglie wavelengths to form wave numbers, which by the null-geodesic axiom obey $K^A K_A = 0$.

Observers who are unaware of the fifth dimension must perforce interpret and test a theory by what is measurable in 4D. Some topics for further investigations are as follows: (a) The canonical metric (1) can embed any vacuum solution of general relativity, and the de Sitter metric has been used to study de Broglie waves in this way [5], but other solutions should be employed to see what physics they imply. (b) The interaction of vacuum with ordinary matter will require more complicated metrics than (1), and these should be investigated. (c) The role of the scalar field $\left(g_{44} \right)$ in causing a change of 5D signature needs to be examined in detail. (d) The properties of virtual particles need closer attention, particularly in how they relate to the waves. (e) The de Broglie version of wave mechanics, in which matter waves have different phase and group velocities, implies that the vacuum has some property akin to dispersion, which may be amenable to experiment.

Perhaps the main reason for considering five dimensions, however, is that four-dimensional quantum mechanics is plagued with inconsistencies. There is no need to list these here, as several books are available that do the job [11 – 15]. As one supporter is obliged to admit, when summing up the concepts of conventional quantum physics: "the barrier to understanding is not their difficulty but their differentness" [15]. By contrast, once the existence of a fifth dimension is admitted, the rest of the theory follows logically.

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[16] In regards to the uncertainty limit $\Delta p \Delta x = h$ of the main text, it should be noted that in 5D dynamics there is generally an extra acceleration, or force per unit mass, which is associated with the extra dimension. For Space-Time-Matter theory [5] it is given by

$$f^{\mu} = -\frac{u^{\mu}}{2}\left(g_{\alpha\beta,4} u^{\alpha} u^{\beta}\right)\frac{dl}{ds} \quad .$$

This acts parallel to the velocity, and as such is a unique signature of a 5D interaction as opposed to 4D one. It can be evaluated for the canonical-type metric (1) by using $g_{\alpha\beta} = \left(l - l_0\right)^2 L^{-2} g_{\alpha\beta}\left(x^{\gamma}\right)$ where $g_{\alpha\beta}\left(x^{\gamma}\right)$ is the metric tensor of spacetime. Then $g_{\alpha\beta,4} = 2\left(l - l_0\right)^{-1} g_{\alpha\beta}$. Assuming that the 4-velocities are normalized via $g_{\alpha\beta} u^{\alpha} u^{\beta} = 1$, the scalar coupling term is $g_{\alpha\beta,4} u^{\alpha} u^{\beta} = 2\left(l - l_0\right)^{-1}$. Then the extra force is

$$f^{\mu} = -\frac{u^{\mu}}{(l-l_0)}\frac{dl}{ds} = \mp\frac{(i)}{L}u^{\mu} \quad .$$

The sign choice here has to do with reversibility, and the (i) arises because the acceleration $f^{\mu} = du^{\mu}/ds$ and the 4-velocity u^{μ} are necessarily out of phase for simple harmonic motion. Neither thing is important for the analysis, so they may be dropped, giving

$$\frac{du^{\mu}}{ds} = \frac{1}{L}\frac{dx^{\mu}}{ds} \text{ or } du^{\mu} = \frac{dx^{\mu}}{L} \quad .$$

This last equation can be employed to form the scalar quantity $du^{\mu}dx_{\mu} = ds^2/L$. This may in turn be re-expressed using the condition $dl/(l-l_0) \to 1$ wherein $ds/L \to 1$ (see the main text). The result is $du^{\mu}dx_{\mu} = L$. Substituting for $L = h/mc$ from before, and replacing the change in velocity by the change in momentum, gives

$$dp^{\mu}dx_{\mu} = h \quad .$$

This is the same type of relation as the uncertainty limit quoted above.